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Lyapunov, Sylvester and Riccati equations with some applications

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Abstract

An overview of the differential and algebraic Lyapunov, Sylvester and Riccati equations is presented. A special attentions is focused on relationship between the equations and their applications in control systems theory. The well-known classical Cayley-Hammilton theorem is extended for the Lyapunov time-varying systems.

Keywords: algebraic, differential, equation, Lyapunov, Sylvester, Riccati, solution, application.

Równania Lapunowa, Sylvestera i Riccatiego oraz ich niektóre zastosowania

Streszczenie

W pracy podano przegląd różniczkowych i algebraicznych równań Lapunowa, Sylvestera i Riccatiego. Szczegolna uwage zwrócono na zwiazki występujące miedzy tymi równaniami oraz ich zastosowaniami w teorii sterowania i systemow. Uogólniono klasyczne twierdzenie Cayleya - Hamiltona na układy Lapunowa o zmiennych w czasie parametrach.

Słowa kluczowe: algebraiczne, różniczkowe, równanie ,Lapunowa, Sylvestera, Riccatiego, rozwiązanie, zastosowanie.

1. Introduction

The Lyapunov stability theory for continuous-time and discrete-time systems is well known for many years [2, 4, 8]. The theory is based on the algebraic Lyapunov equation which has originated the linear matrix inequalities (LMI) approach. The equation is also used in computation of the controllability and observability Gremmians and the H_2-norm of the transfer matrix of the linear system.

The differential and algebraic Sylvester equations play also important roles in control systems theory. It will be shown that the differential Sylvester equation is the starting point for development of the linear continuous-time Lyapunov systems. The algebraic Sylvester equation is used in designing of the Luenberger state observers of linear continuous systems. It is well-known that the differential and algebraic Riccati equations play crucial roles in the optimal control of linear time-varying and time-invariant systems with quadratic performance indices (cost functions).

In this paper an overview of the Lyapunov, Sylvester and Riccati equations will be presented. A special attention will be focused on the relationship between the equations and their applications in control systems theory. The well-known classical Cayley-Hamilton theorem will be extended for the Lyapunov tame-varying systems.

2. Lyapunov equations

Consider the continuous-time linear systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad (2.1)$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$ are the state, input and output vectors and $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$.

The system (1) (or equivalently the pair (A,B)) is controllable if and only if

$$rank\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n \qquad (2.2)$$

The system (1) (or equivalently the pair (A,C)) is observable if and only if

$$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \qquad (2.3)$$

Definition 2.1. The matrix equations

$$XA + A^T X = Q, \quad A \in R^{n\times n}, \; X \in R^{n\times n}, \; Q \in R^{n\times n} \qquad (2.4a)$$

and

$$AX + XA^T = -Q \qquad (2.4b)$$

are called the Lyapunov equations.

Theorem 2.1. Let A be a asymptotically stable matrix and let Q be symmetric, positive definite or semidefinite. Then the unique solution X of (4a) is given by

$$X = \int_0^\infty e^{A^T t} Q e^{At} dt \qquad (T\text{-denotes the transpose}) \qquad (2.5a)$$

and the unique solution X of (4b) is given by

$$X = \int_0^\infty e^{At} Q e^{A^T t} dt \qquad (T\text{-denotes the transpose}) \qquad (2.5b)$$

Theorem 2.2. Let X be the solution of the Lyapunov equation (6)

$$XA + A^T X = -C^T C \qquad (2.6)$$

Then the followings hold:

1. If X is a symmetric positive definite matrix and the pair (A,C) is observable then A is an asymptotically stable matrix
2. If A is an asymptotically stable matrix and pair (A,C) is observable then X is a symmetric positive definite matrix
3. If A is an asymptotically stable matrix and X is a symmetric positive definite matrix then the pair (A,C) is observable.

Dual results we have for a controllable pair (A,B) satisfying the condition (2).

