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Componentwise asymptotic stability and exponential stability of positive discrete-time linear systems with delays

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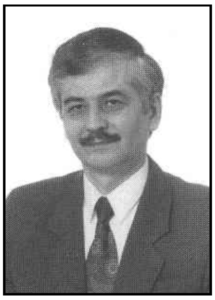

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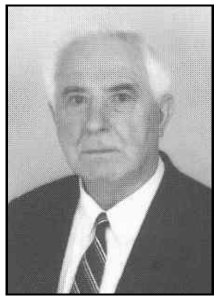

Abstract

Definitions of the componentwise asymptotic stability and of the exponential stability are extended for positive discrete-time linear systems with delays. Necessary and sufficient conditions for the componentwise asymptotic stability and the exponential stability are established.

Keywords: bilinear system, controllability, delay, discrete-time, positive solution

Stabilność asymptotyczna według składowych i stabilność wykładnicza liniowych dodatnich układów dyskretnych z opóźnieniami

Streszczenie

Definicje asymptotycznej stabilności według składowych oraz stabilności wykładniczej rozszerzono na liniowe dodatnie układy dyskretne z opóźnieniami. Podano warunki konieczne i wystarczające asymptotycznej stabilności według składowych oraz stabilności wykładniczej.

Słowa kluczowe: system bilinearny, opóźnienie, czas dyskretny, układ dodatni

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial states and non-negative controls. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. [1, 10, 11].

Recently, conditions for stability and robust stability of positive discrete-time linear systems with delays were given in [3-9, 12].

In this paper we give definitions and necessary and sufficient conditions for the componentwise asymptotic stability and the exponential stability of linear positive discrete-time systems with delays.

To the best author's knowledge, the componentwise asymptotic stability and exponential stability problems of positive discrete-time systems with delays were not considered yet.

2. Problem formulation

Let $\Re^{n\times m}$ be the set of $n\times m$ matrices with entries from the field of real numbers and $\Re^n = \Re^{n\times 1}$. The set of $n\times m$ matrices with real non-negative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider the positive discrete-time linear system with delays described by the homogeneous equation

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-k}, \quad i \in Z_+, \tag{1}$$

with the initial conditions

$$x_{-i} \in \Re_+^n, \quad i = 0,1,\ldots,h, \tag{2}$$

where h is a positive number and $A_k \in \Re_+^{n\times n}$ $(k = 0,1,\ldots,h)$.

If $A_k \in \Re_+^{n\times n}$ $(k = 0,1,\ldots,h)$ then for every $x_{-i} \in \Re_+^n$ $(i = 0,1,\ldots,h)$ we have $x_i \in \Re_+^n$ for $i \in Z_+$ [12].

System (1) is asymptotically stable if and only if all roots $z_1, z_2, \ldots, z_n$ of the characteristic equation

$$\det(zI_n - \sum_{k=0}^{h} A_k z^{-k}) = 0 \tag{3}$$

have moduli less than 1, or equivalently, all roots $z_1, z_2, \ldots, z_{\tilde{n}}$ of the equation

$$\det(z^{h+1}I_n - \sum_{k=0}^{h} A_k z^{h-k}) = z^{\tilde{n}} + a_{\tilde{n}-1}z^{\tilde{n}-1} + \ldots + a_1 z + a_0 = 0 \tag{4}$$

have moduli less than 1, i.e. $|z_k| < 1$ for $k = 1,2,\ldots,\tilde{n} = (h+1)n$ [12].

Let us introduce the matrix

$$A = \begin{bmatrix} A_0 & A_1 & \cdots & A_h \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n}\times\tilde{n}}. \tag{5}$$

Theorem 1. [12]. Positive system with delays (1) is asymptotically stable if and only if the following equivalent conditions hold:

1) all coefficients of characteristic polynomial of the matrix $S = A - I_{\tilde{n}}$, of the form

$$\begin{aligned} \det[(z+1)I_{\tilde{n}} - A] &= \det[(z+1)^{h+1}I_n - \sum_{k=0}^{h} A_k (z+1)^{h-k}] \\ &= z^{\tilde{n}} + \tilde{a}_{\tilde{n}-1}z^{\tilde{n}-1} + \ldots + \tilde{a}_1 z + \tilde{a}_0 \end{aligned} \tag{6}$$

are positive, i.e. $\tilde{a}_i > 0$ for $i = 0,1,\ldots,\tilde{n}-1$,

2) all leading principal minors of the matrix $\bar{A} = I_{\tilde{n}} - A$ are positive.

In this paper we consider the new kinds of stability of positive system (1), namely, the componentwise asymptotic stability and the exponential stability.

By generalisation of definitions of the componentwise asymptotic stability and the exponential stability of positive systems without delays (given in [11]), we obtain the following.

