Zbigniew EMIRSAJŁOW
POLITECHNIKA SZCZECIŃSKA, INSTYTUT AUTOMATYKI PRZEMYSŁOWEJ, ZAKŁAD TEORII STEROWANIA I TECHNIK SYMULACYJNYCH
Stuart TOWNLEY
EXETER UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES

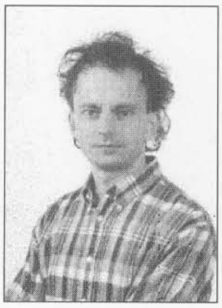
Implemented Semigroup for Infinite-Dimensional Lyapunov Equations

Zbigniew EMIRSAJŁOW

Profesor nadzwyczajny Politechniki Szczecińskiej. Stopień doktora nauk technicznych uzyskał w 1983 roku na Wydziale Elektrycznym Politechniki Szczecińskiej, a stopień doktora habilitowanego w dyscyplinie automatyka i robotyka w 1992 roku w Instytucie Badań Systemowych PAN. Od 1993 roku kieruje Zakładem Teorii Sterowania i Technik Symulacyjnych Instytutu Automatyki Przemysłowej Politechniki Szczecińskiej. Zajmuje się teorią sterowania i systemów, a w szczególności układami nieskończenie wymiarowymi oraz nieskończenie wymiarowymi równaniami Sylvestera, Lapunowa i Riccatiego.

emirsaj@we.ps.pl

Stuart TOWNLEY



Profesor Uniwersytetu w Exeter, Wielka Brytania. Stopień naukowy doktora uzyskał w 1987 roku na Wydziale Matematyki Uniwersytetu Warwick w Wielkiej Brytanii. Od 1999 roku jest profesorem matematyki stosowanej na Wydziale Nauk Matematycznych Uniwersytetu w Exeter, a od 2003 roku jest dziekanem tego wydziału. Zajmuje się teorią sterowania i systemów, a w szczególności układami nieskończenie wymiarowymi, nieliniowymi układami dynamicznymi oraz zastosowania teorii sterowania w ekologii.

townley@maths.ex.ac.uk

Abstract

The main objective of the paper is to show that the concept of the implemented semigroup provides a natural mathematical framework for analysis of the infinite-dimensional differential Lyapunov equation (DLE). Such a Lyapunov equation arises quite naturally in various system-theoretic and control problems in which the time horizon is finite, the state space is infinite-dimensional and the operators involved in the mathematical model of the system are unbounded. As an application we show how this approach can be used to solve a simple decoupling problem arising in optimal control.

Streszczenie

Głównym celem pracy jest pokazanie, że koncepcja półgrupy złożonej jest naturalnym narzędziem matematycznym do analizy nieskończenie wymiarowego różniczkowego równania Lapunowa. Tego typu równania występują w problemach sterowania ze skończonym horyzontem czasowym i modelem matematycznym zawierającym operatory nieograniczone. Podejście oparte na półgrupie złożonej pozwala wyprowadzić warunki konieczne i wystarczające ograniczoności rozwiązania różniczkowego równania Lapunowa w odpowiedniej przestrzeni. Jesteśmy przekonani, że półgrupa złożona może być użytecznym narzędziem matematycznym w nieskończenie wymiarowej teorii sterowania i systemów. Jak przykład zastosowania przedstawionej teorii w pracy pokazano rozwiązanie pewnego problemu rozprzęgania występującego w zadaniach sterowania optymalnego.

Keywords: implemented semigroup, infinite-dimensional differential Lyapunov equation, decoupling
Słowa kluczowe: półgrupa złożona, nieskończenie wymiarowe różniczkowe równanie Lapunowa, rozprzęganie

1. Introduction

In a variety of problems of systems and control theory one encounters the following differential Lyapunov equation (DLE):

$$\dot{X}(t) = AX(t) + X(t)A^* + BB^*, \quad X(0) = X_0, \tag{1}$$

where $X(t)$, A and B are linear operators acting on infinite-dimensional Hilbert spaces. In this paper we present a natural mathematical framework within which a comprehensive analysis of the equation (1) can be carried out. In particular we are able to derive a necessary and sufficient condition on the operator B under which DLE (1) with unbounded operators A and B admits a suitable bounded solution $X(t)$. As the main mathematical tool we will use the concept of the implemented semigroup, e.g. [1], [2], [3], and explore its properties as examined in [4], [5].

2. Motivating example

In order to state a certain optimal control problem which will serve as a motivating example we first need to introduce the following notation and assumptions:

- H, U are Hilbert spaces (identified with their duals) which play the role of the *state* space and the *control* space.

- A is a linear, unbounded operator on H generating a strongly continuous semigroup $T(t) \in \mathscr{L}(H)$, $t \geq 0$, which describes the free system dynamics. $H_1(A) = \mathscr{D}(A)$ is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_{H_1(A)} = \langle (\lambda I - A)(\cdot), (\lambda I - A)(\cdot) \rangle_H$, where $\lambda \in \rho(A)$ and $\rho(A)$ denotes the resolvent set of A. Analogously we define $H_1(A^*) = \mathscr{D}(A^*)$, where A^* is the unbounded adjoint to A.

