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Application of adaptive simulation methods in reliability analysis

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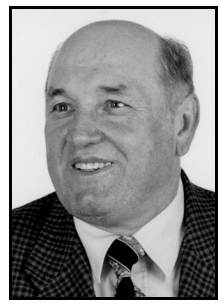

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Abstract

The cross-entropy method is a new approach to estimate rare event probabilities by Monte Carlo simulation. Application of this method for structural reliability analysis is presented. The efficiency of the approach is tested on some benchmark problems typical for reliability analysis.

Keywords: reliability analysis, simulation, Monte Carlo Method

Zastosowanie adaptacyjnych metod symulacyjnych w analizie niezawodności

Streszczenie

Metoda wzajemnej entropii (ang. cross-entropy) jest nową modyfikacją metod Monte Carlo służącą do oszacowania prawdopodobieństwa rzadkich zdarzeń. W pracy przedstawiono zastosowanie tej metody do rozwiązania zagadnienia analizy niezawodności. Jej efektywność jest oceniona na przykładach charakteryzujących się trudnościami typowymi w analizie niezawodności.

Słowa kluczowe: analiza niezawodności, symulacja, metoda Monte Carlo

1. Introduction

Evaluation of the probability of failure is an essential problem in a structural reliability analysis. Probability of failure is defined as the integral of probability density function over the region in the random variable space, for which failure occurs [5]. Due to a usually high number of random variables in real life applications, numerical integration is inefficient for this problem. In practice, first or second order approximation methods (FORM/SORM [5]) are often used to evaluate the probability of failure. However, applicability of these methods is limited only to problems satisfying certain conditions. An alternative is the Monte Carlo integration. Since a failure event is usually rare, it is common to apply importance sampling in order to facilitate calculations. Some well developed algorithms for structural reliability are available, however all of them have certain limitations. Thus, the authors found it interesting to investigate efficiency of the cross-entropy approach applied to structural reliability analysis problems. The cross-entropy method is a recently proposed approach for simulation of rare events [6]. Its application for structural reliability analysis is very simple and requires only a straightforward formulation of the problem. Basically the cross-entropy method is a form of importance sampling. It is interesting because it is based on a very elegant and efficient approach to selection of the sampling distribution by adaptive simulation without need of any additional optimisation algorithm. The results of numerical experiments presented in the paper show that application of the cross-entropy method seems to be a reasonable approach to solving structural reliability problems.

2. Structural reliability analysis by importance sampling

The time invariant structural reliability problem is usually defined as follows [5]. Uncertain structural parameters are represented by a real-valued random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, with joint probability density function $f(\mathbf{x})$. Structural performance with respect to random parameters is reflected by a limit state function $g(\mathbf{x})$. The limit state function is defined to take negative values for parameters for which failure occurs. Thus, the limit state function defines a subset in the random variable space called the failure domain $\Omega_f = \{\mathbf{x} : g(\mathbf{x}) \le 0\}$. Finally, the probability of failure is defined as

$$P_f = \int_{\Omega_f} f(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{1}$$

P_f can be estimated by means of Monte Carlo integration. Since for engineering structures a small probability of failure is desired, the crude Monte Carlo method is inefficient for such problems. Therefore, application of variance reduction techniques, like importance sampling, is usually attempted. The formula for importance sampling in evaluation of P_f is based on (1) rewritten as follows

$$P_f = \int_{\Omega_f} \frac{f(\mathbf{x})}{h(\mathbf{x})} h(\mathbf{x}) \mathrm{d}\mathbf{x} = E_h\left[I(g(\mathbf{X}) \le 0) \frac{f(\mathbf{X})}{h(\mathbf{X})} \right], \tag{2}$$

where $h(\mathbf{x})$ is an importance sampling probability density, $I(\cdot)$ is the indicator function of the failure domain, and E_h denotes the expectation operation with respect to the density $h(\mathbf{x})$. Having n independent samples $\mathbf{X}^{(k)}$, $i = 1, \ldots, n$ from the distribution with density $h(\mathbf{x})$, the expectation in (2) can be estimated from

$$\hat{P}_f = \frac{1}{n} \sum_{i=1}^{n} I\left(g(\mathbf{X}^{(k)} \le 0) \frac{f(\mathbf{X}^{(i)})}{h(\mathbf{X}^{(i)})} \right) \tag{3}$$

The optimal density function $h(\mathbf{x})$ that minimizes variance of this estimator has the following form

$$h^*(\mathbf{x}) = \begin{cases} \dfrac{f(\mathbf{x})}{P_f}, & \text{if } g(\mathbf{x}) \le 0, \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

However, this formula is rather theoretical, since generation of independent random variables requires the knowledge of the term of interest P_f. In practice, the distribution, from which samples are produced, is usually chosen to resemble the distribution with density $h^*(\mathbf{x})$.