Theorem 2.3. Let X be a solution of the Lyapunov equation

$$AX + XA^T = -BB^T \tag{2.7}$$

Then the followings hold:

1. If X is a symmetric positive definite matrix and the pair (A,B) is controllable then A is an asymptotically stable matrix
2. If A is an asymptotically stable matrix and the pair (A,B) is controllable then X is a symmetric positive definite matrix
3. If A is an asymptotically stable matrix and X is a symmetric positive definite matrix then the pair (A,B) is controllable.

Definition 2.2. Let $A \in R^{n\times n}$ be a state matrix. Then the matrix

$$G_C = \int_0^\infty e^{At} BB^T e^{A^T t} dt \tag{2.8}$$

is called the controllability Grammian and the matrix

$$G_O = \int_0^\infty e^{A^T t} C^T C e^{At} dt \tag{2.9}$$

Is called the observabiliy Grammian.

Theorem 2.4. Let A be an asymptotically stable matrix. Then the controllability Grammian (8) satisfies the Lyapunov equation

$$AG_C + G_C A^T = -BB^T \tag{2.10}$$

and is a symmetric positive definite if and only if the pair is controllable.

The observability Grammian G_O satisfies the Lyapunov equation

$$G_O A + A^T G_O = -C^T C \tag{2.11}$$

and is symmetric positive definite if and only if the pair (A,C) is observable.

Using the controllability Grammian (8) or the observability Grammian (9) we can compute the H_2-norm of the transfer matrix of asymptotically state system (1) by solving the Lyapunov equation (10) or (11) and computing either trace ($CG_C C^T$) or trace ($B^T G_O B$) [2, 5].

Similar results can be obtained for discrete-time linear systems [2, 11].

3. Sylvester equations

3.1. Differential equations

Consider the differential time-varying Sylvester equation

$$\dot{X}(t) = AX(t) + X(t)B(t) + C(t) \tag{3.1}$$

with the initial condition

$$X(t_0) = X_0 \tag{3.2}$$

where $A(t)$, $B(t)$, $C(t)$, $X(t)$ are continuous-time $n\times n$ matrices and t_0 is a given initial instant.

Theorem 3.1. The unique solution of (1) satisfying the initial condition (2) has the form

$$X(t) = X_1(t) X_0 X_2(t) + X_1(t)\left[\int_{t_0}^{t} X_1^{-1}(\tau) C(\tau) X_2^{-1}(\tau) d\tau\right] X_2(t) \tag{3.3}$$

where $X_1(t)$ and $X_2(t)$ are solutions of the equations

$$\dot{X}_1(t) = A(t) X_1(t) \quad \text{with } X_1(t_0) = I_n \tag{3.4a}$$

$$\dot{X}_2(t) = X_2(t) B(t) \quad \text{with } X_2(t_0) = I_n \tag{3.4b}$$

Proof is given in [4].

The Sylvester equation (1) is closely related with Lyapunov systems described by the equations

$$\dot{X}(t) = A_0(t) X(t) + X(t) A_1(t) + B(t) U(t) \tag{3.5a}$$

$$Y(t) = C(t) X(t) + D(t) U(t) \tag{3.5b}$$

where $X(t)$ is an $n\times n$ state matrix, $U(t)$ is an $m\times n$ input matrix, $Y(t)$ is a $p\times n$ output matrix and $A_0(t), A_1(t), B(t), C(t), D(t)$ are continuous-time real matrices with appropriate dimensions.

Applying Theorem 1 to (5a) we obtain the following.

