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Recent Developments in Theory of Positive Discrete-Time Linear Systems with Delays – Stability and Robust Stability

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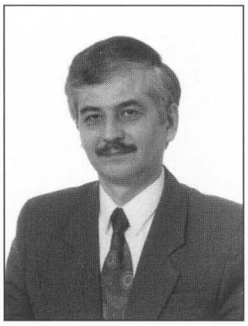

Uzyskał dyplom mgr inż. elektryka w 1974 roku na Wydziale Elektrycznym Politechniki Warszawskiej. Na tym samym wydziale w 1977 uzyskał stopień doktora nauk technicznych, a w 1988 roku doktora habilitowanego. Tytuł profesora uzyskał w 2002 roku. W latach 1988-1999 był prodziekanem i dziekanem Wydziału Elektrycznego Politechniki Białostockiej. Od 1999 roku jest prorektorem ds. nauki Politechniki Białostockiej. Od 1992 roku jest kierownikiem Katedry Automatyki i Elektroniki. Autor trzech monografii oraz ponad 100 artykułów. Zainteresowania naukowe koncentrują się wokół problematyki analizy i syntezy układów regulacji automatycznej z opóźnieniami czasowymi, układów dodatnich oraz układów o niepewnych parametrach.

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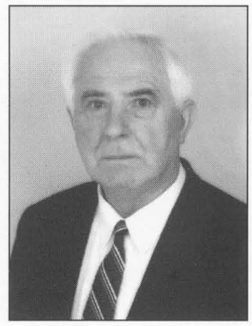

Uzyskał dyplom magistra inżyniera elektryka w roku 1956 na Wydziale Elektrycznym Politechniki Warszawskiej. Na tym samym Wydziale w roku 1962 uzyskał stopień naukowy doktora nauk technicznych, a w roku 1964 – doktora habilitowanego. Tytuł naukowy profesora nadzwyczajnego nadała Mu Rada Państwa w roku 1971, a profesora zwyczajnego w 1974 roku. Członkiem korespondentem PAN został wybrany w 1986 roku, a członkiem rzeczywistym w 1998. Od czerwca 1999 roku jest również członkiem zwyczajnym Akademii Inżynierskiej w Polsce. W latach 1969-1970 był dziekanem Wydziału Elektrycznego, a w latach 1970-1979 prorektorem d/s nauczania Politechniki Warszawskiej. W latach 1970-1981 był dyrektorem Instytutu Sterowania i Elektroniki Przemysłowej Politechniki Warszawskiej. W latach 1988-1991 był dyrektorem Stacji Naukowej PAN w Rzymie. Jest autorem 18 książek, w tym 5-ciu wydanych za granicą oraz ponad 500 artykułów i rozpraw naukowych, opublikowanych w czasopismach krajowych i zagranicznych. Główne kierunki badań naukowych to analiza i synteza układów sterowania i systemów, a w szczególności układy wielowymiarowe, układy singularne i układy dodatnie.

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Abstract

Necessary and sufficient conditions for asymptotic stability of positive linear discrete-time systems with delays and for robust stability of interval positive systems with delays are given. First, the conditions are formulated for positive systems in general case. Next, they are simplified in two special cases: positive scalar systems with delays and positive systems with pure delay.

Streszczenie

Podano warunki konieczne i wystarczające asymptotycznej stabilności dyskretnych dodatnich układów liniowych stacjonarnych z opóźnieniami oraz odpornej stabilności przedziałowych dodatnich układów z opóźnieniami. Najpierw sformułowano warunki stabilności w przypadku ogólnym, a następnie w dwóch przypadkach szczególnych: dodatniego układu skalarnego z opóźnieniami oraz dodatniego układu z czystym opóźnieniem.

Keywords: Linear system, positive, discrete-time, interval, time delays, stability, robust stability.
Słowa kluczowe: Układ liniowy, dodatni, dyskretny, przedziałowy, opóźnienia, stabilność, odporna stabilność.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial states and non-negative controls. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Recently, conditions for stability and robust stability of positive discrete-time systems with delays were given in [2, 3, 5]. The stability problem of linear positive systems with time-delays was not considered yet.

In this paper an overview of recent developments in stability and robust stability of positive discrete-time systems with delays will be presented. Some new results also will be given.

2. Asymptotic stability

Let $\Re^{n\times m}$ be the set of $n\times m$ matrices with entries from the field of real numbers and $\Re^n=\Re^{n\times 1}$. The set of $n\times m$ matrices with real non-negative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n=\Re_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider the positive discrete-time linear system with delays described by the homogeneous equation

$$x_{i+1}=A_0x_i+\sum_{k=1}^{h}A_kx_{i-k}, \tag{1}$$

where h is a positive integer and $A_k\in\Re_+^{n\times n}$ $(k=0,1,...,h)$.