3. Adaptive Importance sampling

In real life applications, the choice of the sampling distribution is usually reduced to a parametric family of distributions for which it is easy to generate independent random samples. Moreover, a common practice is to employ the family $F=\{f(\mathbf{x},\mathbf{v})\}, \mathbf{v}\in V$ ($\mathbf{v}$ is a vector of parameters) including the distribution of the random vector $\mathbf{X}$ for which the reliability problem is defined. The probability density function of $\mathbf{X}$ will be denoted by $f(\mathbf{x},\mathbf{v}_0)$. Obviously, the parameters $\mathbf{v}$ should be selected to facilitate the estimation of probability of failure by the formula (3).

In general, the parameters of the sampling distribution can be obtained by minimizing variance of the importance sampling estimator. This problem can be formulated as follows

$$\min_{v\in V} Var_{f(x,v)}\left[I_{\Omega_f}(\mathbf{X})\frac{f(\mathbf{X},\mathbf{v}_0)}{f(\mathbf{X},\mathbf{v})}\right], \tag{5}$$

or alternatively by

$$\min_{v\in V}\left\{E_{f(x,v)}\left[I_{\Omega_f}(\mathbf{X})\frac{f^2(\mathbf{X},\mathbf{v}_0)}{f^2(\mathbf{X},\mathbf{v})}\right]\right\}. \tag{6}$$

The above optimisation problem can be solved using the following estimate of the expectation

$$\min_{v\in V}\left\{\frac{1}{n}\sum_{i=1}^{n} I_{\Omega_f}(\mathbf{X}^{(i)})\frac{f(\mathbf{X}^{(i)},\mathbf{v}_0)}{f(\mathbf{X}^{(i)},\mathbf{v})}\frac{f(\mathbf{X}^{(i)},\mathbf{v}_0)}{f(\mathbf{X}^{(i)},\mathbf{v}_S)}\right\}, \tag{7}$$

where $f(\mathbf{x},\mathbf{v}_S)$ is the probability density of the random sample $\mathbf{X}^{(i)}$, $i=1,\ldots,n$.

A simple adaptive algorithm for estimation of failure probability based on the formula (7) can be formulated as follows:

1. Take $f(\mathbf{x},\mathbf{v}_S)=f(\mathbf{x},\mathbf{v}_0)$. Generate the sample $\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(n)}$ with density $f(\mathbf{x},\mathbf{v}_S)$ and solve the optimization problem (7). Denote the solution as $\hat{\mathbf{v}}^*$. Assume $\hat{\mathbf{v}}^*$ as the estimate of the optimal parameter vector $\mathbf{v}^*$.
2. Estimate the probability of failure with (3) taking $h(\mathbf{x})=f(\mathbf{x},\hat{\mathbf{v}}^*)$.

In order to obtain more accurate estimate of $\mathbf{v}^*$, take $\mathbf{v}_S=\hat{\mathbf{v}}^*$ and repeat the first step of the algorithm.

4. The Cross-Entropy method

An alternative method for selection of the importance sampling distribution parameters can be formulated using the cross-entropy [6]. The cross-entropy, known also as the Kullback-Leibler distance, of two probability distributions with densities $f(\mathbf{x})$ and $g(\mathbf{x})$ is defined as

$$D(f,g)=\int f(\mathbf{x})\ln\frac{f(\mathbf{x})}{g(\mathbf{x})}d\mathbf{x}. \tag{8}$$

It should be mentioned that the cross-entropy is not a distance in the formal sense, since for instance, it does not satisfy the symmetry requirement $D(f,g)\neq D(g,f)$.