- $H_{-1}(A)$ is the completion of H in the norm

$$\| \cdot \|_{H_{-1}(A)} = \|(\lambda I - A)^{-1}(\cdot)\|_H ,$$

where $\lambda \in \rho(A)$. Analogously we define $H_{-1}(A^*)$. One can show that the Hilbert spaces $H_{-1}(A)$ and $H_{-1}(A^*)$ can be equivalently defined as the duals $H_1(A^*)'$ and $H_1(A)'$, respectively. Moreover, the above definitions imply that $H_1(A) \subset H \subset H_{-1}(A)$ with continuous and dense inclusions. The semigroup $T(t) \in \mathscr{L}(H)$ and its generator A restrict to $H_1(A)$ and extend to $H_{-1}(A)$. For the sake of simplicity we use the same notation for these restrictions and extensions.

- $B \in \mathscr{L}(U, H_{-1}(A))$ is the *control* operator.

With the pair $\{A, B\}$ we associate a *control system* with the *state* $x(t)$, $t \geq 0$, characterized by the usual differential *state equation*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{2}$$

where $u(t) \in U$ is the *control*. Then we assume that $\tau \in (0, \infty)$ is fixed, $x_0, z_1 \in H$, B is an admissible control operator (e.g. [6], and consider the problem of finding a control $u_{opt} \in L^2(0, \tau; U)$ which minimizes the quadratic performance index

$$J(u) = \|x(\tau) - z_1\|_H^2 + \|u\|_{L^2(0,\tau;U)}^2 \tag{3}$$

over the space $L^2(0,\tau;U)$.

Under our assumptions the above optimal control problem has a unique solution u_{opt} and the optimal pair $\{u_{opt}, x_{opt}\} \in L^2(0,\tau;U) \times C([0,\tau];H)$ is uniquely characterized by the following optimality conditions

$$\dot{x}_{opt}(t) = Ax_{opt}(t) + Bu_{opt}(t), \quad x_{opt}(0) = x_0, \tag{4a}$$
$$\dot{p}(t) = -A^*p(t), \quad p(\tau) = z_1 - x_{opt}(\tau), \tag{4b}$$
$$u_{opt}(t) = B^*p(t). \tag{4c}$$

It is clear that for $x_0, z_1 \in H$ and admissible $B \in \mathscr{L}(U, H_{-1}(A))$ solutions of both differential equations (4a) and (4b) have to be understood in the mild sense and the expression (4c) for the optimal control only makes sense as a function in $L^2(0,\tau;U)$.

When we substitute (4c) into (4a) we can see that the problem we are dealing with takes the following form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A & BB^* \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \in [0,\tau], \tag{5}$$

with $x_1(0) = x_0$ and $x_2(\tau) = z_1 - x_1(\tau)$, where $x_{opt} = x_1$ and $u_{opt} = B^*x_2$. This is a two-point boundary value problem whose solution $[x_1(t)\ x_2(t)]^T$, $t \in [0,\tau]$, is awkward to compute. This is because the final condition $x_2(\tau)$ depends on the final condition $x_1(\tau)$ so that the two differential equations are really coupled and what is more the operator BB^* appearing in the first equation is highly unbounded with respect to the state space H (it satisfies $BB^* \in \mathscr{L}(H_1(A^*), H_{-1}(A))$).

We will show that we can introduce new state variables and decouple the system by transforming the operator matrix of the system (5) into the block diagonal form. In order to state our problem properly we replace (5) by considering first the following two-point boundary value problem

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A & BB^* \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(\tau) \end{bmatrix}, \tag{6}$$

with $x_1(0) \in H$ and $x_2(\tau) \in H_2(A^*)$, where we *do not* assume a priori admissibility of the control operator B. The initial-final conditions of (6) differ from those of (5) since we do not now assume that $x_2(\tau)$ depends on $x_1(\tau)$. However, in the end we will return to this case. Since $x_2(\tau) \in H_2(A^*)$ we immediately obtain

$$x_2(\cdot) \in C([0,\tau]; H_2(A^*)) \cap C^1([0,\tau]; H_1(A^*)),$$

where $H_2(A^*)$ is the domain of A^*. The differentiability of $x_2(t)$ together with the assumption $x_1(0) \in H$ imply that

$$x_1(\cdot) \in C[0,\tau]; H) \cap C^1([0,\tau]; H_{-1}(A)).$$

Thus the system of differential equations (6) holds in $H_{-1}(A) \times H_1(A^*)$ for every $t \in [0,\tau]$.

