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Optimal control problem for infinite variables hyperbolic systems with time lags

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In this paper, by using the theorems of [Lions (1971) and Lions & Magenes (1972)], the optimal control problem for distributed hyperbolic systems, involving second order operator with an infinite number of variables, in which constant lags appear both in the state equations and in the boundary conditions is considered. The optimality conditions for Neumann boundary conditions are obtained and the set of inequalities that characterize these conditions is formulated. Also, several mathematical examples for derived optimality conditions are presented. Finally, we consider an optimal distributed control problem for $(n \times n)$-infinite variables hyperbolic systems.

Key words: optimal control, hyperbolic system, time delays, distributed control problems, Neumann conditions, existence and uniqueness of solutions, second order operator with an infinite number of variables.

1. Introduction

Various optimization problems associated with the optimal control of distributed parameter systems with time lags appearing in the boundary conditions have been studied recently by Wang (1975); Knowles (1978); Wong (1987) and Kowalewski (1993a,b, 1995, 1998, 1999, 2000).

Infinite dimensional systems can be used to describe many physical phenomena in the real world. Well-known examples are heat conduction, vibration of elastic material, diffusion-reaction processes, population systems, and many others. Thus, the optimal control theory for infinite dimensional systems has a wide range of applications in engineering, economics, and some other fields.

We refer, for instance, to Imanuvilov (1998), and Lions & Enrique (1955) for the application of similar results in quantum field and as an physical examples. In Imanuvilov (1998) the local controllability problem for the Navier-Stokes equations that described

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by an $n \times n$ systems were established. In Lions & Enrique (1955), the controllability of the motion of a fluid by means of the action of a vibrating shell coupled at the boundary of the fluid is formulated.

The necessary and sufficient conditions of optimality for systems governed by different types of partial differential operators defined on spaces with finite number of variables are discussed for example in [Lions (1971), Lions & Magenes (1972) and Petukhov (1995)].

The optimal control problem of systems governed by different types of operators defined on spaces with an infinite number of variables are initiated and proved in [Gali & El-Saify (1982,1989) and Kotarski (1997)].

In [Bahaa (2008a), El-Saify & Bahaa (2001), and El-Saify, Serag & Bahaa (2000)], we have obtained the set of inequalities that characterize the optimal control for $n \times n$ system governed by elliptic, parabolic and hyperbolic equation of infinite number of variables with different conditions.

In this paper, we consider the optimal control problem for linear hyperbolic systems in which constant time lags appear both in the state equations and in the Neumann boundary conditions involving second order operator with an infinite number of variables. The optimal control is characterized by the adjoint equation. Using this characterization particular properties of the optimal control are proved.

This paper is organized as follows. In section 2, we introduce spaces of functions of infinitely many variables. In section 3 we formulate some facts and new results which enable us to statement the Neumann problem for hyperbolic operator with an infinite number of variables. In section 4, the distributed optimal control problem for this case is formulated, then we give the necessary and sufficient conditions for the control to be an optimal. In the end of this section we present some special cases for derived the optimality conditions. In section 5, we generalized the discussion to two cases, the first case: The optimal control for (2×2) coupled hyperbolic systems with infinite number of variables is studied. The second case: The optimal control for $(n \times n)$ coupled hyperbolic systems with infinite number of variables was be formulated. In section 6, we concluded our results.

2. Sobolev spaces with infinite number of variables

This section covers the basic notations, definitions and properties, which are necessary to present this work (Berezanskii, 1975), (Gali & El-Saify 1982; 1983), (El-Saify & Serag & Bahaa, 2000) and (El-Saify & Bahaa, 2001).

Let $(p_k(t))_{k=1}^{\infty}$ be a sequence of weights, fixed in all that follows, such that;

$$0 < p_k(t) \in C^{\infty}(\mathbb{R}^1), \int_{\mathbb{R}^1} p_k(t)dt = 1,$$

with respect to it we introduce on the region $\mathbb{R}^\infty = \mathbb{R}^1 \times \mathbb{R}^1 \times \dots$, the measure $d\rho(x)$ by setting,

$$d\rho(x) = p_1(x_1)dx_1 \otimes p_2(x_2)dx_2 \otimes \dots, (\mathbb{R}^\infty \ni x = (x_k)_{k=1}^\infty, x_k \in \mathbb{R}^1).$$

On $\mathbb{R}^\infty$ we construct the space $L^2(\mathbb{R}^\infty, d\rho(x))$ with respect to this measure i.e., $L^2(\mathbb{R}^\infty, d\rho(x))$ is the space of quadratic integrable functions on $\mathbb{R}^\infty$. We shall often set $L^2(\mathbb{R}^\infty, d\rho(x)) = L^2(\mathbb{R}^\infty)$.

It is classical result that $L^2(\mathbb{R}^\infty)$ is a Hilbert space for the scalar product

$$(\phi, \psi)_{L^2(\mathbb{R}^\infty)} = \int_{\mathbb{R}^\infty} \phi(x)\psi(x)d\rho(x).$$

We next consider a Sobolev space in the case of an unbounded region. For functions which are $\ell = 1, 2, \dots$ times continuously differentiable up to the boundary Γ of $\mathbb{R}^\infty$ (Γ is meant to be the boundary of the support of the measure $d\rho(x)$) and which vanish in a neighborhood of ∞, we introduce the scalar product

$$(\phi, \psi)_{W^\ell(\mathbb{R}^\infty)} = \sum_{|\alpha| \leqslant \ell} (D^\alpha \phi, D^\alpha \psi)_{L^2(\mathbb{R}^\infty)},$$

where D^α is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2} \dots}, \quad |\alpha| = \sum_{i=1}^\infty \alpha_i,$$

and the differentiation is taken in the sense of generalized functions on $\mathbb{R}^\infty$, and after the completion, we obtain the Sobolev space $W^\ell(\mathbb{R}^\infty)$. So in short, Sobolev space $W^1(\mathbb{R}^\infty)$ is defined by :