The cross-entropy of the distribution h^* given by (4) and the distribution $f(\mathbf{x},\mathbf{v})\in F$ can be expressed as follows

$$\begin{aligned} D(h^*(\mathbf{x}),f(\mathbf{x},\mathbf{v})) &= \int P_f^{-1} I_{\Omega_f}(\mathbf{x}) f(\mathbf{x},\mathbf{v}_0)\ln\frac{P_f^{-1} I_{\Omega_f}(\mathbf{x}) f(\mathbf{x},\mathbf{v}_0)}{f(\mathbf{x},\mathbf{v})}d\mathbf{x} \\ &= \int P_f^{-1} I_{\Omega_f}(\mathbf{x}) f(\mathbf{x},\mathbf{v}_0)\left(\ln\left(P_f^{-1} I_{\Omega_f}(\mathbf{x}) f(\mathbf{x},\mathbf{v}_0)\right)-\ln f(\mathbf{x},\mathbf{v})\right)d\mathbf{x}. \end{aligned} \tag{9}$$

Because the distributions h^* and $f(\mathbf{x},\mathbf{v})$ should be similar, it seems reasonable to require the cross-entropy of distributions h^* and $f(\mathbf{x},\mathbf{v})$ to be minimal. Thus the optimal parameter $\mathbf{v}^*$ according to the cross-entropy criteria is the solution of the following problem

$$\min_{v\in V}\{D(h^*(\mathbf{x}),f(\mathbf{x},\mathbf{v}))\}, \tag{10}$$

or alternatively [4]

$$\max_{v\in V}\{D(\mathbf{v})=E_{f(x,v_0)}[I_{\Omega_f}(\mathbf{X})\ln f(\mathbf{X},\mathbf{v})]\}. \tag{11}$$

The solution of the problem (10) can be approximated by means of importance sampling

$$\max_{v\in V}\left\{\hat{D}_n(\mathbf{v})=\frac{1}{n}\sum_{i=1}^{n} I_{\Omega_f}(\mathbf{X}^{(i)})\frac{f(\mathbf{X}^{(i)},\mathbf{v}_0)}{f(\mathbf{X}^{(i)},\mathbf{v}_S)}\ln f(\mathbf{X}^{(i)},\mathbf{v})\right\}. \tag{12}$$

The problems (11) and (6) aim at the same goal; finding parameters of the importance sampling distribution which are optimal in some sense. Because the variance of the estimate is minimized in (6), it seems that considering the problem (11) is pointless. However the cross-entropy method is based on a much nicer optimisation problem, which can be solved even analytically in some cases [2]. Moreover, according to [4], the proof can be found that the solutions of both problems are equivalent for probability of failure going to zero. Thus, the application of the cross-entropy method is justified if only savings by solving of the easier optimisation problem compensate the use of the sub-optimal sampling density.

Because for typical problems the function D in (11) is convex and differentiable with respect to $\mathbf{v}$, the solution of (12) can be found by solving the following system of equations:

$$\nabla\hat{D}_n(\mathbf{v})=\frac{1}{n}\sum_{i=1}^{n} I_{\Omega_f}(\mathbf{X}^{(i)})\frac{f(\mathbf{X}^{(i)},\mathbf{v}_0)}{f(\mathbf{X}^{(i)},\mathbf{v}_S)}\nabla\ln f(\mathbf{X}^{(i)},\mathbf{v})=\mathbf{0}. \tag{13}$$

The above system of equations takes very simple form for independent random variables. For instance consider a set of independent normal variables with joint probability density functions given by

$$\varphi(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\sigma})=\prod_{i=1}^{n}\frac{1}{\sigma_i\sqrt{2\pi}}\exp\left(-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}\right), \tag{14}$$

where $\boldsymbol{\mu}=\{\mu_1,\ldots,\mu_n\}$ are mean values and $\boldsymbol{\sigma}=\{\sigma_1,\ldots,\sigma_n\}$ are standard deviations of the components. The gradient of the logarithm of the probability density has elements of the following form

$$\frac{\partial\ln\varphi(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\sigma})}{\partial\mu_i}=\frac{\mu_i-x_i}{\sigma_i^2},\qquad i=1,\ldots n, \tag{15}$$

$$\frac{\partial\ln\varphi(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\sigma})}{\partial\sigma_i}=\frac{(x_i-\mu_i)^2-\sigma_i^2}{\sigma_i^3},\qquad i=1,\ldots n. \tag{16}$$