Definition 1. Positive system (1) is called componentwise asymptotically stable if for every initial conditions (2) there exist vectors $\rho_i \in \Re_+^n$ satisfying the condition $\lim_{i\to\infty} \rho_i = 0$ such that

$$x_i \le \rho_i \text{ for all } i = -h, -h+1, \dots, -1, 0, 1, 2, \dots \tag{7}$$

where x_i is the solution of equation (1) with initial conditions (2).

Definition 2. Positive system (1) is called exponentially stable if for every initial conditions (2) there exist a scalar $0 < \beta < 1$ and a vector $\alpha > 0$ such that

$$x_i \le \alpha\beta^i \text{ for all } i = -h, -h+1, \dots, -1, 0, 1, 2, \dots . \tag{8}$$

From definitions 1 and 2 it follows that asymptotic stability is necessary for componentwise asymptotic stability and exponential stability of positive system (1) with delays.

In the paper we give necessary and sufficient conditions for componentwise asymptotic stability and exponential stability of positive discrete-time systems (1). The proposed conditions were obtained by generalisation of suitable conditions given in [11] for positive systems without delays.

3. The main results

Theorem 2. Positive system (1) is componentwise asymptotically stable if and only if the vectors $\rho_i \in \Re_+^n$ satisfy the difference inequality

$$\rho_{i+1} \ge \sum_{k=0}^{h} A_k \rho_{i-k}, \quad i \in Z_+. \tag{9}$$

Proof. First we show that if inequality (9) is satisfied then (7) holds.

Let $v_i = \rho_i - x_i$, $i \in Z_+$. From (9) and (1) we have

$$v_{i+1} = \rho_{i+1} - x_{i+1} \ge \sum_{k=0}^{h} A_k(\rho_{i-k} - x_{i-k}) = \sum_{k=0}^{h} A_k v_{i-k}. \tag{10}$$

Hence,

$$v_{i+1} = \sum_{k=0}^{h} A_k v_{i-k} + w_i, \; i \in Z_+, \tag{11}$$

where $w_i \in \Re_+^n$.

The solution of equation (11) has the form [2]

$$v_i = \Phi(i)v_0 + \sum_{j=-h}^{-1} \sum_{k=1}^{h+j+1} \Phi(i-k)A_{k-1-j}v_j + \sum_{j=0}^{i-1}\Phi(i-1-j)w_j, \tag{12}$$

where $\Phi(i)$ is the state-transition matrix, which satisfies the equation

$$\Phi(i+1) = A_0\Phi(i) + A_1\Phi(i-1) + \dots + A_h\Phi(i-h) \tag{13}$$

with the initial conditions

$$\Phi(0) = I_n, \quad \Phi(i) = 0 \text{ for } i < 0. \tag{14}$$

Because $A_k \in \Re_+^{n\times n}$ $(k = 0,1,\dots,h)$, from (13) and (14) it follows that $\Phi(i) \in \Re_+^n$ for all $i \in Z_+$. Hence, $v_i = \rho_i - x_i \ge 0$ and (7) holds since $w_i \in \Re_+^n$, $i \in Z_+$.

Now, by contradiction we show that (7) is satisfied only if (9) holds.

Assume that (9) is not satisfied for mth component ρ_i^m of the vector ρ_i, i.e.

$$\rho_{i+1}^m < \sum_{j=1}^{n} \sum_{k=0}^{h} a_{mj}^k \rho_{i-k}^j \quad m = 1,2,\dots,n, \tag{15}$$

where a_{mj}^k is the (m, j) entry of A_k.

From (1) for the mth component x_i^m of x_i we have

$$x_{i+1}^k = \sum_{j=1}^{n} \sum_{k=0}^{h} a_{mj}^k x_{i-k}^j. \tag{16}$$

Using (7) we obtain

$$\sum_{j=1}^{n} \sum_{k=0}^{h} a_{mj}^k x_{i-k}^j \le \sum_{j=1}^{n} \sum_{k=0}^{h} a_{mj}^k \rho_{i-k}^j < \rho_{i+1}^m. \tag{17}$$

Hence, we get the contradiction and (9) holds.

Theorem 3. Positive system (1) is exponentially stable if and only if

$$[I\beta - \sum_{k=0}^{h} A_k\beta^{-k}]\alpha \ge 0 \tag{18}$$

and

$$1 > \beta \ge \max_i \frac{1}{\alpha_i} \sum_{k=0}^{h} \beta^{-k} \sum_{j=1}^{n} a_{ij}^k \alpha_j \ge 0. \tag{19}$$

where $A_k = [a_{ij}^k]$ $(k = 0,1,\dots,h)$, $i, j = 1,2,\dots,n$,

and $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$.

Proof. Substitution of $\rho_i = \alpha\beta^i$ into (9) yields

$$\rho_{i+1} = \alpha\beta^{i+1} \ge \sum_{k=0}^{h} A_k\alpha\beta^{i-k}$$

or

$$[I\beta - \sum_{k=0}^{h} A_k\beta^{-k}]\alpha\beta^i \ge 0.$$

After division the last inequality by β^i we obtain inequality (18).