$$W^1(\mathbb{R}^\infty) = \{\phi | \phi, D\phi \in L^2(\mathbb{R}^\infty)\}.$$

As in the case of a bounded region, the space $W^1(\mathbb{R}^\infty)$ form the space with positive norm $||.||_{W^1(\mathbb{R}^\infty)}$. We can construct the space $W^{-1}(\mathbb{R}^\infty) = (W^1(\mathbb{R}^\infty))^*$ with negative norm $||.||_{W^{-1}(\mathbb{R}^\infty)}$ with respect to the space $W^0(\mathbb{R}^\infty) = L^2(\mathbb{R}^\infty)$ with zero norm $||.||_{L^2(\mathbb{R}^\infty)}$, then we have the following equipped,

$$W^1(\mathbb{R}^\infty) \subseteq L^2(\mathbb{R}^\infty) \subseteq W^{-1}(\mathbb{R}^\infty),$$

$$||\phi||_{W^1(\mathbb{R}^\infty)} \geqslant ||\phi||_{L^2(\mathbb{R}^\infty)} \geqslant ||\phi||_{W^{-1}(\mathbb{R}^\infty)}.$$

Let $L^2(0, T; W^1(\mathbb{R}^\infty))$ be the space of square integrable measurable functions $t \to \phi(t)$ of $]0, T[\to W^1(\mathbb{R}^\infty)$, where the variable t denotes the " time "; $t \in]0, T[$, $T < \infty$. This space is a Hilbert space with respect to the scalar product

$$(\phi, \psi)_{L^2(0,T;W^1(\mathbb{R}^\infty))} = \int_0^T (\phi(t), \psi(t))_{W^1(\mathbb{R}^\infty)} dt,$$

and its dual is the space $L^2(0,T;W^{-1}(\mathbb{R}^\infty))$, analogously, we can define the spaces $L^2(0,T;L^2(\mathbb{R}^\infty))$ which we shall denote by $L^2(Q)$.

Let $\Omega \subset \mathbb{R}^\infty$ is a bounded, open set with boundary Γ, which is a C^∞ manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ and denote by $W^1(\Omega,\mathbb{R}^\infty,d\rho(x))$ (briefly $W^1(\Omega,\mathbb{R}^\infty)$) the Sobolev space of vector function $y(x)$ defined on Ω.

The construction of the Cartesian product of n-times to the above Hilbert spaces can be construct, for example

$$(W^1(\Omega,\mathbb{R}^\infty))^n = \underbrace{W^1(\Omega,\mathbb{R}^\infty)\times W^1(\Omega,\mathbb{R}^\infty)\times\cdots\times W^1(\Omega,\mathbb{R}^\infty)}_{n-times} = \prod_{i=1}^{n}(W^1(\Omega,\mathbb{R}^\infty))^i,$$

with norm defined by:

$$||\boldsymbol{\phi}||_{(W^1(\Omega,\mathbb{R}^\infty))^n} = \sum_{i=1}^{n} ||\phi_i||_{W^1(\Omega,\mathbb{R}^\infty)},$$

where $\boldsymbol{\phi} = (\phi_1,\phi_2,...,\phi_n) = (\phi_i)_{i=1}^n$ is a vector function and $\phi_i \in W^1(\Omega,\mathbb{R}^\infty)$.

Finally, we have the following chain:

$$(L^2(0,T;W^1(\Omega,\mathbb{R}^\infty)))^n \subseteq (L^2(Q))^n \subseteq (L^2(0,T;W^{-1}(\Omega,\mathbb{R}^\infty)))^n,$$

where $(L^2(0,T;W^{-1}(\Omega,\mathbb{R}^\infty)))^n$ are the dual spaces of $(L^2(0,T;W^1(\Omega,\mathbb{R}^\infty)))^n$. The spaces considered in this paper are assumed to be real.

3. Mixed Neumann problem for infinite variables hyperbolic system with time lags

The object of this section is to formulate the following mixed initial boundary value problem for the hyperbolic system with time lag which defines the state of the system model (El-Saify & Bahaa, 2001).

$$\frac{\partial^2 y(u)}{\partial t^2} + \mathcal{A}(t)y(u) + b(x,t)y(x,t-h;u) = u, \quad x\in\Omega, t\in(0,T), \tag{1}$$

$$y(x,t') = \Phi_0(x,t'), \quad x\in\Omega, t'\in[-h,0), \tag{2}$$

$$y(x,0) = y_0(x), \quad x\in\Omega, \tag{3}$$

$$\frac{\partial y(x,0)}{\partial t} = y_l(x), \quad x\in\Omega, \tag{4}$$

$$\frac{\partial y}{\partial \nu_{\mathcal{A}}} = c(x,t)y(x,t-h)+v, \quad x\in\Gamma, t\in(0,T), \tag{5}$$

$$y(x,t') = \Psi_0(x,t'), \quad x\in\Gamma, t'\in[-h,0), \tag{6}$$

where $\Omega \subset \mathbb{R}^\infty$ has the same properties as in the above section. Also, we have

$$y \equiv y(x,t;u),\ u \equiv u(x,t),\ v \equiv v(x,t),$$

$$Q = \Omega \times (0,T),\ \overline{Q} = \overline{\Omega} \times [0,T],\ Q_0 = \Omega \times [-h,0),\ \Sigma = \Gamma \times (0,T),\ \Sigma_0 = \Gamma \times [-h,0),$$

h is a time lag, b and c are given real C^∞ functions defined on $\overline{Q}$ ($\overline{Q}$ closure of Q) and on Σ, respectively, and Φ_0, Ψ_0 are initial functions defined on Q_0 and Σ_0 respectively.