Theorem 3.2. The solution of (5a) satisfying the initial condition $X(t_0) = X_0$ has the form

$$X(t) = \Phi_1(t,t_0) X_0 \Phi_2(t_0,t) + \Phi_1(t_0,t_0)\left[\int_{t_0}^{t} \Phi(t_0,\tau) B(\tau) U(\tau) \Phi_2(\tau,t_0) d\tau\right] \Phi_2(t_0,t) \tag{3.6}$$

where $\Phi_1(t,t_0) = X_1(t) X_1^{-1}(t_0)$, $\Phi_1(t_0,t) = X_2^{-1}(t_0) X_2(t)$ and $X_1(t), X_2(t)$ are solution of the equations

$$\begin{aligned} \dot{X}_1(t) &= A_0(t) X_1(t) \quad &\text{with } X_1(t_0) = I_n \\ \dot{X}_2(t) &= X_2(t) A_1(t) \quad &\text{with } X_2(t_0) = I_n \end{aligned} \tag{3.7}$$

In [11] necessary and sufficient conditions for controllability and observability of the Lyapunov systems are established. We shall show that the classical Cayley-Hamilton theorem can be extended for the Lyapunov systems.

The Kronecker product $A \otimes B$ of matrices $A = [a_{ij}] \in R^{m\times n}$ and $B \in R^{p\times q}$ is by definition the block matrix [4]

$$A \otimes B = [a_{ij} B]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \in R^{mp\times nq} \tag{3.8}$$

Using (8) for (5a) we may define the matrix

$$\bar{A}(t) = A_0(t) \otimes I_n + I_n \otimes A_1^T(t) \in R^{\bar{n}\times\bar{n}} \quad (\bar{n} = n^2) \quad (T \text{ denotes the transpose}) \tag{3.9}$$

and its characteristic polynomial

$$p_{\bar{A}}(\lambda) = \det[I_{\bar{n}}\lambda - \bar{A}(t)] = s^{\bar{n}} + a_{\bar{n}-1} s^{\bar{n}-1} + \dots + a_1(t)\lambda + a_0(t) \tag{3.10}$$

Theorem 3.3. Let $A(t) = A_0(t) + A_1(t)$ and $p_A(\lambda) = \det[I_n\lambda - A]$. If

$$I_n \otimes A_1^T(t) = A_1(t) \otimes I_n \tag{3.11}$$

then

$$p_{\bar{A}}(\lambda) = (p_A(\lambda))^n \tag{3.12}$$

and

$$p_{\bar{A}}^{(k)}\left(A(t)\right)=0 \quad \text{for } k=0,1,\dots,n-1 \tag{3.13}$$

where

$$p_{\bar{A}}^{(k)}(\lambda)=\frac{d^k p_{\bar{A}}(\lambda)}{d\lambda^k} \quad \text{for } k=1,2,\dots,n-1$$

Proof. If (11) holds then

$$\bar{A}(t)=A_0(t)\otimes I_n+I_n\otimes A_1^T(t)=\left(A_0(t)+A_1(t)\right)\otimes I_n=A(t)\otimes I_n \tag{3.14}$$

and

$$\begin{aligned} p_{\bar{A}}(\lambda)&=\det\left[I_{\bar{n}}\lambda-\bar{A}(t)\right]=\det\left[\left(I_n\lambda-A(t)\right)\otimes I_n\right]=\\ &=\left(\det\left[I_n\lambda-A(t)\right]\right)^n=\left(p_A(\lambda)\right)^n \end{aligned} \tag{3.15}$$

since $\det\left[A\otimes B\right]=\left(\det A\right)^n\left(\det B\right)^n$ for $A,B\in R^{n\times n}$ [4].

From Cayley-Hamilton theorem applied to the matrix $A(t)$ [4] we have $p_A\left(A(t)\right)=0$ and from (12) we obtain

$$p_{\bar{A}}\left(A(t)\right)=\left(p_A\left(A(t)\right)\right)^n \tag{3.16}$$

The equality (16) implies (13).

Remark 3.1 It is easy to show that the condition (11) is met if and only if the matrix $A_1(t)$ is a scalar matrix, i.e., $A_1(t)=a(t)I_n$, $a(t)\neq 0$.