Thus, the following set of equations for optimal parameters of (14) can be obtained by substituting formulas (15) and (16) into (13):

$$\hat{\mu}_i^* = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i I_{\Omega_f}\left(\mathbf{X}^{(i)}\right)\frac{\varphi\left(\mathbf{X}^{(i)},\mathbf{0},\mathbf{1}\right)}{\varphi\left(\mathbf{X}^{(i)},\boldsymbol{\mu}_S,\boldsymbol{\sigma}_S\right)}}{\frac{1}{n}\sum_{i=1}^{n} I_{\Omega_f}\left(\mathbf{X}^{(i)}\right)\frac{\varphi\left(\mathbf{X}^{(i)},\mathbf{0},\mathbf{1}\right)}{\varphi\left(\mathbf{X}^{(i)},\boldsymbol{\mu}_S,\boldsymbol{\sigma}_S\right)}}, \tag{17}$$

$$\hat{\sigma}_i^{*2} = \frac{\frac{1}{n}\sum_{i=1}^{n} \left(x_i-\mu_i\right)^2 I_{\Omega_f}\left(\mathbf{X}^{(i)}\right)\frac{\varphi\left(\mathbf{X}^{(i)},\mathbf{0},\mathbf{1}\right)}{\varphi\left(\mathbf{X}^{(i)},\boldsymbol{\mu}_S,\boldsymbol{\sigma}_S\right)}}{\frac{1}{n}\sum_{i=1}^{n} I_{\Omega_f}\left(\mathbf{X}^{(i)}\right)\frac{\varphi\left(\mathbf{X}^{(i)},\mathbf{0},\mathbf{1}\right)}{\varphi\left(\mathbf{X}^{(i)},\boldsymbol{\mu}_S,\boldsymbol{\sigma}_S\right)}}. \tag{18}$$

With the set of equations (13), the algorithm proposed for minimum variance criteria can be adapted easily for the probability of failure estimation using the cross-entropy optimal parameters.

It should be mentioned that results (17) and (18) are equivalent to the well-known approach employing the sampling distribution with the same first or second moments as distribution (4) [1].

5. Adaptive algorithm

When the probability of failure is vary small the basic cross-entropy algorithm is inefective and the following adaptive algorithm proposed in [4] should be employed. Here it is presented with notation used in structural reliability problems.

The parameters needed to be specified for this algorithm are ρ, $\alpha > 1$, $\delta > 0$ and n which is the number of simulations made in each iteration. The algorithm proceeds as follows:

1. Initialization of the algorithm. Set $\rho_0 = \rho$. For a sample $\mathbf{X} = \mathbf{X}^{(1)},\ldots,\mathbf{X}^{(n)}$ from the distribution with probability density $f\left(\mathbf{x},\mathbf{v}_0\right)$, ($\mathbf{v}_0 = \hat{\mathbf{v}}_0$) evaluate a sample quantile ρ of the random variable $Y_i = g\left(\mathbf{X}^{(i)}\right)$, and denote it by $\hat{\gamma}_0$:

$$P\left[g\left(\mathbf{X}\right) \le \hat{\gamma}_0\right] \le \rho_0, \tag{19}$$

Set $t = 1$.

2. Use the same sample $\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(n)}$ to solve the optimisation problem:

$$\hat{\mathbf{v}}_t = \arg\max_{v\in V}\left\{\frac{1}{n}\sum_{i=1}^{n} I_{\{g(\mathbf{x})\le\hat{\gamma}_{t-1}\}}\left(\mathbf{X}^{(i)}\right)\frac{f\left(\mathbf{X}^{(i)},\mathbf{v}_0\right)}{f\left(\mathbf{X}^{(i)},\hat{\mathbf{v}}_{t-1}\right)}\ln f\left(\mathbf{X}^{(i)},\mathbf{v}\right)\right\}, \tag{20}$$

where $I_{\{g(\mathbf{x})\le\hat{\gamma}_{t-1}\}}$ is the indicator function of the set for which $g\left(\mathbf{x}\right) \le \hat{\gamma}_{t-1}$.

3. Generate a new sample $\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(n)}$ from probability density function $f\left(\mathbf{x},\hat{\mathbf{v}}_t\right)$, and set $\rho_t = \rho$.

4. For the current sample $\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(n)}$, evaluate quantile ρ_t of the random variable $Y_i = g\left(\mathbf{X}^{(i)}\right)$ and denote it by $\hat{\gamma}_t$

5. If $\hat{\gamma}_t \le 0$ set $\hat{\gamma}_t = 0$ and find the estimate of the parameter $\mathbf{v}^*$ denoted by $\hat{\mathbf{v}}_t^*$, by solving the following problem

$$\hat{\mathbf{v}}_t^* = \arg\max_{v\in V}\left\{\frac{1}{n}\sum_{i=1}^{n} I_{\Omega_f}\left(\mathbf{X}^{(i)}\right)\frac{f\left(\mathbf{X}^{(i)},\mathbf{v}_0\right)}{f\left(\mathbf{X}^{(i)},\hat{\mathbf{v}}_{t-1}\right)}\ln f\left(\mathbf{X}^{(i)},\mathbf{v}\right)\right\}. \tag{21}$$

Go to step 7.