As the next step we would like to introduce new state variables $[w_1(t)\ w_2(t)]^T$ defined by

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} I & -M(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \in [0,\tau], \tag{7}$$

such that

$$\begin{bmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \quad t \in [0,\tau], \tag{8}$$

with $w_1(0) = x_1(0)$ and $w_2(\tau) = x_2(\tau)$. Ideally we would want the operator $M(t)$ to have the following nice properties:

- $M(t)$ is bounded on the state space H, i.e.

$$M(t) \in \mathscr{L}(H), \quad t \in [0,\tau], \tag{9}$$

and continuous in time in the strong operator topology of $\mathscr{L}(H)$.

- $\dot{M}(t)$ is well-defined at least in $\mathscr{L}(H_1(A^*), H_{-1}(A))$, i.e.

$$\dot{M}(t) \in \mathscr{L}(H_1(A^*), H_{-1}(A)), \quad t \in [0,\tau], \tag{10}$$

and is continuous in time in the strong operator topology of this space.

- In order to ensure that the initial-final conditions have the form as in (8) we also need

$$M(0) = 0. \tag{11}$$

One can check that for every $t \in [0,\tau]$ the operator matrix (if it exists) transforming the original state variables $[x_1(t)\ x_2(t)]^T$ into the new state variables $[w_1(t)\ w_2(t)]^T$ (see (7)) is boundedly invertible for every $t \in [0,\tau]$ and the original variables can be recovered by means of the formula

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I & M(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \quad t \in [0,\tau]. \tag{12}$$

We are now ready to proceed and in order to find out how $M(t)$ should look like let us differentiate (7) to obtain

$$\begin{bmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \end{bmatrix} = \begin{bmatrix} I & -\dot{M}(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} I & -M(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}. \tag{13}$$

Under our assumptions on $x_1(0)$, $x_2(\tau)$ and $M(t)$ the system of equations (13) is well-defined in $H_{-1}(A) \times H_1(A^*)$ for every $t \in [0,\tau]$. After the substitutions of (6) and (12) followed by simple manipulations we finally arrive at

$$\begin{bmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \end{bmatrix} = \begin{bmatrix} A & -\dot{M}(t) + AM(t) + M(t)A^* + BB^* \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix},$$

which again makes sense in $H_{-1}(A) \times H_1(A^*)$ for every $t \in [0,\tau]$. Thus in order to justify (8) we have to answer the question whether there exists a solution $M(t) \in \mathscr{L}(H)$, $t \in [0,\tau]$, to the differential Lyapunov equation (DLE)

$$\dot{M}(t) = AM(t) + M(t)A^* + BB^*, \quad M(0) = 0, \tag{14}$$

where the equality is understood in $\mathscr{L}(H_1(A^*), H_{-1}(A))$. It turns out that the natural mathematical framework for analysis of (14) is the implemented semigroup concept.

3. Implemented semigroup

In this section we keep all the notation and assumptions introduced at the beginning of Section 2. Using the strongly continuous semigroups $T(t) \in \mathscr{L}(H)$ and $T^*(t) \in \mathscr{L}(H)$ generated by A and its adjoint A^*, respectively, we can define another semigroup.

Definition 1. The family $\mathcal{U}(t) \in \mathscr{L}(\mathscr{L}(H))$, $t \geq 0$, defined as follows

$$\mathcal{U}(t)X = T(t)XT^*(t), \quad X \in \mathscr{L}(H), \quad t \geq 0, \tag{15}$$

is called the *implemented semigroup*.

Using this definition we can easily check the following properties of the family $\mathcal{U}(t)$:

(a) The family $\mathcal{U}(t) \in \mathscr{L}(\mathscr{L}(H))$, $t \geq 0$, is a semigroup, i.e.

$$\begin{aligned} \mathcal{U}(0)X &= X, \quad X \in \mathscr{L}(H), \\ \mathcal{U}(t+s)X &= \mathcal{U}(t)(\mathcal{U}(s)X) = \mathcal{U}(s)(\mathcal{U}(t)X). \end{aligned}$$

(b) $\mathcal{U}(t) \in \mathscr{L}(\mathscr{L}(H))$ is continuous in time at every $t \geq 0$ in the strong operator topology of $\mathscr{L}(H)$, i.e.

$$\lim_{\Delta \to 0} \|(\mathcal{U}(t+\Delta)X)h - (\mathcal{U}(t)X)h\|_H = 0, \quad h \in H. \qquad (16)$$

The main difference between the usual strongly continuous semigroup (see [2]) and the implemented semigroup is that in general for $X \in \mathscr{L}(H)$ we cannot expect the operator $\mathcal{U}(t)X \in \mathscr{L}(H)$ to be continuous in time in the (natural) uniform operator topology of $\mathscr{L}(H)$ unless the semigroups $T(t), T^*(t) \in \mathscr{L}(H)$ are uniformly continuous. However, this is true only if their generators A and A^* are bounded operators on H.

