10.2478/v10170-010-0030-7

Archives of Control Sciences
Volume 21(LVII), 2011
No. 1, *pages 25–36*

Stability of continuous-discrete linear systems with delays in state vector

TADEUSZ KACZOREK and ŁUKASZ SAJEWSKI

A new class of positive continuous-discrete linear systems with delays in state vector described by the model based on 2D general model is addressed. Necessary and sufficient conditions for the positivity and asymptotic stability of this class of linear systems are established. A procedure for checking the asymptotic stability is proposed. The effectiveness of the procedure is demonstrated on a numerical example.

Key words: stability, positivity, continuous-discrete, delays, state vector

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems is given in the monographs [6, 9].

2D hybrid system is dynamic systems that incorporate both continuous-time and discrete-time dynamics. It means that state vector, input and output vectors of 2D hybrid system depend on the continuous time t and the discrete variable i. Examples of hybrid systems include systems with relays, switches and hysteresis, transmissions, and other motion controllers, constrained robotic systems, automated highway systems, flight control, management systems and analog/digital circuit. The positive continuous-discrete 2D linear systems have been introduced in [8], positive hybrid linear systems in [10] and the positive fractional 2D hybrid systems in [11]. Different methods of solvability of 2D hybrid linear systems have been discussed in [14] and the solution to singular 2D

The Authors are with Faculty of Electrical Engineering, Bialystok University of Technology, Poland. E-mails: kaczorek@isep.pw.edu.pl, l.sajewski@pb.edu.pl

This work was supported by Ministry of Science and Higher Education of Poland under grant No. NN514 1939 33.

Received 29.12.2010.

hybrids linear systems has been derived in [16]. The realization problem for positive 2D hybrid systems has been addressed in [12]. Some problems of dynamics and control of 2D hybrid systems have been considered in [5, 7]. The problems of stability and robust stability of 2D continuous-discrete linear systems have been investigated in [1-4, 17-19]. The stability of positive continuous-time linear systems with delays has been addressed in [13]. Recently the stability and robust stability of Fornasini-Marchesini type model and of Roesser type model of scalar continuous-discrete linear systems have been analyzed by Buslowicz in [2-4].

The main goal of this paper is to present a new class of positive continuous-discrete linear systems with delays in state vector. Necessary and sufficient conditions for positivity and asymptotic stability of this class of continuous-discrete linear systems with delays will be established and a procedure for checking the stability will be proposed.

The paper is organized as follows. In section 2 the conditions for the positivity of continuous-discrete time systems with delays are established. Some preliminaries on asymptotic stability of positive linear systems are also recalled in section 2. The main result is presented in section 3 where the necessary and sufficient conditions for the asymptotic stability are formulated and proved. A procedure for the checking the asymptotic stability and illustrating example are also given. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ – the set of real numbers, Z_+ – the set of nonnegative integers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$. M_n is the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries) and I_n is the $n\times n$ identity matrix. A strictly positive vector $\lambda\in\Re_+^n$ will be denoted by $\lambda>0$ and strictly negative $\lambda<0$.

2. Positivity of continuous-discrete time systems with delays

Positivity of continuous-discrete time systems with delays

$$\dot{x}(t,i+1)=\sum_{k=0}^{q}A_0^k x(t-kd,i-k)+\sum_{k=0}^{q}A_1^k\dot{x}(t,i-k)+\sum_{k=0}^{q}A_2^k x(t-kd,i+1) \tag{1a}$$
$$+B_0u(t,i)+B_1\dot{u}(t,i)+B_2u(t,i+1)$$

$$y(t,i)=Cx(t,i)+Du(t,i)\quad t\in\Re_+=[0,+\infty]\quad i\in Z_+=\{0,1,\dots\} \tag{1b}$$

where $\dot{x}(t,i)=\dfrac{\partial x(t,i)}{\partial t}$, $x(t,i)\in\Re^n$, $u(t,i)\in\Re^m$, $y(t,i)\in\Re^p$ and $A_l^k\in\Re^{n\times n}$, $l=0,1,2,$ $k=0,1,\dots,q$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$ are the real matrices, $d>0$ is a delay.

Boundary conditions for (1a) have the form

$$x_{0i}(t,i),\ t\in[-qd,0],\ i\in Z_+,\ x_{t0}(t,i),\ \dot{x}_{t0}(t,i),\ i\in[-q,0],\ t\in\Re_+ \tag{2a}$$

$$x_{0i}(t,0),\ t\in[-qd,0],\ x_{t0}(0,i)=\dot{x}_{t0}(0,i)=0,\ i\in[-q,0]. \tag{2b}$$

Definition 1 *The hybrid system with delays* (1) *is called (internally) positive if* $x(t,i) \in \Re_+^n$ *and* $y(t,i) \in \Re_+^p$, $t \in \Re_+$, $i \in Z_+$ *for arbitrary boundary conditions* $x_{0i}(t,i) \in \Re_+^n$, $t \in [-qd, 0]$, *and* $x_{t0}(t,i) \in \Re_+^n$, $\dot{x}_{t0}(t,i) \in \Re_+^n$, $i \in [-q, 0]$, $t \in \Re_+$ *and all inputs* $u(t,i) \in \Re_+^m$, $t \in \Re_+$, $i \in Z_+$.

