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Hardware Implementation of Simplified Representation of Convolutive Numbers

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Abstract

In the article, simplified representation of convolutive representation is presented. The arithmetic operators for addition and multiplication were defined. On the ground of addition and multiplication, the operators for subtraction and division were educed. The comparison between arithmetic operators implementation and simplified representation was carried.

Keywords: convolutive representation, convolutive number, convolution.

Sprzętowa implementacja reprezentacji uproszczonej liczb splotowych

Streszczenie

W artykule przestawiono formę uproszczoną reprezentacji splotowej składającej się z dwóch elementów wartości średniej i wariancji. Zdefiniowano dla niej operatory arytmetyczne dodawania i mnożenia. Na bazie operatorów dodawania i mnożenia wyprowadzono operatory odejmowania i dzielenia. Przeprowadzono porównanie możliwości implementacji operatorów arytmetycznych liczb rzeczywistych oraz reprezentacji uproszczonej w układzie Xilinx Spartan 3. W porównaniu z liczbami rzeczywistymi implementacja sprzętowa reprezentacji splotowej wymaga trzy razy więcej elementów, przy czym jak wykazała implementacja iloczynu skalarnego dla różnych wymiarów przestrzeni, stosunek ten nie zależy od liczby koniecznych do przeprowadzenia obliczeń.

Słowa kluczowe: reprezentacja splotowa, liczby splotowe, splot.

1. Introduction

Convolution operator is used in many fileds of knowlegde. Among other things in operator calculation [1], in differential equations theory, approximation theory [2], and the like. It has great importance in statistics as well, where it is used to count the sum of independent random variables [3]. Connection of the convolution and random variables is the reason to use the convolution in fuzzy arithmetic. Mareŝ was the first, who brought to our attention the possibility of using convolution as an addition operator in fuzzy arithmetic, originally in integral form [5], and later in sum form [6]. A description of convolution as an addition operator may also be found in [7, 8].

Mareŝ has demonstrated the group properties of convolution using an equivalence relation. This indicates the existence of opposite numbers – "reductors". He also presented a problem with the definition of multiplication in convolutive representation, in conformity with the principle of extension, and gave a formula for alternative multiplication by multiple additions.

Any number in convolutive representation can be described with the aid of two parameters: average value and variance. Unfortunately neither of these parameters indicates symmetry of distribution, which is why we are able to split the variance into the sum of two parameters characterizing symmetry. These parameters together with average value will constitute the simplified representation of convolutive representation.

Simplified represenation, thanks to the limitation of array length to three elements needed to remember convolutive number, simplifies the computation a lot. Additionaly, what would be shown below, it simplifies the way of performing of addition, subtraction, multiplication and division. Thanks to this property, the quantity of resources needed to hardware realization reduces in significant way.

2. Number addition

A normal distribution curve can be described with two parameters: average value $\bar{x}$ and variance σ^2. Observing the behaviour of the average value when adding two numbers in convolutive representation, we can note that the average value of resulting number equals the sum of the average values of these numbers.

This means that we can replace a normally distributed number by its average value and variance, creating the ordered pair $(\bar{x};\sigma^2)$. The variance can be replaced by two parameters α and β, with their sum of equal variance: $\alpha+\beta=\sigma^2$. This gives us the ordered triple: $(\bar{x};\alpha;\beta)$. Parameter α determines the value of variance from the left side, and value β from the right side of average value.

Addition for this number can be defined as follows:

$$\left[(\bar{x}_a;\alpha_a;\beta_a)+(\bar{x}_b;\alpha_b;\beta_b)\right]\equiv(\bar{x}_a+\bar{x}_b;\alpha_a+\alpha_b;\beta_a+\beta_b) \quad (2)$$

Here $\bar{x}$, α and β could be any real numbers. Due to this, commutativity along with associativity of this operation result from the commutativity and associativity of real number addition. The neutral element is (0;0;0), and the inverse element is $(-\bar{x};-\alpha;-\beta)$. Hence the algebraic structure $(+;(-\bar{x};-\alpha;-\beta))$ is close to the commutative group.

An ordered triple $(\bar{x};\alpha;\beta)$ is a simplified form of convolutive representation. We can note that in simplified form we may have ordered triples with the sum of parameters α i β resulting in a negative value of variance. In convolutive representation given by Mareŝ this kind of number is absent; however, he demonstrated their existence. Such numbers could be determined when looking for opposite elements in accordance with the principles of algebra.

3. Number multiplication

There are two multiplications in convolutive representation as proposed by Mareŝ. One is consistent with the extension principle and the other could be referred to as multiplication by multiple addition [6]. The first multiplication is not distributive over addition. Using this multiplication would significantly limit the possibility to use convolutive representation.

Similarly, we can define the operator of integer numbers multiplication by fuzzy numbers, as the multiple addition of fuzzy numbers only. The properties of commutativity and associativity of such a defined multiplication operator stem from commutativity and associativity of addition. The neutral element is the value one. Moreover, multiplication is distributive over addition. The multiplication operator has, however, one serious limitation. In a set of integers it is impossible to calculate the inverse element, although this does not mean that division is impossible.