Definition 2. The *infinitesimal generator* $\mathcal{A}$ of the implemented semigroup $\mathcal{U}(t) \in \mathscr{L}(\mathscr{L}(H))$, $t \geq 0$, is defined as the limit

$$(\mathcal{A}X)h = \lim_{t \to 0^+} \frac{(\mathcal{U}(t)X)h - Xh}{t}, \quad X \in \mathscr{D}(\mathcal{A}), \quad h \in H, \qquad (17)$$

where $\mathscr{D}(\mathcal{A}) \subset \mathscr{L}(H)$ is the *domain* of $\mathcal{A}$ defined as follows

$$\mathscr{D}(\mathcal{A}) = \{X \in \mathscr{L}(H) : \lim_{t \to 0^+} \frac{(\mathcal{U}(t)X)h - Xh}{t} \text{ exists}\}. \qquad (18)$$

In order to get more understanding what the domain $\mathscr{D}(\mathcal{A}) \in \mathscr{L}(H)$ and the generator $\mathcal{A}$ look like we provide the following results:

(c) $X \in \mathscr{L}(H)$ belongs to the domain $\mathscr{D}(\mathcal{A})$ if and only if the restriction of X to $H_1(A^*)$ belongs to $\mathscr{L}(H_1(A^*), H_1(A))$, i.e.

$$\mathscr{D}(\mathcal{A}) \subset \mathscr{L}(H) \cap \mathscr{L}(H_1(A^*), H_1(A)), \qquad (19)$$

and an extension of $(AX + XA^*) \in \mathscr{L}(H_1(A^*), H)$ to H belongs to $\mathscr{L}(H)$.

(d) The operator $\mathcal{A}$ has the following explicit representation

$$(\mathcal{A}X)h = AXh + XA^*h, \quad X \in \mathscr{D}(\mathcal{A}), \quad h \in H_1(A^*), \qquad (20)$$

where by (c) the right hand side of the equality (20) is well-defined in H.

The basic properties of the implemented semigroup resemble to a great extent the corresponding properties of strongly continuous semigroups and can be summarized as follows:

(e) For $X \in \mathscr{L}(H)$ the following relations hold

$$\int_0^t \mathcal{U}(r)X\,dr \in \mathscr{D}(\mathcal{A}), \quad t \geq 0, \qquad (21)$$

where the integrals are convergent in the strong operator topology.

(f) For $X \in \mathscr{D}(\mathcal{A})$ and $t \geq 0$ we have $\mathcal{U}(t)X \in \mathscr{D}(\mathcal{A})$ and

$$\frac{d}{dt}(\mathcal{U}(t)X) = (\mathcal{A}(\mathcal{U}(t)X)) = (\mathcal{U}(t)(\mathcal{A}X)). \qquad (22)$$

(g) $\mathscr{D}(\mathcal{A})$ is dense in $\mathscr{L}(H)$ in the strong operator topology.

(h) The operator $\mathcal{A}$ is closed on $\mathscr{L}(H)$ in the uniform operator topology and hence also in the strong operator one.

(i) The following inequality holds

$$\|\mathcal{U}(t)\|_{\mathscr{L}(\mathscr{L}(H))} = \|T(t)\|^2_{\mathscr{L}(H)} \leq M\,e^{\omega t}, \quad t \geq 0, \qquad (23)$$

and for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 2\omega$ we can define a family of operators $\mathcal{R}(\lambda) \in \mathscr{L}(\mathscr{L}(H))$

$$\mathcal{R}(\lambda)X = \int_0^\infty e^{-\lambda t}\mathcal{U}(t)X\,dt, \quad X \in \mathscr{L}(H), \qquad (24)$$

where the integrals are convergent in the strong operator topology.

(j) The following inclusion holds $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 2\omega\} \subset \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$.

(k) If $\operatorname{Re}\lambda > 2\omega$, then the family $\mathcal{R}(\lambda)$ coincides with the resolvent $\mathcal{R}(\lambda, \mathcal{A})$ of the operator $\mathcal{A}$, i.e.

$$\mathcal{R}(\lambda) = \mathcal{R}(\lambda, \mathcal{A}) = (\lambda\mathcal{I} - \mathcal{A})^{-1} \in \mathscr{L}(\mathscr{L}(H)) \qquad (25)$$

and $\mathscr{R}(\mathcal{R}(\lambda)) = \mathscr{R}(\mathcal{R}(\lambda, \mathcal{A})) = \mathscr{D}(\mathcal{A})$.

It immediately follows from property (f) of the implemented semigroup that for every $X_0 \in \mathscr{D}(\mathcal{A})$ the expression

$$X(t) = \mathcal{U}(t)X_0 = T(t)X_0T^*(t), \quad t \geq 0,$$

satisfies the conditions

$$X(t) \in \mathscr{D}(\mathcal{A}), \quad \dot{X}(t) = \mathcal{A}X(t) \in \mathscr{L}(H), \quad X(0) = X_0.$$

So $X(t)$ is a solution to the homogeneous Cauchy problem

$$\dot{X}(t) = \mathcal{A}X(t) \in \mathscr{L}(H), \quad t \geq 0, \quad X(0) = X_0. \qquad (26)$$

Let us now notice that by property (d) the differential equation (26) can be written in the form

$$\dot{X}(t)h = AX(t)h + X(t)A^*h, \quad h \in H_1(A^*), \quad X(0) = X_0,$$

and thus it becomes the homogeneous Cauchy problem corresponding to DLE. However we are interested in the non-homogeneous Cauchy problem for this equation. This problem is dealt with in the next section.