The system (1) is asymptotically stable if and only if all roots $z_1,z_2,...,z_{\tilde{n}}$ of the characteristic equation $w(z)=0$ have moduli less than 1, where

$$w(z)=\det(z^{h+1}I_n-\sum_{k=0}^{h}A_kz^{h-k}). \tag{2}$$

The positive system without delays equivalent to (1) has the form

$$\tilde{x}_{i+1}=A\tilde{x}_i,\quad i\in Z_+ \tag{3}$$

where $\tilde{x}_i\in\Re_+^{\tilde{n}}$ with $\tilde{n}=(h+1)n$ and

$$A=\begin{bmatrix}A_0 & A_1 & \cdots & A_h\\ I_n & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & I_n & 0\end{bmatrix}\in\Re_+^{\tilde{n}\times\tilde{n}}. \tag{4}$$

The positive system (3) is asymptotically stable if and only if

$$w_A(z)=\det(zI_{\tilde{n}}-A)\neq 0 \text{ for } |z|\geq 1. \tag{5}$$

In [5] it was shown that $w(z)=w_A(z)$. Hence, asymptotic stability of positive system (1) (with delays) is equivalent to asymptotic stability of positive system (3) (without delays).

Theorem 1 [5]. The positive system with time-delays (1) is asymptotically stable if and only if the following equivalent conditions hold:

1) all coefficients of polynomial $w(z+1)$ are positive, where has the form (2),
2) all principal (leading) minors Δ_i $(i=1,2,...,\tilde{n})$ of the matrix $\bar{A}=I_{\tilde{n}}-A$ of the form

$$\overline{A} = \begin{bmatrix} I_n - A_0 & -A_1 & \cdots & -A_h \\ -I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -I_n & I_n \end{bmatrix} \tag{6}$$

are positive.

Lemma 1 [5]. The positive time-delays system (1) is unstable if
1) the positive system $x_{i+1} = A_0 x_i$ (without delays) is unstable, or
2) at least one diagonal entry of the matrix A_0 is greater than 1.

From the above it follows that asymptotic stability of positive system (1) with delays depends of the matrices A_k ($k = 0,1,...,h$) but it does not depend of delay $h > 0$.

2.1. Stability of scalar systems with delays

Consider the positive system with delays described by the homogeneous equation

$$x_{i+1} = a_0 x_i + \sum_{k=1}^{h} a_k x_{i-k},\ i \in Z_+, \tag{7}$$

where h is a positive integer and $a_k \geq 0$ (with assumption that at least one coefficient a_k is positive).

The system (7) is asymptotically stable if and only if $w_1(z) \neq 0$ for $|z| \geq 1$, where

$$w_1(z) = z^{h+1} - a_0 z^h - ... - a_{h-1} z - a_h. \tag{8}$$

The positive system without delays equivalent to (7) is described by the equation

$$\tilde{x}_{i+1} = A_s x_i,\ i \in Z_+, \tag{9}$$

where $A_s \in R_+^{n \times n}$, $n = h + 1$, and

$$A_s = \begin{bmatrix} a_0 & a_1 & \cdots & a_h \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{10}$$

From Theorem 1 and Lemma 1 we have the following.

Theorem 2. The positive system with time-delays (7) is asymptotically stable if and only if the following equivalent conditions hold:
1) all coefficients of polynomial $w_1(z+1)$ are positive, where $w_1(z)$ has the form (8),
2) all principal minors of the matrix $\overline{A}_s = I_n - A_s$ of the form

$$\overline{A}_s = \begin{bmatrix} 1-a_0 & -a_1 & \cdots & -a_h \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{11}$$

are positive.

Lemma 2. The positive time-delays system (7) is unstable if the positive system (without delays) $x_{i+1} = a_0 x_i$, $i \in Z_+$, is unstable, that is $a_0 \geq 1$.

The condition 2) of Theorem 2 can be written in the form

$$\Delta_i = 1 - \sum_{k=0}^{i-1} a_k > 0 \text{ for } i = 1,2,...,n = h+1 \tag{12}$$

Because $a_k \geq 0$ for $k=0,1,...,n-1=h$, inequalities (12) hold if and only if

$$\Delta_n = 1 - \sum_{k=0}^{h} a_k > 0. \tag{13}$$

Hence, we have the following theorem.