The hyperbolic operator $\frac{\partial^2}{\partial t^2} + \mathcal{A}(t)$ in the state equation (1) is a second order hyperbolic operator with infinite number of variables and $\mathcal{A}(t)$ (Berezanskii, 1975), (Gali & El-Saify, 1982; 1983) and (Kotarski & El-Saify & Bahaa, 2002b) is given by:

$$\begin{aligned} \mathcal{A}(t)y(x) &= \left(-\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k(x_k,t)}} \frac{\partial^2}{\partial x_k^2} \sqrt{p_k(x_k,t)} + q(x,t) \right) y(x) \\ &= -\sum_{k=1}^{\infty} D_k^2 y(x) + q(x,t)y(x), \end{aligned} \tag{7}$$

where

$$D_k y_(x) = \frac{1}{\sqrt{p_k(x_k,t)}} \frac{\partial}{\partial x_k} \sqrt{p_k(x_k,t)} y(x), \tag{8}$$

and $q(x,t)$ is a real-valued function in x which is a bounded and measurable on $\Omega \subset \mathbb{R}^\infty$, such that $q(x,t) \geqslant c_0 > 1$, c_0 is a constant. The operator $\mathcal{A}(t)$ is a bounded second order self-adjoint elliptic partial differential operator with an infinite number of variables maps $W^1(\Omega,\mathbb{R}^\infty)$ onto $W^{-1}(\Omega,\mathbb{R}^\infty)$.

For this operator we define the bilinear form as follows:

Definition 3.1 *For each $t \in (0,T)$, we define a family of bilinear forms on $W^1(\Omega,\mathbb{R}^\infty)$ by:*

$$\pi(t;y,\phi) = (\mathcal{A}(t)y,\phi)_{L^2(\Omega,\mathbb{R}^\infty)}, \quad y,\phi \in W^1(\Omega,\mathbb{R}^\infty), \tag{9}$$

where $\mathcal{A}(t)$ maps $W^1(\Omega,\mathbb{R}^\infty)$ onto $W^{-1}(\Omega,\mathbb{R}^\infty)$ and takes the above form. Then

$$\begin{aligned} \pi(t;y,\phi) &= \Big(\mathcal{A}(t)y,\phi \Big)_{L^2(\Omega,\mathbb{R}^\infty)} \\ &= \left(-\sum_{k=1}^{\infty} D_k^2 y(x) + q(x,t)y(x), \phi(x) \right)_{L^2(\Omega,\mathbb{R}^\infty)} \\ &= \int_\Omega \sum_{k=1}^{\infty} D_k y(x) D_k \phi(x)\, d\rho(x) + \int_\Omega q(x,t) y(x) \phi(x)\, d\rho(x). \end{aligned}$$

Lemma 3.1 *The bilinear form $\pi(t;y,\phi)$ is coercive on $W^1(\Omega,\mathbb{R}^\infty)$, that is*

$$\pi(t;y,y) \geqslant \lambda||y||^2_{W^1(\Omega,\mathbb{R}^\infty)}, \quad \lambda > 0. \tag{10}$$

Proof It is well known that the ellipticity of $\mathcal{A}(t)$ is sufficient for the coerciveness of $\pi(t;y,\phi)$ on $W^1(\Omega,\mathbb{R}^\infty)$.

$$\pi(t;\phi,\psi) = \int_\Omega \sum_{k=1}^{\infty} D_k\phi(x)D_k\psi(x)\,d\rho + \int_\Omega q(x,t)\phi(x)\psi(x)\,d\rho.$$

Then

$$\begin{aligned}
\pi(t;y,y) &= \int_\Omega \sum_{k=1}^{\infty} |D_k y(x)|^2\,d\rho(x) + \int_\Omega q(x,t)|y(x)|^2\,d\rho(x) \\
&\geqslant \sum_{k=1}^{\infty} ||D_k y(x)||^2_{L^2(\Omega,\mathbb{R}^\infty)} + c_0||y(x)||^2_{L^2(\Omega,\mathbb{R}^\infty)} \\
&= ||y(x)||^2_{W^1(\Omega,\mathbb{R}^\infty)} + c_0||y(x)||^2_{L^2(\Omega,\mathbb{R}^\infty)} \\
&\geqslant ||y(x)||^2_{W^1(\Omega,\mathbb{R}^\infty)} \\
&= \lambda||y||^2_{W^1(\Omega,\mathbb{R}^\infty)}, \quad \lambda > 0.
\end{aligned}$$

□

Also we have:

$$\left.\begin{array}{l}
\forall y,\phi \in W^1(\Omega,\mathbb{R}^\infty) \text{ the function} t \to \pi(t;y,\phi) \\
\qquad\qquad \text{is continuously differentiable in } (0,T) \text{ and} \\
\pi(t;y,\phi) = \pi(t;\phi,y)
\end{array}\right\} \tag{11}$$

Then the left-hand side of the boundary condition (5) may be written in the following form:

$$\frac{\partial y(u)}{\partial \nu_{\mathcal{A}}} = \sum_{k=1}^{\infty} (D_k y(u))\cos(n,x_k) = d(x,t), \tag{12}$$

where $\dfrac{\partial}{\partial \nu_{\mathcal{A}}}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n,x_k)$ is the $k-th$ direction cosine of n, with n being the normal at Γ exterior to Ω, and

$$d(x,t) = c(x,t)y(x,t-h) + v(x,t) \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma). \tag{13}$$

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial boundary value problem (1)–(6) for the case where the control u belong to $L^2(Q)$.

To this purpose, for any pair of real numbers $r,s \geqslant 0$, we introduction the Sobolev space $W^{r,s}(Q)$ (Lions and Magenes, 1972, Vol. 2, p. 6) defined by

$$W^{r,s}(Q) = L^2\left(0,T;W^r(\Omega,\mathbb{R}^\infty)\right) \cap W^s\left(0,T;L^2(\Omega,\mathbb{R}^\infty)\right) \tag{14}$$

which is a Hilbert space normed by

$$\left(\int_0^T ||y(t)||^2_{W^r(\Omega,\mathbb{R}^\infty)}dt + ||y||^2_{W^s(0,T;L^2(\Omega,\mathbb{R}^\infty))}\right)^{1/2}, \tag{15}$$

where $W^s\left(0,T;L^2(\Omega,\mathbb{R}^\infty)\right)$ denotes the Sobolev space of order s of functions defined on $(0,T)$ and taking values in $L^2(\Omega,\mathbb{R}^\infty)$.

For simplicity, we introduce the following notation:

$$E_j \triangleq ((j-1)h, jh), \quad Q_j = \Omega \times E_j, \quad \Sigma_j = \Gamma \times E_j,\ j = 1,2,\ldots. \tag{16}$$

The existence of a unique solution for the mixed initial-boundary value problem (1)–(6) was verified in Kowalewski (1993a) and in Kotarski, El-Saify & Bahaa (2002a,b).