4. DLE with a bounded input

In this section we consider the non-homogeneous Cauchy problem of the form

$$\dot{X}(t) = \mathcal{A}X(t) + F, \quad t \geq 0, \quad X(0) = X_0, \qquad (27)$$

where $X_0 \in \mathscr{D}(\mathcal{A})$ and $F \in \mathscr{L}(H)$. In order to state the main result for the equation (27) it is convenient to work with the following notation and assumptions:

- Throughout the rest of the paper we assume $\lambda \in \mathbb{C}$ satisfies the condition $\operatorname{Re}\lambda > 2\omega$.

- $\mathcal{H} = \mathscr{L}(H)$ denotes the Banach space equipped with the usual norm $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_{\mathscr{L}(H)}$. In the space $\mathcal{H}$ we distinguish the uniform operator topology generated by the norm and the strong operator topology.

- $\mathcal{H}_1 = \mathscr{D}(\mathcal{A})$ denotes the Banach space with the norm

$$\|\cdot\|_{\mathcal{H}_1} = \|(\lambda\mathcal{I} - \mathcal{A})(\cdot)\|_{\mathcal{H}}. \qquad (28)$$

In $\mathcal{H}_1$ we distinguish the uniform operator topology generated by the norm and the strong operator topology which is defined as follows: Every sequence $\{X_k\}_{k=1}^\infty$, where $X_k \in \mathcal{H}_1$, is convergent to $X \in \mathcal{H}_1$ in the strong operator topology if the sequence $\{(\lambda\mathcal{I} - \mathcal{A})X_k\}_{k=1}^\infty$, where $(\lambda\mathcal{I} - \mathcal{A})X_k \in \mathcal{H}$, is convergent to $(\lambda\mathcal{I} - \mathcal{A})X \in \mathcal{H}$ in the strong operator topology.

As we already know (see (20)) the differential equation (27) can be also written explicitly as DLE

$$\dot{X}(t)h = AX(t)h + X(t)A^*h + Fh, \quad h \in H_1(A^*), \qquad (29)$$

with $X(0) = X_0 \in \mathcal{H}_1$ and $F \in \mathcal{H}$.

Lemma 3. *If $F \in \mathcal{H}$, then DLE (29) has a unique strong solution $X(t) \in \mathcal{H}_1$, $t \geq 0$, which is continuous in time in the strong operator topology of $\mathcal{H}_1$ and continuously differentiable in the strong operator topology of $\mathcal{H}$. This solution is explicitly given by the expression*

$$\begin{aligned} X(t) &= \mathcal{U}(t)X_0 + \int_0^t \mathcal{U}(t-r)F\,dr \\ &= T(t)X_0T^*(t) + \int_0^t T(t-r)FT^*(t-r)\,dr\,. \quad (30) \end{aligned}$$

5. DLE with an unbounded input

In this section we consider the non-homogeneous Cauchy problem of the form

$$\dot{X}(t) = \mathcal{A}X(t) + F\,, \quad t \geq 0\,, \quad X(0) = X_0\,, \qquad (31)$$

where $X_0 \in \mathcal{H}$ and $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$. We need to introduce the following extra notation and assumptions:

- $\mathcal{H}_{-1}$ is defined as the completion of $\mathcal{H} = \mathscr{L}(H)$ in the following sense: Every sequence $\{Y_k\}_{k=1}^{\infty}$, where $Y_k \in \mathcal{H}$, is a Cauchy sequence in the strong operator topology of $\mathcal{H}_{-1}$ if the sequence $\{X_k\}_{k=1}^{\infty}$, where $X_k = \mathcal{R}(\lambda, \mathcal{A})Y_k \in \mathcal{H}_1$, is a Cauchy sequence in the strong operator topology of $\mathcal{H}$. This allows us to extend the resolvent $\mathcal{R}(\lambda, \mathcal{A})$ to $\overline{\mathcal{R}}(\lambda, \mathcal{A}) : \mathcal{H}_{-1} \to \mathcal{H}$ such that $\mathscr{R}(\overline{\mathcal{R}}(\lambda, \mathcal{A})) = \mathcal{H}$ and also allows us to define a norm in $\mathcal{H}_{-1}$ as follows

$$\|X\|_{\mathcal{H}_{-1}} = \|\overline{\mathcal{R}}(\lambda, \mathcal{A})X\|_{\mathcal{H}}\,, \quad X \in \mathcal{H}_{-1}\,. \qquad (32)$$