Theorem 3. The positive system with time-delays (7) is asymptotically stable if and only if (13) holds.

From Theorem 3 it follows that asymptotic stability of positive scalar time-delays system (7) does not depend of delays.

Example 1. Consider the system (7) with $h=1$ and $a_0=0.4$, $a_1=0.5$. From (13) we have $\Delta_2=1-a_0-a_1=0.1>0$. Hence, the system is asymptotically stable. Notice that any system described by the equation $x_{i+1} = a_0 x_i + a_1 x_{i-h}$ with $h \geq 1$ is also asymptotically stable.

2.2. Stability of systems with pure delay

The system (1) is the system with pure delay if $A_k \equiv 0$ for $k = 0,1,...,h-1$. In such case this system is described by the homogeneous equation

$$x_{i+1} = A_h x_{i-h},\ i \in Z_+. \tag{14}$$

From (4) it follows that the matrix A_p of the equivalent system $\tilde{x}_{i+1} = A_p x_i$ without delays has the form (with $\tilde{n} = (h+1)n$)

$$A_p = \begin{bmatrix} 0 & 0 & \cdots & A_h \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n} \times \tilde{n}}. \tag{15}$$

The system (14) is asymptotically stable if and only if $w_h(z) \neq 0$ for $|z| \geq 1$, where

$$w_h(z) = \det(z^{h+1} I_n - A_h). \tag{16}$$

From Theorem 1 we have the following.

Theorem 4 [3]. The positive system (14) with pure delay is asymptotically stable if and only if the following equivalent conditions hold:
1) all coefficients of the polynomial $w_h(z+1)$ are positive, where $w_h(z)$ has the form (16),
2) all principal minors of the matrix $\overline{A}_p = I_{\tilde{n}} - A_p$ of the form

$$\overline{A}_p = \begin{bmatrix} I_n & 0 & \cdots & -A_h \\ -I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -I_n & I_n \end{bmatrix} \tag{17}$$

are positive.

From structure of the matrix (17) it follows that all principal minors of order from 1 to nh of $\overline{A}_p$ are always positive. Moreover, all principal minors of $\overline{A}_p$ of order from $nh + 1$ to $(h+1)n$ are positive if and only if are positive all principal minors of the matrix

$$D = I_n - A_h. \tag{18}$$

From the above it follows that if the system (14) with fixed delay $h>0$ (h is a positive integer) is asymptotically stable, than the system $x_{i+1} = A_h x_{i-p}$, where p is any positive integer, is also asymptotically stable. Hence, asymptotic stability of positive system (14) with pure delay does not depend of delay.

Positivity of all principal minors of (18) is necessary and sufficient for asymptotic stability of the positive system without delay, described by the equation [4]

$$x_{i+1} = A_h x_i,\ i \in Z_+, \tag{19}$$

It is well known [4] that the system (19) is asymptotically stable if and only if all eigenvalues of the matrix A_h have moduli less than 1.

From the above and [4] we have the following.

Theorem 5 [3]. The positive system (14) with pure delay is asymptotically stable if and only if the following equivalent conditions hold:
1) all principal minors of the matrix (18) are positive,
2) all coefficients of the polynomial

$$\det[(z+1)I_n - A_h] = z^n + \overline{a}_{n-1} z^{n-1} + ... + \overline{a}_0 \tag{20}$$

are positive, i.e. $\overline{a}_i > 0$ for $i = 0,1,...,n-1$

Lemma 3 [3]. The positive system (14) is not stable if at least one diagonal entry of the matrix $A_h = [a_{hij}]$ is greater than 1, i.e. $a_{hkk} > 1$ for some $k \in (1,2,...,n)$.

Example 2. Consider the positive system (14) with

$$A_h = \begin{bmatrix} a & 0.2 & 0 \\ 0.4 & 0.1 & 0.1 \\ 1 & 0.3 & b \end{bmatrix}. \quad (21)$$

Find values of the parameters $a \geq 0$ and $b \geq 0$ for which the system is asymptotically stable. In this case matrix (18) has the form

$$D = \begin{bmatrix} 1-a & -0.2 & 0 \\ -0.4 & 0.9 & -0.1 \\ -1 & -0.3 & 1-b \end{bmatrix}. \quad (22)$$

Computing all principal minors of (22), from condition 1) of Theorem 5 we obtain: $\Delta_1 = 1 - a > 0$, $\Delta_2 = 0.82 - 0.9a > 0$, $\Delta_3 = 0.77 - 0.87a - 0.82b + 0.9ab > 0$. These inequalities can be written in the form

$$a < 0.9111,\ 0.77 - 0.87a - 0.82b + 0.9ab > 0. \quad (23)$$

Hence, the system is asymptotically stable for a and b satisfying (23) and for any fixed delay (**h** = 1,2,...).