Example 3.1. For the matrices

$$A_0(t)=\begin{bmatrix}0 & 1\\ t & -e^{-t}\end{bmatrix},\quad A_1(t)=\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-t}\end{bmatrix} \tag{3.17}$$

the condition (11) is satisfied since

$$I_n\otimes A_1^T(t)=A_1(t)\otimes I_n=\begin{bmatrix}e^{-t} & 0 & 1 & 0\\ 0 & e^{-t} & 0 & 1\\ t & 0 & 0 & 0\\ 0 & t & 0 & 0\end{bmatrix}$$

In this case the characteristic polynomial $p_A(\lambda)$ and $p_{\bar{A}}(\lambda)$ have

$$p_A(\lambda)=\det\left[I_n\lambda-A(t)\right]=\begin{vmatrix}\lambda-e^{-t} & -1\\ -t & \lambda\end{vmatrix}=\lambda^2-e^{-t}\lambda t$$

$$\begin{aligned} p_{\bar{A}}(\lambda)&=\det\left[I_n\lambda-\bar{A}(t)\right]=\begin{bmatrix}\lambda-e^{-t} & 0 & -1 & 0\\ 0 & \lambda-e^{-t} & 0 & -1\\ -t & 0 & \lambda & 0\\ 0 & -t & 0 & \lambda\end{bmatrix}=\left(\lambda^2-e^{-t}\lambda t\right)^2=\\ &=\lambda^4-2e^{-t}\lambda^3+\left(e^{-2t}-2t\right)\lambda^2+2te^{-t}\lambda+t^2 \end{aligned} \tag{3.18}$$

Using (13) and (18) and taking into account that

$$A(t)=\begin{bmatrix}e^{-t} & 1\\ t & 0\end{bmatrix},\quad A^2(t)=\begin{bmatrix}e^{-2t}+t & e^{-t}\\ te^{-t} & t\end{bmatrix},\quad A^3(t)=\begin{bmatrix}e^{-3t}+2te^{-t} & e^{-2t}+t\\ te^{-2t}+t^2 & te^{-t}\end{bmatrix}$$

$$A^4(t)=\begin{bmatrix}e^{-4t}+3te^{-2t}+t^2 & e^{-3t}+2te^{-t}\\ te^{-3t}+2t^2e^{-t} & te^{-2t}+t^2\end{bmatrix}$$

we obtain

$$p_A\left(A(t)\right)=\left(A(t)\right)^2-e^{-t}A(t)-tI_2=\begin{bmatrix}e^{-t2}+t & e^{-t}\\ te^{-t} & t\end{bmatrix}-e^{-t}\begin{bmatrix}e^{-t} & 1\\ t & 0\end{bmatrix}-t\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{aligned} p_A\left(A(t)\right)&=\left(A(t)\right)^4-2e^{-t}\left(A(t)\right)^3+\left(e^{-2t}-2t\right)\left(A(t)\right)^2+2te^{-t}A(t)+t^2I_2=\\ &=\begin{bmatrix}e^{-4t}+3te^{-2t}+t^2 & e^{-3t}+2te^{-t}\\ te^{-3t}+2t^2e^{-t} & te^{-2t}+t^2\end{bmatrix}-2e^{-t}\begin{bmatrix}e^{-3t}+2te^{-t} & e^{-2t}+t\\ te^{-2t}+t^2 & te^{-t}\end{bmatrix}+\left(e^{-2t}-2t\right)\begin{bmatrix}e^{-2t}+t & e^{-t}\\ te^{-t} & t\end{bmatrix}+\\ &+2te^{-t}\begin{bmatrix}e^{-t} & 1\\ t & 0\end{bmatrix}+t^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \end{aligned}$$

$$\begin{aligned} p_{\bar{A}}^1\left(A(t)\right)&=4\left(A(t)\right)^3-6e^{-t}\left(A(t)\right)^2+2\left(e^{-2t}-2t\right)A(t)+2te^{-t}I_2=\\ &=4\begin{bmatrix}e^{-3t}+2te^{-t} & e^{-2t}+t\\ te^{-2t}+t^2 & te^{-t}\end{bmatrix}-6e^{-t}\begin{bmatrix}e^{-2t}+t & e^{-t}\\ te^{-t} & t\end{bmatrix}+2\left(e^{-2t}-2t\right)\begin{bmatrix}e^{-t} & 1\\ t & 0\end{bmatrix}2te^{-t}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \end{aligned}$$

Remark 3.2. Let σ_A $(\sigma_{\bar{A}})$ be the spectrum of the matrix A $(\bar{A})$. If $\sigma_A\subset\sigma_{\bar{A}}$ then $p_{\bar{A}}(\lambda)=p_A(\lambda)p(\lambda)$ and $p_{\bar{A}}(A)=0$.