Theorem 3.2 *Let y_0, y_1 ,Φ_0,Ψ_0, v and u be given with $y_0 \in W^2(\Omega,\mathbb{R}^\infty)$, $y_1 \in W^{3/2}(\Omega,\mathbb{R}^\infty), \Phi_0 \in W^{2,2}(Q_0), \Psi_0 \in W^{3/2,3/2}(\Sigma_0)$, $v \in W^{3/2,3/2}(\Sigma)$, $u \in W^{0,1}(Q)$ and the following compatibility relations:*

$$\frac{\partial y_0}{\partial \nu_{\mathcal{A}}}(x,0) = d_1(x,0), \quad on\ \Gamma, \tag{17}$$

$$\frac{\partial y_1}{\partial \nu_{\mathcal{A}}}(x,0) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \nu_{\mathcal{A}}}\right)\right) y_0(x,0) = \frac{\partial}{\partial t} d_1(x,0), \quad on\ \Gamma. \tag{18}$$

Then, there exists a unique solution $y \in W^{2,2}(Q)$ for the mixed initial-boundary value problem (1)–(6) *with $y(\cdot,h) \in W^2(\Omega,\mathbb{R}^\infty)$ and $y'(\cdot,h) \in W^{3/2}(\Omega,\mathbb{R}^\infty)$ for $j = 1,2,\ldots$*

4. Problem formulation. Optimization theorems

Now, we formulate the optimal control problem for (1)–(6) in the context of the Theorem 3.2, that is $u \in L^2(Q)$.

Let us denote by $U = L^2(Q)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \int_Q [y(x,t;v) - z_d]^2\, d\rho\, dt + \lambda_2 \int_Q (Nv)v\, d\rho\, dt, \tag{19}$$

where $\lambda_i \geqslant 0$, $\lambda_1+\lambda_2>0$, z_d is a given element in $L^2(Q)$ and N is a positive linear operator on $L^2(Q)$ into $L^2(Q)$.

Control constraints: We define the set of admissible controls U_{ad} such that

$$U_{ad} \text{ is closed, convex subset of } U=L^2(Q). \tag{20}$$

Let $y(x,t;u)$ denote the solution of the mixed initial-boundary value problem (1)–(6) at (x,t) corresponding to a given control $u\in U_{ad}$. We note from Theorem 3.2 that for any $u\in U_{ad}$ the performance functional (19) is well-defined since $y(u)\in W^{2,2}(Q)\subset L^2(Q)$.

Making use of the Loins's scheme we shall derive the necessary and sufficient conditions of optimality for the optimization problem (1)–(6), (19) and (20). The solving of the formulated optimal control problem is equivalent to seeking a $u^*\in U_{ad}$ such that

$$I(u^*)\leqslant I(u),\quad \forall u\in U_{ad}.$$

From the Lion's scheme (Theorem 1.3 of Lions, 1971, p.10), it follows that for $\lambda_2>0$ a unique optimal control u^* exists. Moreover, u^* is characterized by the following condition:

$$I'(u^*)(u-u^*)\leqslant 0,\quad \forall u\in U_{ad}. \tag{21}$$

For the performance functional of form (19) the relation (21) can be expressed as

$$-\lambda_1\int_Q (z_d-y(u^*))[y(u)-y(u^*)]\,d\rho\,dt+\lambda_2\int_Q Nu^*(u-u^*)\,d\rho\,dt\leqslant 0,\quad \forall u\in U_{ad}. \tag{22}$$

We shall apply Theorem 3.2 to the control problem of (1)–(6). To simplify (21), we introduce the adjoint equation and for every $u\in U_{ad}$, we define the adjoint variable $p=p(u)=p(x,t;u)$ as the solution of the system

$$\begin{aligned}\frac{\partial^2 p(u)}{\partial t^2}&+\mathcal{A}^*(t)p(u)+b(x,t+h)p(x,t+h;u)\\ &=\lambda_1(y(u)-z_d),\quad x\in\Omega,\ t\in(0,T-h),\end{aligned} \tag{23}$$

$$\frac{\partial^2 p(u)}{\partial t^2}+\mathcal{A}^*(t)p(u)=\lambda_1(y(u)-z_d),\quad x\in\Omega,\ t\in(T-h,T), \tag{24}$$

$$p(x,T,u)=0,\quad x\in\Omega, \tag{25}$$

$$\frac{\partial p(x,T;u)}{\partial t}=0,\quad x\in\Omega, \tag{26}$$

$$\frac{\partial p}{\partial\nu_{\mathcal{A}^*}}(x,t)=c(x,t+h)p(x,t+h;u),\quad x\in\Gamma,\ t\in(0,T-h), \tag{27}$$

$$\frac{\partial p(u)}{\partial\nu_{\mathcal{A}^*}}(x,t)=0,\quad x\in\Gamma,\ t\in(T-h,T). \tag{28}$$

The existence of a unique solution for the problem (23)–(28) on the cylinder $\Omega \times (0,T)$ can be proved using a constructive method. It is easy to notice that for given z_d and u, problem (23)–(28) can be solved backwards in time starting from $t = T$, i.e., first solving (23)–(28) on the sub-cylinder Q_k and in turn on Q_{k-1}, etc. until the procedure covers the whole cylinder $\Omega \times (0,T)$. For this purpose, we may apply Theorem 3.2 (with an obvious change of variables).

Hence, using Theorem 3.2, the following result can be proved.

Lemma 4.1 *Let the hypothesis of Theorem* 3.2 *be satisfied. Then for given $z_d \in L^2(\Omega)$ and any $u \in W^{0,1}(Q)$, there exists a unique solution $p(u) \in W^{2,2}(Q))$ for the adjoint problem* (23)–(28).

Now, we have the main result.

Theorem 4.2 *If the assumptions concerning system* (1)–(6) *and controllability condition* (20) *are satisfied, then optimal control u^* exists and is characterized by the following condition*

$$\int_0^T \int_\Omega (p(u^*) + \lambda_2 N u^*)(u - u^*)\, d\rho\, dt \leqslant 0, \quad \forall u \in U_{ad}, \tag{29}$$

where $p(u^*)$ is the solution of the adjoint system (23)–(28).