$\mathcal{H}_{-1}$ is complete in this norm and we distinguish the strong operator topology defined above and the uniform operator topology defined by the norm (32). For the sake of simplicity we will use the notation $\mathcal{R}(\lambda, \mathcal{A})$ also for the extension $\overline{\mathcal{R}}(\lambda, \mathcal{A})$. We can restrict the implemented semigroup $\mathcal{U}(t) \in \mathcal{H}$, $t \geq 0$, and its generator $\mathcal{A}$ to $\mathcal{H}_1$ and extend them to $\mathcal{H}_{-1}$ in the strong operator sense. We will use the notation $\mathcal{U}(t)$ and $\mathcal{A}$ also for these restrictions and extensions. The operator $\mathcal{A}$ on $\mathcal{H}_{-1}$ satisfies

$$\mathscr{D}(\mathcal{A}) = \mathcal{H}\,, \quad \mathcal{R}(\lambda, \mathcal{A}) \in \mathscr{L}(\mathcal{H}_{-1})\,, \quad \mathscr{R}(\mathcal{R}(\lambda, \mathcal{A})) = \mathcal{H}\,.$$

Every operator $Y \in \mathcal{H}_{-1}$ can be identified with an operator

$$Y \in \mathscr{L}(H_1(A^*), H_{-1}(A))$$

and we have

$$\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathscr{L}(H_1(A^*), H_{-1}(A)) \qquad (33)$$

with dense and continuous inclusions in the strong sense.

The formula

$$\overline{\mathcal{U}}(t)X = T(t)XT^*(t)\,, \quad X \in \mathscr{L}(H_1(A^*), H_{-1}(A))\,,$$

where $T(t) \in \mathscr{L}(H_{-1}(A))$ and $T^*(t) \in \mathscr{L}(H_1(A^*))$, defines an implemented semigroup on $\mathscr{L}(H_1(A^*), H_{-1}(A))$ which is continuous in time in the strong operator topology. The domain $\mathscr{D}(\overline{\mathcal{A}})$ of the inifinitesimal generator $\overline{\mathcal{A}}$ contains $\mathcal{H}$, i.e.

$$\mathcal{H} \subset \mathscr{D}(\overline{\mathcal{A}}) \subset \mathscr{L}(H_1(A^*), H_{-1}(A))\,, \qquad (34)$$

and the generator $\overline{\mathcal{A}}$ admits the following explicit representation

$$(\overline{\mathcal{A}}X)h = AXh + XA^*h\,, \quad X \in \mathcal{H}\,, \quad h \in H_1(A^*)\,, \qquad (35)$$

where this equality is understood in $H_{-1}(A)$.

- $\mathcal{A}, \mathcal{U}(t)$ understood as operators on $\mathcal{H}_{-1}$ and $\overline{\mathcal{A}}, \overline{\mathcal{U}}(t)$ understood as operators on $\mathscr{L}(H_1(A^*), H_{-1}(A))$ are related as follows:

$$\begin{aligned} (\mathcal{A}X)h &= (\overline{\mathcal{A}}X)h = AXh + XA^*h,\ X \in \mathcal{H}\,, \qquad (36) \\ (\mathcal{U}(t)Y)h &= (\overline{\mathcal{U}}(t)Y)h = T(t)YT^*(t)h,\ Y \in \mathcal{H}_{-1}\,, \end{aligned}$$

where $h \in H_1(A^*)$ and all equalities understood in $H_{-1}(A)$. We regard these relations as justification of the notation $\mathcal{A}$ and $\mathcal{U}(t)$ used for $\overline{\mathcal{A}}$ and $\overline{\mathcal{U}}(t)$ in the remaining part of the paper.

Lemma 4. *Let $\mathrm{Re}\,\lambda > 2\omega$. Then $Y \in \mathscr{L}(H_1(A^*), H_{-1}(A))$ satisfies the condition*

$$Y \in \mathcal{H}_{-1} \qquad (37)$$

if and only if the following algebraic Lyapunov equation (ALE) has a solution $X \in \mathcal{H}$

$$\lambda\langle Xh, g\rangle_H - \langle Xh, A^*g\rangle_H - \langle XA^*h, g\rangle_H = \langle Yh, g\rangle_{H_1' \times H_1}\,,$$

where $h, g \in H_1(A^*)$ and $\langle\cdot,\cdot\rangle_{H_1' \times H_1}$ denotes the duality pairing between $H_1(A^*)$ and $H_1(A^*)'$.

We can now consider the differential equation

$$\dot{X}(t) = \mathcal{A}X(t) + F\,, \quad t \geq 0\,, \quad X(0) = X_0\,, \qquad (38)$$

where $X_0 \in \mathcal{H}$ and $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$. In view of the relations (36) the above equation can be rewritten in the more explicit form

$$\dot{X}(t)h = AX(t)h + X(t)A^*h + Fh\,, \quad h \in H_1(A^*)\,, \qquad (39)$$

where $X(0) = X_0 \in \mathcal{H}$, $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$ and the equality in equation (39) understood in $H_{-1}(A)$.