In [6, 7] the considerations have been extended for positive discrete-time and continuous-time Lyapunov systems.

3.2. Algebraic equations

Consider the algebraic Sylvester equation

$$XA+BX=C \tag{3.19}$$

where $A\in R^{n\times n}$, $B\in R^{m\times m}$, $C\in R^{m\times n}$ are given and $X\in R^{m\times n}$ is unknown. The Lyapunov equation (4a) is a particular case of the Sylvester equation (19) for $B=A^T$ and $C=-Q$.

Theorem 3.4. Let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of the matrix A, and $\mu_1,\dots,\mu_m$ be the eigenvalues of the matrix B. Then the Sylvester equation (19) has a unique solution X if and only if $\lambda_i+\mu_j\neq 0$ for $i=1,\dots,n$ and $j=1,\dots,m$.

Using the Kronecker product we may reduce the Sylvester equation (19) to the equivalent one

$$Px=c \tag{3.20}$$

where

$P=I_n\otimes B+A^T\otimes I_m\in R^{nm\times nm}$

$x=\left[X_1\ \ X_2\ \ \dots\ \ X_m\right]\in R^{nm}$, $c=\left[C_1\ \ C_2\ \ \dots\ \ C_m\right]\in R^{nm}$

and $X_i\left(C_i\right)$ is the ith $(i=1,\dots,m)$ row of the matrix $X(C)$

From (20) it follows that the equation (19) has a unique solution if and only if A and $-B$ do not have a common eigenvalue.

In particular case the Lyapunov equation (4a) has a unique solution if and only if A and $-A$ do not have a common eigenvalue.

Following [2] we shall apply the Sylvester equation (19) to design he Lyapunov state observer

$$\dot{z}(t)=Fz(t)+Gu(t)+Hy(t) \tag{3.21}$$

of the continuous-time systems

$$\begin{aligned} \dot{x}(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t) \end{aligned} \tag{3.22}$$

where $A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}$ are given $u(t)\in R^m$, $y(t)\in R^p$ are the known input and output vectors, $F\in R^{n\times n}$, $G\in R^{n\times m}$, $H\in R^{n\times p}$ are unknown and $x\in R^n$, $z\in R^n$ are the state vectors of the system and of the observer.

Knowing A,B,C $u(t)$ and $y(t)$ find F,G,H and nonsingular matrix $X\in R^{n\times n}$ in such a way that the error vector

$$e(t) = z(t) - Xx(t) \to 0$$

for all $x(0), z(0)$ and for every $u(t)$.

The vector $z(t)$ is an estimate of $Xx(t)$.

It is easy to show that the system (21) is a state observer of the system (22) if the following conditions are met.

$$XA - FX = HC \tag{3.23}$$

$$G = XB \tag{3.24}$$

and F is asymptotically stable.

Definition 3.1. The matrix equation (23) is called the Sylvester observer equation.

The Luenberger state observatory (31) exists for the system (22) if the pare (A, C) is observable. The observer can be compute via Sylvester-observer equation (23) by the us of the following two step procedure.

Procedure 3.1.

Step 1: Choose an asymptotically stable matrix F and a matrix H in such a way that the solution X of (23) is nonsingular

Step 2: Compute $G = XB$.