Proof We simplify (22) using the adjoint equation (23)–(28). For this purpose setting $u = u^*$ in (23)–(28), multiplying both sides of (23) and (24) by $y(u) - y(u^*)$, then integrating over $\Omega \times (0, T-h)$ and $\Omega \times (T-h, T)$ respectively, and then adding both sides of (23) and (24) we get

$$\begin{aligned}
&-\lambda_1 \int_Q (z_d - y(x,t;u^*))(y(x,t;u) - y(x,t;u^*))\, d\rho\, dt \\
&= \int_0^T \int_\Omega \left(\frac{\partial^2 p(u^*)}{\partial t^2} + \mathcal{A}^*(t) p(u^*) \right)(y(u) - y(u^*))\, d\rho\, dt \\
&\quad + \int_0^{T-h} \int_\Omega b(x,t+h) p(x,t+h;u^*)[y(x,t;u) - y(x,t;u^*)]\, d\rho\, dt \\
&= \int_\Omega p'(x,t.;u^*)(y(x,t;u) - y(x,t;u^*))\, d\rho \\
&\quad + \int_0^T \int_\Omega p(u^*) \frac{\partial^2}{\partial t^2}(y(u) - y(u^*))\, d\rho\, dt + \int_0^T \int_\Omega \mathcal{A}^*(t) p(u^*)(y(u) - y(u^*))\, d\rho\, dt
\end{aligned} \tag{30}$$

$$+\int_0^{T-h}\int_\Omega b(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,d\rho\,dt.$$

Then, applying (26), formula (30) can be expressed as

$$-\lambda_1\int_0^T\int_\Omega (z_d-y(x,t;u^*))(y(x,t;u)-y(x,t;u^*))\,d\rho\,dt = \tag{31}$$

$$\int_0^T\int_\Omega p(u^*)\frac{\partial^2}{\partial t^2}(y(u)-y(u^*))\,d\rho\,dt+\int_0^T\int_\Omega \mathcal{A}^*(t)p(u^*)(y(u)-y(u^*))\,d\rho\,dt$$

$$+\int_0^{T-h}\int_\Omega b(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,d\rho\,dt.$$

Using (1), the first integral on the right-hand side of (31) can be rewritten as

$$\begin{aligned}
&\int_0^T\int_\Omega p(u^*)\frac{\partial^2}{\partial t^2}(y(u)-y(u^*))\,d\rho\,dt\\
&=\int_0^T\int_\Omega p(u^*)(u-u^*)\,d\rho\,dt-\int_0^T\int_\Omega p(u^*)\mathcal{A}(t)(y(u)-y(u^*))\,d\rho\,dt\\
&\quad-\int_0^T\int_\Omega p(x,t;u^*)b(x,t)(y(x,t-h;u)y(x,t-h;u^*))\,d\rho\,dt \qquad (32)\\
&=\int_0^T\int_\Omega p(u^*)(u-u^*)\,d\rho\,dt-\int_0^T\int_\Omega p(u^*)\mathcal{A}(t)(y(u)-y(u^*))\,d\rho\,dt\\
&\quad-\int_h^{T-h}\int_\Omega p(x,t'+h;u^*)b(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,d\rho\,dt'.
\end{aligned}$$

The second integral on the right-hand side of (31), in view of Green's formula, can be expressed as

$$\begin{aligned}
&\int_0^T\int_\Omega \mathcal{A}^*(t)p(u^*)(y(u)-y(u^*))\,d\rho\,dt=\int_0^T\int_\Omega p(u^*)\mathcal{A}(t)(y(u)-y(u^*))\,d\rho\,dt\\
&\quad+\int_0^T\int_\Gamma p(u^*)\left(\frac{\partial y(u)}{\partial\nu_{\mathcal{A}}}-\frac{\partial y(u^*)}{\partial\nu_{\mathcal{A}}}\right)d\Gamma\,dt-\int_0^T\int_\Gamma\frac{\partial p(u^*)}{\partial\nu_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt.
\end{aligned} \tag{33}$$

Using the boundary condition (5), the second component on the right-hand side of (33) can be written as

$$\int_0^T \int_\Gamma p(u^*)\left(\frac{\partial y(u)}{\partial \nu_{\mathcal{A}}} - \frac{\partial y(u^*)}{\partial \nu_{\mathcal{A}}}\right) d\Gamma\, dt$$
$$= \int_0^T \int_\Gamma p(x,t;u^*)c(x,t)(y(x,t-h;u) - y(x,t-h;u^*))\, d\Gamma\, dt \tag{34}$$
$$= \int_{-h}^{T-h} \int_\Gamma p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u) - y(x,t';u^*))\, d\Gamma\, dt'.$$

The last component in (33) can be rewritten as

$$\int_0^T \int_\Gamma \frac{\partial p(u^*)}{\partial \nu_{\mathcal{A}^*}}(y(u) - y(u^*))\, d\Gamma\, dt = \int_0^{T-h} \int_\Gamma \frac{\partial p(u^*)}{\partial \nu_{\mathcal{A}^*}}(y(u) - y(u^*))\, d\Gamma\, dt$$
$$+ \int_{T-h}^{T} \int_\Gamma \frac{\partial p(u^*)}{\partial \nu_{\mathcal{A}^*}}(y(u) - y(u^*))\, d\Gamma\, dt. \tag{35}$$