Multiplication of numbers through addition can be expressed also in representation $(\bar{x};\alpha;\beta)$:

$$a\cdot(\bar{x};\alpha;\beta)=\underbrace{(\bar{x};\alpha;\beta)+(\bar{x};\alpha;\beta)+\ldots+(\bar{x};\alpha;\beta)}_{a}=(a\bar{x};\mathrm{a}\alpha;\mathrm{a}\beta) \quad (4)$$

where $a\in Z$. Taking into account the form of notation, we can easily extend the multiplication operation from the set of integer numbers onto the set of real numbers. This operation can be extended for multiplication by any number in the representation $(\bar{x};\alpha;\beta)$:

$$(\bar{x}_1;\alpha_1;\beta_1)\cdot(\bar{x}_2;\alpha_2;\beta_2)=(\bar{x}_1\bar{x}_2;\bar{x}_1\alpha_2+\bar{x}_2\alpha_1-\alpha_1\alpha_2;\bar{x}_1\beta_2+\bar{x}_2\beta_1+\beta_1\beta_2) \quad (6)$$

Multiplication satisfy the following conditions: commutativity, associavity, neutral element and inverse element.

Algebraic structure containing of set of a ordered triples $(\bar{x};\alpha;\beta)$ and multiplication operations is a monoid. It can be observed, that all elements of numbers' set in simplified representation, except: $(0;0;0)$, $(a;a;c)$, $(a;b;-a)$ and $(a;a;-a)$, have inverse element. For these numbers assigned inverse element will contain at least one infinity. For example, the inverse element for $(a;a;-a)$ will be $\left(\frac{1}{a};-\infty;\infty\right)$.

From deliberation above follows that the algebraic structure consisting of a set of ordered triples $(\bar{x};\alpha;\beta)$ and the multiplication operation is close to an abelian group. Moreover, the multiplication operator is distributive over addition $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$. As a consequence the set of all ordered triples $(\bar{x};\alpha;\beta)$ with the operations of addition and multiplication is close to the field.

4. Comparison of hardware implementation of common arithmetic operators with the operators in simplified representation

Thanks to the simplification of arithmetic operations it is possible to realize the addition in the simplified representation by performing only three additions on fixed point numbers, and the multiplication by performing seven multiplications, three additions and one subtraction. These are numbers of operation which are significantly lower than number of operations necessary to perform in the convolutive representation, where to addition n^2 multiplications and n^2-n additions are needed, n is the length of the array of values that describes convolutive number. An assumption has to be made that the addition can be realized by using cycle convolution. For example, for n = 5 we have 25 multiplications and 20 additions. But it has to be remarked that n hardly ever reaches so small values.

To compare the common arithmetic operators with simplified representation operators the number of NAND gates and flip-flops, which are necessary to hardware realization of these operators, was examined.

To achieve this goal Agility Compiler by Celoxica was used. Compilation was done for Xilinx Spartan 3. The result is shown in table 1.

Tab. 1. Comparison of number of the elements needed to perform arithmetic operations
Tab. 1. Porównanie liczby elementów niezbędnych dla wykonania działań arytmetycznych

No.	Operator	Fixed point numbers			Simplified representation numbers		
		NAND gates	flip flops	ALUs	NAND gates	flip flops	ALUs
1	+	404	36	0	1148	100	0
2	-	404	36	0	1148	100	0
3	*	272	36	1	1280	100	7
4	/	4438	36	0	12670	100	8

To examine the dependencies of the proper number of elements needed to realize the arithmetic operation in he common operators and simplified representation operators scalar product of two vectors was used.

The formula for scalar product is one of the most common in the methods based on vector calculus. The ratio of NAND gates number essential to scalar product realization in the convolutive representation to the NAND gates number necessary to realize the scalar product for fixed point numbers. The similar ratio was introduced for the flip-flops and the maximal number of logic levels between the flip-flops. The result is shown on the figure 2.

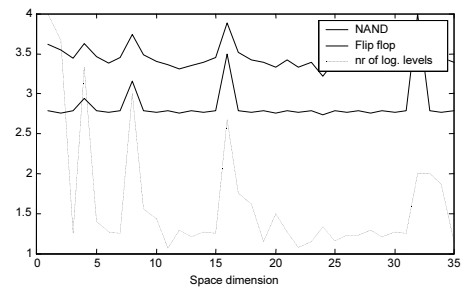

Fig. 2. Dependency of ratio of NAND gates, flip-flops and logic levels number to the space dimension
Rys. 2. Zależność stosunku liczby bramek NAND, przerzutników Flip flop, oraz liczby poziomów logicznych od wymiaru przestrzeni

The number of NAND gates essential to realize the arithmetic operations in simplfied representation is only 3,5 times higher. Similar situation is for flip-flops – this number is 3 times higher. For the number of logic levels it is less, but bigger oscillation can be observed.

Such oscillation is caused by the use of optimalization algorithms. For the space dimensions, which are powers of 2, these algorithms act 4 times more efficient for fixed point numbers and only 2 times more efficient for numbers in the convolutive representation.

5. Summary

In the article, the simplified way of notation and performing arithmetic operations on convolutive numbers was presented. Thanks to this approach, the number of elements essential to the realization of computations is multiple lower. In comparison to real numbers implementation this number does not differ significantly. The number of elements in this operation is only 4 times higher than number of operations on fixed point numbers. The compilation of a program which computes the scalar product of two vectors for Xilinx Spartan 3 was performed.

6. References

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Artykuł recenzowany