3. Robust stability of interval systems

Let us consider a family of positive discrete-time systems with delays

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-k},\ A_k \in [A_k^-, A_k^+] \subset \Re_+^{n\times n}, \quad (24)$$

where $a_{kij} \in [a_{kij}^-, a_{kij}^+]$, $a_{kij}^- \leq a_{kij}^+$, with $A_k^- = [a_{kij}^-]$, $A_k^+ = [a_{kij}^+]$ for k=0,1,...,h.

The family (24) is called as an interval family or an interval system with delays.

The interval positive system (24) is called robustly stable if the system (1) is asymptotically stable for all $A_k \in [A_k^-, A_k^+]$ (k=0,1,...,h).

If $A_k \in [A_k^-, A_k^+]$, k=0,1,...,h, then for the equivalent system (3) we have $A \in A_I$, where A is of the form (4), $A_I = [A^-, A^+]$ and

$$A^- = \begin{bmatrix} A_0^- & A_1^- & \cdots & A_h^- \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix}, \quad (25)$$

Theorem 6 [2]. The interval positive time-delays system (24) is robustly stable if and only if the positive system without delays $\tilde{x}_{i+1} = A^+\tilde{x}_i$, $i \in Z_+$, is asymptotically stable or, equivalently, is asymptotically stable the positive system with delays

$$x_{i+1} = A_0^+ x_0 + \sum_{k=1}^{h} A_k^+ x_{i-k},\ i \in Z_+. \quad (26)$$

Proof. The proof follows directly from the fact that all eigenvalues of any non-negative matrix $A \in [A^-, A^+]$ have moduli less than 1 if and only if all eigenvalues of A^+ have moduli less than 1 [1].

From Theorem 6 it follows that robust stability of interval system (24) does not depend of the matrices $A_k^- \in \Re_+^{n\times n}$, k = 0,1,...,h. Therefore, may be $A_k^- = 0$ for k = 0,1,...,h. Moreover, if the system (1) is asymptotically stable for any fixed $A_k = A_{kf} \in \Re_+^{n\times n}$, k = 0,1,...,h, then this system is also asymptotically stable for all $A_k \in [0, A_{kf}]$, k = 0,1,...,h.

From the above and Theorem 1 and Lemma 1 we have the following theorem and lemma.

Theorem 7 [3]. The interval positive time-delays system (24) is robustly stable if and only if the following equivalent conditions hold:
1) all coefficients of the polynomial $w^+(z+1)$ are positive, where

$$w^+(z+1) = \det[(z+1)^{h+1} I_n - \sum_{k=0}^{h} A_k^+ (z+1)^{h-k}], \quad (27)$$

2) all principal minors of the matrix $\bar{A}^+ = I_{\bar{n}} - A^+$ of the form

$$\bar{A}^+ = \begin{bmatrix} I_n - A_0^+ & -A_1^+ & \cdots & -A_h^+ \\ -I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -I_n & I_n \end{bmatrix} \quad (28)$$

are positive.

Lemma 4 [3]. The interval positive time-delays system (24) is not robustly stable if the positive system (without delays) $x_{i+1} = A_0^+ x_i$ is unstable, or at least one diagonal entry of the matrix A_0^+ is greater than 1.

3.1. Robust stability of scalar interval systems with delays

Consider a family of positive discrete-time linear systems with delays

$$x_{i+1} = \sum_{k=0}^{h} a_k x_{i-k},\ a_k \in [a_k^-, a_k^+], \quad (29)$$

where $0 \leq a_k^-$ and $a_k^- \leq a_k^+$ for k = 0,1,...,h.

The positive interval system without delays equivalent to (29) is described by

$$\tilde{x}_{i+1} = A_s x_i,\ A_s \in [A_s^-, A_s^+] \subset \Re_+^{n\times n},\ n = h+1. \quad (30)$$

From Theorems 6 and 3 we have the following.