Substituting (35), (34) into (33) and then (33), (32) into (31) we obtain

$$-\lambda_1 \int_0^T \int_Q (z_d - y(t^*;u^*))(y(t^*;u) - y(t^*;u^*))\, d\rho\, dt =$$
$$\int_0^T \int_\Omega p(u^*)(u - u^*)\, d\rho\, dt - \int_0^T \int_\Omega p(u^*)\mathcal{A}(t)(y(u) - y(u^*))\, d\rho\, dt$$
$$- \int_{-h}^{0} \int_\Omega b(x,t+h)p(x,t+h;u^*)(y(x,t;u) - y(x,t;u^*))\, d\rho\, dt$$
$$- \int_0^{T-h} \int_\Omega b(x,t+h)p(x,t+h;u^*)(y(x,t;u) - y(x,t;u^*))\, d\rho\, dt$$
$$+ \int_0^T \int_\Omega p(u^*)\mathcal{A}(t)(y(u) - y(u^*))\, d\rho\, dt \tag{36}$$
$$+ \int_{-h}^{0} \int_\Gamma c(x,t+h)p(x,t+h;u^*)(y(x,t;u) - y(x,t;u^*))\, d\Gamma\, dt$$

$$+\int_0^{T-h}\int_\Gamma c(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,d\Gamma\,dt$$
$$-\int_0^{T-h}\int_\Gamma \frac{\partial p(u^*)}{\partial \nu_{\mathcal{A}^*}}(y(x,t;u)-y(x,t;u^*))\,d\Gamma\,dt$$
$$-\int_{T-h}^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial \nu_{\mathcal{A}^*}}(y(x,t;u)-y(x,t;u^*))\,d\Gamma\,dt$$
$$+\int_0^{T-h}\int_\Omega b(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,d\rho\,dt.$$

Afterwards using the fact that $y(x,t;u)=y(x,t;u^*)=\Phi_0(x,t)$ for $x\in\Omega$ and $t\in[-h,0)$ and $y(x,t;u)=y(x,t;u^*)=\Phi_0(x,t)$ for $x\in\Gamma$ and $t\in[-h,0)$ we obtain

$$-\lambda_1\int_Q (z_d-y(x,t;u^*))(y(x,t;u)-y(x,t;u^*))\,d\rho\,dt=\int_0^T\int_\Omega p(u^*)(u-u^*)\,d\rho\,dt. \quad (37)$$

Substituting (37) into (22) gives (29). □

Mathematical Examples

Example 4.1 *Consider now the particular case where $U_{ad}=U=L^2(Q)$ (no constraints case). Thus the minimum condition* (29) *is satisfied when*

$$u^*=-\lambda_2 N^{-1}p(u^*).$$

If N is the identity operator on $L^2(Q)$, then from the Lemma 4.1 *follows that $u^*\in W^{2,2}(Q)$.*

Example 4.2 *We can also consider an analogous optimal control problem where the performance functional is given by:*

$$I(u)=\lambda_1\int_\Sigma [y|_\Sigma(x,t;u)-z_{\Sigma d}]^2 d\Gamma dt+\lambda_2\int_Q (Nu)u\,d\rho\,dt, \quad (38)$$

where $z_{\Sigma d}\in L^2(\Sigma)$.

From Theorem 3.2 *and the Trace Theorem (Lions & Magenes, 1972, Vol 2, p.9), for each $u\in L^2(Q)$, there exists a unique solution $y(u)\in W^{2,2}(Q)$ with $y|_\Sigma\in L^2(\Sigma)$. Thus, I* *is well defined. Then, the optimal control u^* is characterized by:*

$$\lambda_1 \int_\Sigma (y(u^*) - z_{\Sigma d})[y(u) - y(u^*)]d\Gamma dt + \lambda_2 \int_Q Nu^*(u-u^*)\,d\rho\,dt \leqslant 0, \quad \forall u \in U_{ad}. \quad (39)$$

The above inequality can be simplified by introducing an adjoint equation, we define the adjoint variable $p = p(u) = p(x,t;u)$ as the solution of the equation

$$\frac{\partial^2 p(u)}{\partial t^2} + \mathcal{A}^*(t)p(u) + b(x,t+h)p(x,t+h;u) = 0, \quad x \in \Omega, t \in (0, T-h), \quad (40)$$

$$\frac{\partial^2 p(u)}{\partial t^2} + \mathcal{A}^*(t)p(u) = 0, \quad x \in \Omega, t \in (T-h, T), \quad (41)$$

$$p(x,T,u) = 0, \quad x \in \Omega, \quad (42)$$

$$\frac{\partial p(x,T;u)}{\partial t} = 0, \quad x \in \Omega, \quad (43)$$

$$\frac{\partial p}{\partial \nu_{\mathcal{A}^*}}(x,t) = c(x,t+h)p(x,t+h;u) + \lambda_1(y(u) - z_{\Sigma d}), \quad x \in \Gamma, t \in (0, T-h), \quad (44)$$

$$\frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}}(x,t) = \lambda_1(y(u) - z_{\Sigma d}), \quad x \in \Gamma, t \in (T-h, T). \quad (45)$$

Then using Theorem 3.2 *we can establish the existence of a unique solution $p = p(u^*) = p(x,t;u^*) \in W^{2,2}(Q)$ for* (40)–(45).

As in the above section, we have the following result.

Lemma 4.3 *Let the hypothesis of Theorem* 3.2 *be satisfied. Then, for given $z_{\Sigma d} \in L^2(\Sigma)$ and any $u \in L^2(Q)$, there exists a unique solution $p(u^*) \in W^{2,2}(Q)$ to the adjoint problem* (40)–(45).

Using the adjoint equations (40)–(45) *in this case, the condition* (39) *can also be written in the following form*

$$\int_Q (p(u^*) + \lambda_2 Nu^*)(u-u^*)\,d\rho\,dt \leqslant 0, \quad \forall u \in U_{ad}. \quad (46)$$

The following result is now summarized.

Theorem 4.4 *For the problem* (1)–(6) *with the performance function* (38) *with $z_{\Sigma d} \in L^2(\Sigma)$ and $\lambda_2 > 0$, and with constraint* (20)*, and with adjoint equations* (40)–(45)*, there exists a unique optimal control u^* which satisfies the minimum condition* (46).

5. Generalization

Optimal control problem presented her can be extended to certain different two cases. Case 1: Optimal control problem for (2×2) coupled hyperbolic systems with infinite number of variables and with time lags. Case 2: Optimal control problem for $(n\times n)$ coupled hyperbolic systems with infinite number of variables and with time lags.