4. Riccati equations

4.1. Differential equations

Consider differential time-varying Riccati equation

$$\dot{X}(t) = X(t)A_{11}(t) + A_{22}(t)X(t) + X(t)A_{12}(t) + A_{21}(t) \tag{4.1}$$

with the initial condition

$$X(t_0) = X_0 \tag{4.2}$$

where $X(t), A_{11}(t), A_{12}(t), A_{21}(t), A_{22}(t)$ are continuous-time $n \times n$ matrices and t_0 is a given initial condition. Note that from (1) for $A_{22}(t) = A(t)$, $A_{11}(t) = B(t)$, $A_{21}(t) = C(t)$ and $A_{22}(t) = 0$ we obtain the differential Sylvester equation (3.1).

Theorem 4.1. The solution of (1) satisfying the initial condition (2) has the form

$$X(t) = Y_2(t)Y_1^{-1}(t) \tag{4.3}$$

where $Y_1(t)$ and $Y_2(t)$ are solutions of the equation

$$\begin{bmatrix} \dot{Y}_1(t) \\ \dot{Y}_2(t) \end{bmatrix} = \begin{bmatrix} -A_{11}(t) & -A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \tag{4.4}$$

with the initial condition

$$\begin{bmatrix} Y_1(t_0) \\ Y_2(t_0) \end{bmatrix} = \begin{bmatrix} I_n \\ X_0 \end{bmatrix} \tag{4.5}$$

Proof is given in [4].

Let

$$\Phi(t, t_0) = \begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ \Phi_{21}(t, t_0) & \Phi_{22}(t, t_0) \end{bmatrix} \tag{4.6}$$

Be the fundamental matrix of (4). Then the solution of (1) satisfying the initial condition (3) is given by

$$X(t) = \left[\Phi_{21}(t, t_0) + \Phi_{22}(t, t_0)\right]\left[\Phi_{11}(t, t_0) + \Phi_{12}(t, t_0)X_0\right]^{-1} \tag{4.7}$$

In particular case the differential time-varying Riccati equation has the form

$$X(t) = X(t) + A_{11} + A_{22}X(t) + X(t)A_{12}X(t) + A_{21} \tag{4.8}$$

and its solution is given by (7) and the fundamental matrix (6) can be found using the formula

$$\begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ \Phi_{21}(t, t_0) & \Phi_{22}(t, t_0) \end{bmatrix} = e^{A(t-t_0)} \tag{4.9}$$

where

$$A = \begin{bmatrix} -A_{11} & -A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{4.10}$$

For computation of (9) the well-known [4] Sylvester formula can be used.

It is well-known [2, 8, 11] that he differential Riccati equations used in the optimal control of linear time-varying systems with quadratic performance indices. In particular case for time-invariant linear systems we obtain the Riccati equation (8).

Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{4.11}$$

with the initial condition

$$x(t_0) = x_0 \tag{4.12}$$

where $x(t) \in R^n$ is the state vector and $u(t) \in R^m$ is the input vector.

Let the quadratic performance index (cost function) have the form

$$I = \frac{1}{2}x^T(t_f)Qx(t_f) + \frac{1}{2}\int_{t_0}^{t}\left[x^T(t)R_1(t)x(t) + u^T(t)R(t)u(t)\right]dt \tag{4.13}$$

where $Q \in R^{n \times n}$ is a positive semidefinite constant matrix, $R_1(t) \in R^{n \times n}$ is a positive semidefinite time-varying matrix, $R(t) \in R^{n \times n}$ is a positive definite time-varying matrix and t_0 and t_1 are the given initial and final time instants.

It is well-known [8] that the optimal input $\hat{u}(t)$ minimizing the performance index (13) is given by

$$\hat{u}(t) = R^{-1}(t)B^T(t)P(t)x(t) \tag{4.14}$$

and the matrix $P(t)$ satisfies the Riccati equation

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + R_1(t) \tag{4.15}$$

with the final condition $P(t_f) = -Q$

4.2. Algebraic equations

If $A_{11}(t) = A$, $A_{22}(t) = A^T$, $A_{12}(t) = A_{12}$ and $\dot{X}(t) = 0$ then from the differential equation (1) we obtain the following algebraic Riccati equation

$$XA + A^T X + XA_{12}X + A_{21} = 0 \tag{4.16}$$

where $A \in R^{n\times n}$, $A_{12} \in R^{n\times n}$, $A_{21} \in R^{n\times n}$ and $X \in R^{n\times n}$ are time-invariant (constant) matrices.