6. If $\hat{\gamma}_t > 0$ check if there exist $\overline{\rho}$ such that $\overline{\gamma}$, being a $\overline{\rho}$-quantile of the random sample $g\left(\mathbf{X}^{(i)}\right),\ldots,g\left(\mathbf{X}^{(i)}\right)$, satisfies $\overline{\gamma} \le \max\left\{0,\hat{\gamma}_{t-1}-\delta\right\}$:

   If $\overline{\rho}$ exists and $\overline{\rho} = \rho_t$, then set $t = t+1$ and repeat the iterations from step 2;

   If $\overline{\rho}$ exists and $\overline{\rho} < \rho_t$, then set $\rho_t = \overline{\rho}$ and return to the step 4;

   Otherwise (for example when $\overline{\rho}$ does not exist), increase the number of simulations in the step as $n = \alpha n$ and return to the step 3.

7. Estimate the probability of failure with importance sampling using the sampling density function $f\left(\mathbf{x},\hat{\mathbf{v}}_t^*\right)$.

In [4] it was proven that the algorithm converges in a finite number of iterations to the solution of the problem (11) with probability 1.

6. Numerical examples

The efficiency of the algorithm presented in the preceding section was tested on the benchmark problems used in [3] to evaluate the performance of various importance sampling algorithms. The limit state functions used in the benchmark problems were selected in order to evaluate the algorithms with respect to difficulties characteristic for reliability analysis.

The following examples were computed in the standard normal space. The sampling distributions were chosen from multivariate normal family with the identity covariance matrix. The mean values of the components were adjusted according to (17). Compared to the algorithm presented above, a slightly modified computational scheme was used. The parameter ρ was kept constant during the execution of the algorithm. In each iteration the simulations were performed until distribution parameters were estimated with required accuracy $\hat{e}_v$. For each problem some initial runs of the algorithm were performed in order to select a combination of ρ and $\hat{e}_v$ allowing to estimate the probability of failure with reasonable computational effort. However, the aim of the initial runs was the selection of a reasonable set of parameters, not the optimisation of them. Then, each problem was solved several hundred of times and a mean number $\overline{n}$ of the samples required for estimation of the probability of failure with 0.1 coefficient of variation was calculated. Some of the numbers presented in the following may seem to indicate that the considered algorithm is inefficient. However it should be remembered that the given number of simulations shows the total effort required for solving the problem, including estimation of the sampling distribution parameters.

Example 1. High number of variables and probability levels

The aim of this example is to evaluate performance of the algorithm for various levels of probability and for different number of random variables. The limit state function used in this example is the n-dimensional hypersurface:

$$g_1 = \sqrt{m}\beta - \sum_{j=1}^{m} U_j, \tag{22}$$

where U_j, $j = 1,\ldots,m$ are the standard normal random variables and β is the reliability index.

The numerical experiments were performed for the following values of the parameters: $\beta = 1.0$, $\beta = 5.0$, $\beta = 10.0$ and $m = 2$, $m = 10$, $m = 50$. The results of the experiments are presented in the Table 1. As we can see the required numerical effort increases with the number of random variables and the value of reliability index. The reported limit exceedance (l.e.) for $\beta = 10.0$ and $m = 10{,}50$ means that satisfied accuracy of the estimate was not obtained after the allowed number of simulations.

Tab. 1. Algorithm parameters and total number of simulations. Example 1
Tab. 1. Parametry algorytmu i liczba symulacji. Przykład 1

		β=1.0	β=5.0	β=10.0
m=2	ρ	0.3	0.3	0.3
	$\hat{e}_v$	0.0389	0.0473	0.5527
	$\overline{n}$	230	884	1912
m=10	ρ	0.5	0.5	0.45
	$\hat{e}_v$	0.5	0.5	0.05
	$\overline{n}$	360	1375	3181 16% l.e.
m=50	ρ	0.5	0.5	0.45
	$\hat{e}_v$	0.1	0.1	0.05
	$\overline{n}$	882	2968	8569 38% l.e.