5.1. Optimal control for (2×2) coupled hyperbolic systems with infinite number of variables and with time lags

We can extend the discussions to study the optimal control for (2×2) coupled hyperbolic systems with infinite number of variables and with time lags. We consider the case where $u=(u_1,u_2)\in L^2(Q)\times L^2(Q)$, the performance functional is given by:

$$I(u)=I_1(u)+I_2(u)=\sum_{i=1}^{2}\left(\lambda_1\int_Q [y_i(x,t;u)-z_{id}]^2\,d\rho\,dt+\lambda_2\int_Q (N_iu_i)u_i\,d\rho\,dt\right), \quad (47)$$

where $z_d=(z_{1d},z_{2d})\in (L^2(Q))^2$ and N_i is a positive linear operator on $L^2(Q)$ into $L^2(Q)$, $i=1,2$.

Then the optimality conditions are given by:

State equations:

$$\frac{\partial^2 y_1(u)}{\partial t^2}+\left(-\sum_{k=1}^{\infty}D_k^2+q(x,t)\right)y_1(u)+b_1(x,t)y_1(x,t-h;u)+y_1(u)-y_2(u)=u_1, \quad \text{in}\, Q, \quad (48)$$

$$\frac{\partial^2 y_2(u)}{\partial t^2}+\left(-\sum_{k=1}^{\infty}D_k^2+q(x,t)\right)y_2(u)+b_2(x,t)y_2(x,t-h;u)+y_1(u)+y_2(u)=u_2, \quad \text{in}\, Q, \quad (49)$$

$$y_1(x,t',u)=\Phi_{0,1}(x,t'),\quad y_2(x,t';u)=\Phi_{0,2}(x,t'),\quad x\in\Omega,\, t'\in[-h,0), \quad (50)$$

$$y_1(x,0;u)=y_{0,1}(x),\quad y_2(x,0;u)=y_{0,2}(x),\quad x\in\Omega, \quad (51)$$

$$\frac{\partial y_1(x,0;u)}{\partial t}=y_{1,1}(x),\quad \frac{\partial y_2(x,0;u)}{\partial t}=y_{1,2}(x),\quad x\in\Omega, \quad (52)$$

$$\frac{\partial y_1(u)}{\partial \nu_{\mathcal{A}}}=c_1(x,t)y_1(x,t-h)+v_1,\quad \frac{\partial y_2(u)}{\partial \nu_{\mathcal{A}}}=c_2(x,t)y_2(x,t-h)+v_2, \quad x\in\Gamma,\, t\in(0,T), \quad (53)$$

$$y_1(x,t',u) = \Psi_{0,1}(x), \quad y_2(x,t';u) = \Psi_{0,2}(x), \quad x \in \Gamma, t' \in [-h,0), \tag{54}$$

where $\forall i, i = 1,2$:

$$y_i \equiv y_i(x,t;u), u_i \equiv u_i(x,t), v_i \equiv v_i(x,t),$$

b_i is a given real C^∞ function defined on $\overline{Q}$($\overline{Q}$ closure of Q), c_i is a given real C^∞ function defined on Σ, $\Phi_{0,i}$, $\Psi_{0,i}$ are initial functions defined on Q_0 and Σ_0 respectively.

Adjoint equations:

$$\begin{aligned}\frac{\partial^2 p_1(u)}{\partial t^2} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right) p_1(u) + b_1(x,t+h)p_1(x,t+h:u)\\ +p_1(u) + p_2(u) = \lambda_1(y_1(u) - z_{1d}), \qquad x \in \Omega, t \in (0,T-h),\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\partial^2 p_2(u)}{\partial t^2} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right) p_2(u) + b_2(x,t+h)p_2(x,t+h:u)\\ -p_1(u) + p_2(u) = \lambda_1(y_2(u) - z_{2d}), \qquad x \in \Omega, t \in (0,T-h),\end{aligned} \tag{56}$$

$$\begin{aligned}\frac{\partial^2 p_1(u)}{\partial t^2} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right) p_1(u) + p_1(u) + p_2(u) = \lambda_1(y_1(u) - z_{1d}),\\ x \in \Omega, t \in (T-h,T),\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{\partial^2 p_2(u)}{\partial t^2} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right) p_2(u) - p_1(u) + p_2(u) = \lambda_1(y_2(u) - z_{2d}),\\ x \in \Omega, t \in (T-h,T),\end{aligned} \tag{58}$$

$$p_1(x,T;u) = 0, \quad p_2(x,T;u) = 0, \quad x \in \Omega, \tag{59}$$

$$\frac{\partial p_1(x,t;u)}{\partial t} = 0, \quad \frac{\partial p_2(x,t;u)}{\partial t} = 0, \quad x \in \Omega, \tag{60}$$

$$\begin{aligned}\frac{\partial p_1(x,t)}{\partial \nu_{\mathcal{A}^*}} = c_1(x,t+h)p_1(x,t+h;u), \quad \frac{\partial p_2(x,t)}{\partial \nu_{\mathcal{A}^*}} = c_2(x,t+h)p_2(x,t+h;u),\\ x \in \Gamma, t \in (0,T-h),\end{aligned} \tag{61}$$

$$\frac{\partial p_1(x,t)}{\partial \nu_{\mathcal{A}^*}} = 0, \quad \frac{\partial p_2(x,t)}{\partial \nu_{\mathcal{A}^*}} = 0, \quad x \in \Gamma, t \in (T-h,T). \tag{62}$$

Minimum conditions

$$\int_0^T \int_\Omega \Big([p_1(u^*) + \lambda_2 N_1 u_1^*](u_1 - u_1^*) + [p_2(u^*) + \lambda_2 N_2 u_2^*](u_2 - u_2^*) \Big) d\rho\, dt \leqslant 0, \qquad \forall u = (u_1, u_2) \in (U_{ad})^2, \tag{63}$$

where $u^* = (u_1^*, u_2^*) \in (U_{ad})^2$ is the optimal control and $p(u) = (p_1(u), p_2(u))$ is the adjoint state.

The following result is now summarized.

Theorem 5.1 *For the problem* (48)–(54) *with the performance function* (47), $z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$, $\lambda_2 > 0$, (20)*, and with adjoint equations* (55)–(62)*, there exists a unique optimal control* u^* *which satisfies the minimum condition* (63).