From (18) for the ith component α_i of α we have

$$\beta\alpha_i \ge \sum_{k=0}^{h} \beta^{-k} \sum_{j=1}^{n} a_{ij}^k \alpha_j, \; i = 1,2,\dots,n. \tag{20}$$

After division (20) by α_i we obtain

$$\beta \ge \frac{1}{\alpha_i} \sum_{k=0}^{h} \beta^{-k} \sum_{j=1}^{n} a_{ij}^k \alpha_j \ge 0, \; i = 1,2,\dots,n. \tag{21}$$

Hence, (19) holds.

By generalisation of the results given in [11] for positive systems without delays we obtain the following remark.

Remark 1. In the general case β in Theorem 3 can be chosen so that real roots of (4) are less than β.

4. Example

Consider the positive system (1) with the matrices

$$A_0 = \begin{bmatrix} 0.5 & 0.1 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ 0.4 & 0.2 \end{bmatrix} \quad (22)$$

and the initial conditions

$$x_{-2} = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix},\ x_{-1} = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix},\ x_0 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}. \quad (23)$$

Equation (4) of the form

$$\det(z^3 I_2 - \sum_{k=0}^{2} A_k z^{2-k}) = z^6 - 0.5z^5 - 0.1z^4 - 0.21z^3 + 0.06z^2 + 0.02z = 0 \quad (24)$$

has the following real roots: 0.7900, 0.4091, 0, -0.1953. Hence, according to Remark 1 we can choose $\beta = 0.8$.

Condition (18) for $\beta = 0.8$ has the form

$$[I\beta - \sum_{k=0}^{2} A_k \beta^{-k}]\alpha = \begin{bmatrix} 0.1750 & -0.1 \\ -0.75 & 0.4875 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (25)$$

From (25) we have the following inequalities

$$\alpha_2 \leq 1.75\alpha_1 \text{ and } \alpha_2 \geq 1.5385\alpha_1. \quad (26)$$

Among values of α_1 and α_2 satisfying (26) we chose those that satisfy (8) for $i = -2$, $i = -1$ and $i = 0$, i.e. $\alpha \geq x_{-2}\beta^2$, $\alpha \geq x_{-1}\beta^1$ and $\alpha \geq x_0$.

For $\beta = 0.8$ from the above inequalities we obtain

$$\alpha_1 \geq 0.6120,\ \alpha_2 \geq 0.3440. \quad (27)$$

From (26) and (27) it follows that we can choose $\alpha_1 = 0.7$, $\alpha_2 = 1.2$.

Checking the conditions of Theorem 3 for $\beta = 0.8$ and $\alpha = [0.7, 1.2]^T$ we obtain

$$[I\beta - \sum_{k=0}^{2} A_k \beta^{-k}]\alpha = \begin{bmatrix} 0.0025 \\ 0.06 \end{bmatrix}$$

and

$$1 > \beta \geq \max\left\{ \frac{1}{\alpha_1} \sum_{k=0}^{2} \beta^{-k} \sum_{j=1}^{2} a_{1j}^k \alpha_j, \frac{1}{\alpha_2} \sum_{k=0}^{2} \beta^{-k} \sum_{j=1}^{2} a_{2j}^k \alpha_j \right\} = \max\{0.7964, 0.75\} = 0.7964.$$

Hence, for $\beta = 0.8$ and $\alpha = [0.7, 1.2]^T$ the conditions of Theorem 3 are satisfied and the system is exponentially stable.

For the system the vector $\rho_i = \alpha\beta^i$ has the form

$$\rho_i = \begin{bmatrix} \rho_i^1 \\ \rho_i^2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 1.2 \end{bmatrix} 0.8^i \text{ for } i = -2, -1, 0, 1, 2, \ldots. \quad (28)$$

Plots of x_i^1, x_i^2 and ρ_i^1, ρ_i^2 for $i = -2, -1, 0, 1, 2, \ldots, 10$ are shown in Figure 1. From this figure it follows that condition (8) of Definition 2 holds.

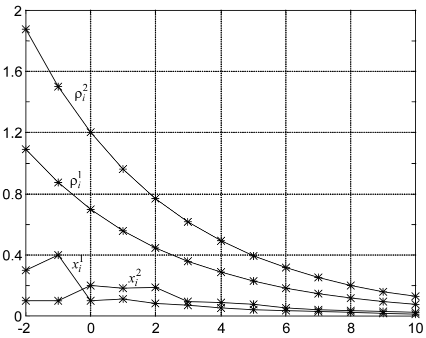

Fig. 1. Plots of x_i^1, x_i^2 and ρ_i^1, ρ_i^2 for $i = -2, -1, 0, 1, 2, \ldots, 10$

5. Concluding remarks

Definitions of the componentwise asymptotic stability and of the exponential stability are extended for positive discrete-time linear systems (1) with delays. Necessary and sufficient conditions for the componentwise asymptotic stability and the exponential stability have been given.

The considerations can be extended for linear positive continuous-time systems with delays.

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