Lemma 5. *If $F \in \mathcal{H}_{-1}$, then DLE (39) has a unique strong solution $X(t) \in \mathcal{H}$, $t \geq 0$, which is continuous in time in the strong operator topology of $\mathcal{H}$ and continuously differentiable in the strong operator topology of $\mathcal{H}_{-1}$. This solution is explicitly given by*

$$X(t) = \mathcal{U}(t)X_0 + \int_0^t \mathcal{U}(t-r)F\,dr\,, \quad t \geq 0\,. \qquad (40)$$

It follows from Lemma 5 that a sufficient condition on $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$ which guarantees that DLE (39) has a solution $X(t) \in \mathcal{H}$, is simply $F \in \mathcal{H}_{-1}$. It turns out this condition is also necessary. In order to show this necessity we introduce

$$(\mathcal{M}F)(t) \in \mathscr{L}(H_1(A^*), H_{-1}(A))\,, \quad t \geq 0\,, \qquad (41)$$

defined as follows

$$(\mathcal{M}F)(t) = \int_0^t \mathcal{U}(t-r)F\,dr\,, \quad t \geq 0\,, \qquad (42)$$

where $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$.

Definition 6. $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$ is said to be an admissible input element for the implemented semigroup $\mathcal{U}(t) \in \mathscr{L}(\mathcal{H})$ if for some $t_1 > 0$ and every $t \in (0, t_1]$ there exists a constant $m(t) > 0$ such that

$$|\langle(\mathcal{M}F)(t)h, g\rangle_{H_1' \times H_1}| \leq m(t)\|h\|_H\|g\|_H \qquad (43)$$

for $h, g \in H_1(A^*)$ (i.e. $(\mathcal{M}F)(t) \in \mathcal{H}$ for $t \in (0, t_1]$) and

$$\lim_{t \to 0^+} \|(\mathcal{M}F)(t)h\|_H = 0\,, \quad h \in H_1(A^*)\,. \qquad (44)$$

Lemma 7. *If $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$ is an admissible input element for the implemented semigroup, then $(\mathcal{M}F)(t) \in \mathcal{H}$, $t \geq 0$, and is continuous in the strong operator topology of $\mathcal{H}$.*

The admissibility of the input operator F is simply equivalent to the requirement that (39) has a solution $X(t) \in \mathcal{H}$, $t \geq 0$, which is continuous in time in the strong operator topology. It follows easily from Lemma 5 and Definition 6 that every operator $F \in \mathcal{H}_{-1}$ is an admissible input element for the implemented semigroup. The main result of [3] shows that the converse is also true.

Theorem 8. *$F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$ is an admissible input element for the implemented semigroup if and only if it satisfies $F \in \mathcal{H}_{-1}$.*

Corollary 9. *DLE (39) has a unique strong solution $X(t) \in \mathcal{H}$, $t \geq 0$, which is continuous in time in the strong operator topology of $\mathcal{H}$ and continuously differentiable in the strong operator topology of $\mathcal{H}_{-1}$ (hence also in the strong operator topology of $\mathscr{L}(H_1(A^*), H_{-1}(A))$) if and only if the input operator $F \in \mathscr{L}(H_1(A^*), H_{-1}(A))$ satisfies the condition $F \in \mathcal{H}_{-1}$. This solution is explicitly given by the expression*

$$\begin{aligned} X(t) &= \mathcal{U}(t)X_0 + \int_0^t \mathcal{U}(t-r)F\,dr \\ &= T(t)X_0T^*(t) + \int_0^t T(t-r)FT^*(t-r)\,dr\,. \quad (45) \end{aligned}$$

6. Example continued

By applying Corollary 9 and Lemma 4 to DLE (14) we immediately obtain the result we are after.

Corollary 10. *The differential Lyapunov equation*

$$\dot{M}(t) = AM(t) + M(t)A^* + BB^*\,, \quad M(0) = 0\,, \quad (46)$$

has a unique strong solution $M(t) \in \mathscr{L}(H)$, $t \geq 0$, which is continuous in time in the strong operator topology of $\mathscr{L}(H)$ and continuously differentiable in the strong operator topology of $\mathcal{H}_{-1}$ (hence also in the strong operator topology of $\mathscr{L}(H_1(A^*), H_{-1}(A))$), if and only if

$$BB^* \in \mathcal{H}_{-1}\,. \quad (47)$$

This solution is explicitly given by the expression

$$\begin{aligned} M(t) &= \int_0^t \mathcal{U}(t-r)(BB^*)\,dr \\ &= \int_0^t T(t-r)BB^*T^*(t-r)\,dr\,, \quad t \geq 0\,, \quad (48) \end{aligned}$$

and the differential equation (46) holds in $\mathcal{H}_{-1}$ for every $t \geq 0$. The condition (47) is equivalent to the fact that for $\mathrm{Re}\,\lambda > 2\omega$ the algebraic Lyapunov equation

$$\lambda\langle Xh, g\rangle_H - \langle Xh, A^*g\rangle_H - \langle XA^*h, g\rangle_H = \langle B^*h, B^*g\rangle_U\,,$$

where $h,\ g \in H_1(A^*)$, has a solution $X \in \mathscr{L}(H)$.