Theorem 8. The interval positive system (29) with delays is robustly stable if and only if the positive system without delays

$$x_{i+1} = A_s^+ x_i,\ i \in Z_+, \quad (31)$$

where

$$A_s^+ = \begin{bmatrix} a_0^+ & a_1^+ & \cdots & a_h^+ \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (32)$$

is asymptotically stable or, equivalently, is asymptotically stable the positive time-delays system $x_{i+1} = \sum_{k=0}^{h} a_k^+ x_{i-k}$, that is (according to Theorem 3)

$$\Delta_n^+ = 1 - \sum_{k=0}^{h} a_k^+ > 0. \quad (33)$$

3.2. Robust stability of interval systems with pure delay

Let us consider interval positive system with pure delay

$$x_{i+1} = A_h x_{i-h},\ A_h \in [A_h^-, A_h^+] \subset \Re_+^{n\times n}. \quad (34)$$

Theorem 9 [3]. The interval positive system (34) with pure delay is robustly stable if and only if the positive time-delay system

$$x_{i+1} = A_h^+ x_{i-h},\ i \in Z_+, \quad (35)$$

is asymptotically stable or, equivalently, is asymptotically stable the positive system without delays

$$x_{i+1} = A_h^+ x_i,\ i \in Z_+, \quad (36)$$

From Theorem 9 it follows that robust stability of interval system (34) does not depend on the matrix $A_h^- \in \Re_+^{n\times n}$. Therefore, may be $A_h^- = 0$.

From the above and Theorem 5 we have the following theorem.

Theorem 10 [3]. The interval positive system (34) with pure delay is robustly stable if and only if the following equivalent conditions hold:

1) all principal minors of the matrix

$$D^+ = I_n - A_h^+ \quad (37)$$

are positive,

2) all coefficients of the polynomial

$$\det[(z+1)I_n - A_h^+] = z^n + \hat{a}_{n-1}z^{n-1} + ... + \hat{a}_0, \quad (38)$$

are positive.

Lemma 5 [3]. The positive interval system (34) is not robustly stable if at least one diagonal entry of the matrix $A_h^+ = [a_{hij}^+]$ is greater than 1, i.e. $a_{hkk}^+ > 1$ for some $k \in (1,2,...,n)$.

4. Conclusions

An overview of the recent developments in stability and robust stability theory of positive linear discrete-time systems with delays has been presented. First, the necessary and sufficient conditions for asymptotic stability of the general case of such systems, scalar systems and for systems with pure delay are given. Next, the conditions for robust stability of interval systems are derived. Moreover, it is shown that if the systems under considerations are asymptotically stable than there are asymptotically stable independent of delay.

An open problem is an extension of the considerations for positive continuos-time linear systems with delays and for singular positive discrete-time and continuos-time systems with delays.

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Tytuł: Wybrane nowe wyniki w teorii liniowych dodatnich układów dyskretnych z opóźnieniami – Stabilność i odporna stabilność

Artykuł recenzowany

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Recent Developments in Theory of Positive Discrete-Time Linear Systems with Delays – Rechability, Minimum Energy Control and Realization Problem

Abstract

Notion of the positive linear discrete-time systems with multiple delays is introduced. Necessary and sufficient conditions for positivity, reachability and minimum energy control are given. Conditions for the solvability of the realization problem are established.

Streszczenie

W pracy podano warunki, jakie musi spełniać dyskretny układ liniowy stacjonarny z opóźnieniami, aby był on układem dodatnim. Sformułowano warunki konieczne i wystarczające osiągalności oraz sterowania z minimalną energią. Podano warunki, przy spełnieniu których problem realizacji rozpatrywanej klasy układów ma rozwiązanie.

Keywords: Linear systems, positive, discrete-time, time-delays, reachability, minimum energy control, positive realization.
Słowa kluczowe: Układ liniowy, dodatni, dyskretny, opóźnienia, osiągalność, sterowanie z minimalną energią, dodatnia realizacja.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial states and non-negative controls. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [6, 7]. Recent developments in positive systems theory and some new results are given in [8].

Recently some known result for standard linear positive systems have been extended for positive systems with time-delays. The reachability of positive discrete-time systems with one delay has been considered in [2, 17] and the minimum energy control of the same class of positive systems has been studied in [3, 12]. Conditions for controllability and minimum energy control of linear positive continuous-time systems with delays were given in [16]. The stability and minimal realization problems was considered in [4, 5, 9] and [10, 11, 13], respectively.

In this paper we give an overview of recent developments in reachability, minimum energy control and realization problem of positive discrete-time systems with delays. Some new results will be also presented.

2. Preliminaries

Let $\Re^{n\times m}$ be the set of $n \times m$ matrices with entries from the field of real numbers and $\Re^n = \Re^{n\times 1}$. The set of $n \times m$ matrices with real non-negative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider the discrete-time linear system with delays described by the equations

$$x_{i+1} = A_0 x_i + \sum_{k=1}^{h} A_k x_{i-k} + Bu_i, \quad (1a)$$