5.2. Optimal control for $(n \times n)$ coupled hyperbolic systems with infinite number of variables and with time lags

We can extend the discussion to $(n \times n)$ coupled hyperbolic systems. We consider the case where $u = (u_1, u_2, \ldots, u_n) \in (L^2(Q))^n$, the performance functional is given by:

$$I(u) = \sum_{i=1}^{n} \Big(\lambda_1 \int_Q [y_i(x,t;u) - z_{id}]^2 \, d\rho\, dt + \lambda_2 \int_Q (N_i u_i) u_i \, d\rho\, dt \Big), \tag{64}$$

where $z_d = (z_{1d}, z_{2d}, \ldots, z_{nd}) \in (L^2(Q))^n$ and N_i is a positive linear operator on $L^2(Q)$ into $L^2(Q)$, $i = 1, 2, \ldots, n$.

Then the optimality conditions are given by:

The state equations:

$$\begin{cases} \dfrac{\partial^2 y_i(u)}{\partial t^2} + \mathcal{S}(t) y_i(u) + b_i(x,t) y_i(x, t-h; u) = u_i, & x \in \Omega, t \in (0,T), \\ y_i(x, t') = \Phi_{i,0}(x, t') & x \in \Omega, t' \in [-h, 0), \\ y_i(x, 0) = y_{i,0}(x), & x \in \Omega, \\ \dfrac{\partial y_i(x,0)}{\partial t} = y_{1,i}(x), & x \in \Omega, \\ \dfrac{\partial y_i}{\partial \nu_S} = c_i(x,t) y_i(x, t-h) + v_i, & x \in \Gamma, t \in (0,T) \\ y_i(x, t') = \Psi_{0,i}(x, t'), & x \in \Gamma, t' \in [-h, 0), \end{cases} \tag{65}$$

where $\forall i, i = 1, 2, \ldots, n$:

$$y_i \equiv y_i(x,t;u), u_i \equiv u_i(x,t), v_i \equiv v_i(x,t),$$

b_i is a given real C^∞ function defined on $\overline{Q}(\overline{Q}$ closure of Q), c_i is a given real C^∞ function defined on Σ, $\Phi_{0,i}$, $\Psi_{0,i}$ are initial functions defined on Q_0 and Σ_0 respectively, and the operator $\mathcal{S}(t)$ is given by

$$\mathcal{S}(t)y_i(x) = \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right) y_i(x) + \sum_{j=1}^{n} a_{ij} y_j(x) \quad \forall i = 1,2,\ldots,n, \tag{66}$$

$$a_{ij} = \begin{cases} 1, & i \geqslant j; \\ -1, & i < j. \end{cases} \tag{67}$$

The operator $\mathcal{S}(t)$ is $(n \times n)$ matrix takes the form [El-Saify & Bahaa (2000), (2001), (2002), (2003)]

$$\mathcal{S}(t) = \begin{pmatrix} -\sum_{k=1}^{\infty} D_k^2 + q + 1 & -1 & \cdots & -1 \\ 1 & -\sum_{k=1}^{\infty} D_k^2 + q + 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & -\sum_{k=1}^{\infty} D_k^2 + q + 1 \end{pmatrix}_{n \times n}. \tag{68}$$

The adjoint equations:

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2} + \mathcal{S}^*(t) p_i(u) + b_i(x,t+h) p_i(x,t+h;u) = \lambda_1 (y_i(u) - z_{id}), \\ \qquad\qquad\qquad\qquad x \in \Omega,\ t \in (0, T-h), \\ \dfrac{\partial^2 p_i(u)}{\partial t^2} + \mathcal{S}^*(t) p_i(u) = \lambda_1 (y_i(u) - z_{id}), \quad x \in \Omega,\ t \in (T-h, T), \\ p_i(x,T,u) = 0, \quad x \in \Omega, \\ \dfrac{\partial p_i(x,T;u)}{\partial t} = 0, \quad x \in \Omega, \\ \dfrac{\partial p_i}{\partial \nu_{\mathcal{S}^*}}(x,t) = c_i(x,t+h) p_i(x,t+h;u), \quad x \in \Gamma,\ t \in (0,T-h), \\ \dfrac{\partial p_i(u)}{\partial \nu_{\mathcal{S}^*}}(x,t) = 0, \quad x \in \Gamma,\ t \in (T-h,T). \end{cases} \tag{69}$$

The following result is now summarized.

Theorem 5.2 *For the problem* (65) *with the performance function* (64)*,* $z_{id} \in L^2(Q)$*,* $i = 1,2,\ldots,n$*,* $\lambda_2 > 0$*,* (20) *and with adjoint equations* (69))*, there exists a unique optimal* *control u^* which satisfies the minimum condition*

$$\sum_{i=1}^{n}\int_{0}^{T}\int_{\Omega}\left([p_i(u^*)+\lambda_2 N_i u_i^*](u_i-u_i^*)\right)d\rho(x)dt \leqslant 0$$
$$\forall u=(u_1,u_2,....,u_n)\in(U_{ad})^n, \tag{70}$$

where $u^*=(u_1^*,u_2^*,.....,u_n^*)\in(U_{ad})^n$ is the optimal control and $p(u)=(p_1(u),\dots,p_n(u))$ is the adjoint state.

6. Conclusions

The optimization problem presented in the paper constitutes a generalization of the optimal control problem for second order hyperbolic systems, which consists of one equation, involving second order operator with finite number of variables and with Neumann boundary condition involving constant time lag appearing both in the state equation and in the Neumann boundary conditions considered in Knowles (1978), (Kowalewski 1993a,b; 1995; 1998; 1999; 2000), (El-Saify, 2005; 2006), (El-Saify & Bahaa, 2002), (Kotarski & Bahaa, 2005; 2007) and (Kotarski & El-Saify & Bahaa, 2002a) onto the case of hyperbolic systems involving second order operator with infinite number of variables with constant retarded arguments appearing in the state equations and in the Neumann boundary condition.

The main result of the paper contains necessary and sufficient conditions of optimality for hyperbolic systems involving second order operator with infinite number of variables that give characterization of optimal control.

Also, in this paper, we considered the distributed optimal control problem for $(n\times n)$ hyperbolic systems involving second order operator with infinite number of variables with constant time delays appearing in both in the state equations and in the Neumann boundary condition.

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