Example 2. Nonlinearity of the limit state function and probability levels

In this example, the limit state function is given by the following formula:

$$g_2 = \pm\sum_{j=1}^{m} X_j \mp C, \tag{23}$$

where the random variables X_j, $j = 1,\ldots,m$ are independent and exponentially distributed with the parameter λ. After transformation to the standard normal space, the considered limit state function becomes highly nonlinear

$$G_2 = \pm\frac{1}{\lambda}\sum_{j=1}^{m} \ln\left(\Phi\left(-U_j\right)\right) \pm C, \tag{24}$$

where U_j, $j = 1,\ldots,m$ are standard normal random variables, and Φ is the inverse of the normal cumulative distribution function. The problem was computed for $\lambda = 1$ and $m = 20$ and different values of C, which are presented in Table 2.

For negative values of C the limit state function has negative curvature. Results presented in Tab. 2 shows that the numerical effort grows rapidly with increasing value of C. It is due to the fact that when C is increasing, then the distance of the failure domain from the origin grows making the iteration process longer, and the important region of the failure domain shrinks causing estimation of the distribution parameters more difficult. Taking this into account the presented results can be considered satisfying.

Tab. 2. Algorithm parameters and total number of simulations. Example 2
Tab. 2. Parametry algorytmu i liczba symulacji. Przykład 2

C	-16.175	-11.077	-8.951	-7.453	-6.277
P_f	0.20	10^{-2}	10^{-3}	10^{-4}	10^{-5}
β	0.841	2.328	3.093	3.722	4.268
ρ	0.475	0.3	0.3	0.3	0.2
$\hat{e}_v$	0.313	0.024	0.015	0.01	0.005
$\overline{n}$	493	668	3014	5512	8505 1.0% l.e.
$\overline{P}_f$	0.20	$1.0\cdot10^{-2}$	$0.97\cdot10^{-3}$	$0.98\cdot10^{-4}$	$0.97\cdot10^{-5}$

C	25.90	31.856	36.720	41.050
P_f	10^{-1}	10^{-2}	10^{-3}	10^{-4}
β	1.282	2.328	3.093	3.722
ρ	0.5	0.4	0.3	0.3
$\hat{e}_v$	0.398	0.0316	0.0562	0.0562
$\overline{n}$	1107	1408	8020 4.0% l.e.	13922 13.3% l.e.
$\overline{P}_f$	0.20	$1.0\cdot10^{-2}$	$0.97\cdot10^{-3}$	$0.98\cdot10^{-4}$

If C is positive then the failure surface has positive curvature. In this case significant mass of the probability is highly spread. Importance sampling based on unimodal distributions is ineffective for this class of problems. Results of the experiments (Tab. 2) show that cross-entropy method is not effective for this problems as well. Estimates with assumed accueracy required very high number of simulations, morever some of the results are biased. The cross entropy method in implemented form should not be employed for problems exihibiting similar properties.

Example 3. Noisy limit state function

The limit state function in this example is defined by:

$$g_3 = X_1 + 2X_2 + 2X_3 + X_4 - 5X_5 - 5X_6 + 0.001\sum_{i=1}^{6}\sin\left(X_i\right), \tag{24}$$

where all random variables are independent and log-normally distributed. Random variables X_1, X_2, X_3, X_4, have mean values 120.0 and standard deviations 12.0. X_5 has mean 50.0 and standard deviation 15.0, and X_6 has mean 40.0 and standard deviation 12.0. Probability of failure defined by g_3 is $P_f = 1.23\cdot10^{-2}$. The last term in (24) introduces small perturbation, so called noise. The noisy limit state surface is higly irregular, thus gradient approximation methods like FOR/SORM have limited use in this case. For this kind of problems application of the adaptive simulation algorithms is the default approach. In practice, noise is present very often, because it occurs not only due to properties of the problem but also as result of numerical errors. In this example analysis were performed with the following values of the algorithm parameters: ρ=0.4 and $\hat{e}_v$ =0.398. The average number of simulations necessary for estimation with 10% coefficient of variation was 710, of which 564 was used for final estimation of failure probability. Average value of estimated probability of failure was P_f=$1.21\cdot10^{-2}$. The number of simulations required for final estimation is comparable with number 500 which is given in [3], while the number of 150 simulations required for estimation of sampling distribution parameters seems acceptable.

7. Conclusion

In the paper the application of the cross-entropy to estimation of structural failure probability was outlined. The performance of the algorithm was tested on problems with difficulties typical for reliability analysis. The obtained results show that if the numerical effort is considered, the cross-entropy is not an alternative for importance sampling methods using the design point (compare with results in (Engelund & Rackwitz (1993)). However, for problems where the design point cannot be found easily, this method seems to be a reasonable approach. The numerical effort is high but still acceptable and it seems that it can be rewarded by the ease of implementation of the method for a specific problem.

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