From (16) for $A_{11} = 0$ and $A_{21} = Q$ we obtain the Lyapunov equation (2.4a). with the Riccati equation (16) we may associated the following Hamiltonian matrix

$$H = \begin{bmatrix} A & -A_{12} \\ -A_{21} & -A^T \end{bmatrix} \tag{4.17}$$

It is well-known [2, 8] that for each eigenvalues λ of (17), $-\bar{\lambda}$ is also an eigenvalue of the matrix with the same geometric and algebraic multiplicity, where $\bar{\lambda}$ denotes the conjugate eigenvalue.

Theorem 4.4 [2] A matrix X is a solution of the equation (16) if and only if the columns of $\begin{bmatrix} I \\ X \end{bmatrix}$ span an *n*-dimensional invariant subspace of the matrix (17).

The algebraic Riccati equation is closely related to the following continuous-time linear quadratic regulator (LQR) problem.

Consider the linear system

$$\dot{x} = Ax + Bu, \quad x \in R^n, \; u \in R^m, \; x(0) = x_0 \tag{4.18}$$

and the quadratic performance index (cost function)

$$I(x) = \int_0^\infty \left[x_t(t) Qx(t) + u^T(t) Ru(t) \right] dt \tag{4.19}$$

where $Q \in R^{n\times n}$ is symmetric positive semidefinite and $R \in R^{m\times m}$ is symmetric positive definite.

Given matrix A, B, Q, R find an input $u(t)$ such that the performance index (19) is minimized, subject to (18).

The pair (A, B) is stabilizable if and only if

$$rank\left[I_n s - A \quad B \right] = n \text{ for all } s \in C, \quad \mathrm{Re}\, s \geq 0$$

and the pair (A, Q) is detectable if and only if

$$rank \begin{bmatrix} I_n s - A \\ Q \end{bmatrix} = n \text{ for all } s \in C, \quad \mathrm{Re}\, s \geq 0$$

where C is the field of complex numbers.

Theorem 4.5 [2] The pair (A, B) be stabilizable and the pair (A, Q) be detectable. Then the exists a unique optimal input $\hat{u}(t)$ which minimizes (19). The optimal input is given by

$$\hat{u}(t) = R^{-1}B^T Xx(t) \tag{4.20}$$

and X is the unique positive semidefinite solution of the equation

$$XA + A^T X - XBR^{-1}B^T X + Q = 0 \tag{4.21}$$

Furthermore, the close-loop matrix $A - BK$ is asymptotically stable and the minimum value of (19) is equal to $x_0^T X x_0$.

Theorem 4.6 [2, 10] The equation (21) has a unique symmetric positive semidefinite solution X if and only if the pair (A, B) is stabilizable and the associated Hamiltonian matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

has no pure imaginary eigenvalues.

The algebraic Riccati equations play also important role in state-space solution of H_∞ and in robust control problem.

In the literature [2] there exist many computational methods for solving the algebraic Riccati equations, for example, the eigenvector methods, the Schur methods, the inverse-free generalized methods, the Newton's methods, the matrix sign function methods, etc.

An interesting comparison of the computational method is given in [2].

5. Concluding remarks

An overview of the Lyapunov, Sylvester and Riccati equations has been presented. A special attention has been focused on the relationship between the equations and their applications in control systems theory. It has been shown that the differential Sylvester equation plies the crucial role in the development of the theory of the Lyapunov systems. The well-known classical Cayley-Hamilton theorem has been extended for the linear time-varying Lyapunov systems. In this paper the linear continuous-time systems has been only considered. With slight modification the considerations can be also extended for discrete-time linear systems.

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