Lemma 11. *The condition $BB^* \in \mathcal{H}_{-1}$ holds if and only if $B \in \mathscr{L}(U, H_{-1}(A))$ is an admissible control operator for the semigroup $T(t) \in \mathscr{L}(H)$, $t \geq 0$.*

Summing up, the condition (47) (or one of its equivalent forms) is a necessary and sufficient condition for the existence of transformation (7) with the required properties (9)-(11). Under this condition $M(t) \in \mathscr{L}(H)$, $t \geq 0$, can be found by solving (46).

Thus under appropriate conditions we can arrive at two-point boundary value problem (12) with $w_1(0) = x_1(0) \in H$ and $w_2(\tau) = x_2(\tau) \in H_2(A^*)$. It is clear that $w_1(\cdot) \in C([0,\tau]; H) \cap C^1([0,\tau]; H_{-1}(A))$ and if we extend the final condition to $w_2(\tau) = x_2(\tau) \in H$ then also $w_2(\cdot) \in C([0,\tau]; H) \cap C^1([0,\tau]; H_{-1}(A^*))$. Consequently, it follows from the relation (12) that for $x_1(0) \in H$ and $x_2(\tau) \in H$ we have $x_1(\cdot) \in C([0,\tau]; H)$ and $x_2(\cdot) \in C([0,\tau]; H)$, where the pair $[x_1(t)\ x_2(t)]^T$ is understood as a mild solution to the two-point boundary value problem (6).

Now we have to take into consideration the fact that in the system (5) the final condition $x_2(\tau)$ depends on $x_1(\tau)$, namely $x_2(\tau) = z_1 - x_1(\tau)$. It follows from (12) for $t = \tau$ that

$$\begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} = \begin{bmatrix} w_1(\tau) + M(\tau)w_2(\tau) \\ w_2(\tau) \end{bmatrix}, \quad (49)$$

and, after simple manipulations, we get

$$w_2(\tau) = z_1 - (I + M(\tau))^{-1}(w_1(\tau) + M(\tau)z_1)\,. \quad (50)$$

Thus we finally obtain the system of differential equations

$$\begin{bmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \quad t \in [0,\tau]\,, \quad (51)$$

with the following initial-final conditions

$$\begin{bmatrix} w_1(0) \\ w_2(\tau) \end{bmatrix} = \begin{bmatrix} x_0 \\ z_1 - (I + M(\tau))^{-1}(w_1(\tau) + M(\tau)z_1) \end{bmatrix},$$

where $x_0 \in H$ and $z_1 \in H$. It is easy to see that to compute $w_1(t)$ we have to solve the first differential equation of (51) forward in time and hence also obtain $w_1(\tau)$. Then having computed $M(\tau)$ and $w_2(\tau)$ we solve the second differential equation of (51) backward in time and obtain $w_2(t)$. So the mild solution to the original two-point boundary value problem (5) with $x_0,\ z_1 \in H$, is obtained via the relation (12). This relation together with Corollary 10 imply that for every pair $x_0,\ z_1 \in H$ the two-point boundary problem (5) has a unique mild solution

$$\begin{bmatrix} x_1(\cdot) \\ x_2(\cdot) \end{bmatrix} \in C([0,\tau]; H) \times C([0,\tau]; H) \quad (52)$$

which continuously depends on the data x_0 and z_1.

7. Final remarks

In this paper we have only dealt with the finite time case but it is also possible to extend the results in a natural way to cover infinite time horizon. Of course, in the latter case the corresponding Lyapunov equation becomes an algebraic one. Although this is a very interesting topic it is beyond the scope of the current paper.

Acknowledgement. This work was supported by the Polish State Committee for Scientific Research under grant no. 4/T11A/007/23.

References

[1] J. Alber: On implemented semigroups. Semigroup Forum 2001, vol. 63.
[2] K.-J. Engel, R. Nagel: One-Parameter Semigroups for Linear Evolution Equations. Springer-Verlag, Berlin 2000.
[3] Z. Emirsajłow: On admissible input elements for the implemented semigroup. Semigroup Forum 2003 (submitted).
[4] Z. Emirsajłow: Implemented Semigroup for Infinite-Dimensional Sylvester and Lyapunov Equations - Systems and Control Theory Point of View (book in preparation).
[5] Z. Emirsajłow, S. Townley: On application of the implemented semigroup to a problem arising in optimal control. International Journal of Control 2003 (submitted).
[6] Z. Emirsajłow, S. Townley: From PDEs with a boundary control to the abstract state equation with an unbounded input operator. European Journal of Control 2000, vol. 7.

Tytuł: Półgrupa złożona dla nieskończenie wymiarowych równań Lapunowa

Artykuł recenzowany