Theorem 1 *The hybrid system with delays* (1) *is internally positive if and only if*

$$\begin{aligned} &A_2^0 \in M_n, \quad A_0^k, A_1^k, \in \Re_+^{n\times n}, \quad k = 0, 1, ..., q; \quad A_2^k, \in \Re_+^{n\times n}, \quad k = 1, 2, ..., q; \\ &A_0 + A_1 A_2 \in \Re_+^{n\times n}, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \end{aligned} \tag{3}$$

Proof

Necessity. Let $q = 1$ and e_i be the i-th ($i = 1, \dots, n$) column of the identity matrix I_n. From (1) for $t \in \Re_+$, $i = 0$ and $x(t,1) = e_i$, $x(t,0) = 0$, $\dot{x}(t,0) = 0$, $x(t-d,1) = 0$, $\dot{x}(t,-1) = 0$, $x(t-d,-1) = 0$ and inputs $u(t,0) = \dot{u}(t,0) = u(t,1) = 0$ we have $\dot{x}(t,1) = A_2^0 e_i$. The trajectory does not live the orthant $\Re_+^n$ only if $A_2^0 e_i \geqslant 0$, what implies $a_{ij} \geqslant 0$, $i \neq j$. Therefore, the matrix A_2 has to be the Metzler matrix. For the same reasons for $x(0,1) = 0$, $x(0,0) = 0$, $\dot{x}(0,0) = 0$, $x(-d,1) = 0$, $\dot{x}(0,-1) = 0$, $x(t-d,-1) = 0$, $\dot{x}(0,1) = B_0 u(0,0) \in \Re_+^n$, what implies $B_0 \in \Re_+^{n\times m}$ since $u(0,0) \in \Re_+^m$ may be arbitrary and $\dot{u}(0,0) = u(0,1) = 0$. Similarly, for $t \in \Re_+$, $i = 0$ and $x(t,1) = 0$, $x(t,0) = 0$, $x(t-d,1) = 0$, $\dot{x}(t,-1) = 0$, $x(t-d,-1) = 0$ we have $\dot{x}(t,1) = A_1^0 \dot{x}(t,0) \in \Re_+^n$ what implies $A_1^0 \in \Re_+^{n\times n}$ since $\dot{x}(t,0) \in \Re_+^n$ may be arbitrary. For the same reasons for $x(0,1) = 0$, $\dot{x}(0,0) = 0$, $x(-d,1) = 0$, $\dot{x}(0,1) = 0$, $x(t-d,-1) = 0$, $\dot{x}(0,1) = A_0^0 x(0,0) \in \Re_+^n$, what implies $A_0^0 \in \Re_+^{n\times n}$ since $x(0,0) \in \Re_+^n$ may be arbitrary. Continuing this procedure for A_0^1, A_1^1, A_2^1 and B_1, B_2 we may show that the hybrid system with delays (1) is internally positive only if the conditions (3) are satisfied. From (1b) for $u(0,0) = 0$ we have $y(0,0) = Cx(0,0) \in \Re_+^p$, what implies $C \in \Re_+^{p\times n}$, since may be arbitrary. For the same reasons for we have , what implies $C \in \Re_+^{p\times n}$, since $x(0,0) \in \Re_+^n$ may be arbitrary. For the same reasons for $x(0,0) = 0$ we have $y(0,0) = Du(0,0) \in \Re_+^p$, what implies $D \in \Re_+^{p\times m}$, since $u(0,0) \in \Re_+^m$ may be arbitrary.

Sufficiency. From (1) for $q = 1$, $i = 0$ and $t \in [0,d]$ we have

$$\dot{x}(t,1) = A_2^0 x(t,1) + A_2^1 x(t-d,1) + F(t,0) \tag{4a}$$

where

$$\begin{aligned} F(t,0) = A_0^0 x(t,0) + A_0^1 x(t-d,-1) + A_1^0 \dot{x}(t,0) + A_1^1 \dot{x}(t,-1) + B_0 u(t,0) \\ + B_1 \dot{u}(t,0) + B_2 u(t,1). \end{aligned} \tag{4b}$$

For given nonnegative initial conditions $x(t,0)$, $\dot{x}(t,0)$, $\dot{x}(t,-1)$, $x(t-d,-1)$, $x(t-d,1) \approx x_{0i}(0,1)$ and $u(t,0)$, $\dot{u}(t,0)$, $u(t,1) \in \Re_+^m$, $t \in [0,d]$ we obtain $F(t,0) \in \Re_+^n$, $t \in [0,d]$ if $A_0^0, A_0^1, A_1^0, A_1^1 \in \Re_+^{n\times n}$, $B_0, B_1, B_2 \in \Re_+^{n\times m}$.

The solution of the equation (4a) has the form

$$x(t,1)=e^{A_2^0 t}x(0,1)+\int_0^t e^{A_2^0(t-\tau)}\left(A_2^1 x(\tau-d,1)+F(\tau,0)\right)d\tau \tag{5}$$

and is nonnegative since $e^{A_2^0 t}\in\Re_+^{n\times n}$ for $t\in[0,d]$ if and only if A_2^0 is the Metzler matrix and $A_2^1\in\Re_+^{n\times n}$. Knowing $x(t,1)$ for $t\in[0,d]$ in a similar way we can find $x(t,1)$ for $t\in[d,2d]$, $t\in[2d,3d]\dots$.

From (1a) for $q=1$, $i=1$ we have

$$\dot{x}(t,2)=A_2^0 x(t,2)+A_2^1 x(t-d,2)+F(t,1) \tag{6a}$$

where

$$\begin{aligned}F(t,1)=A_0^0 x(t,1)+A_0^1 x(t-d,0)+A_1^0\dot{x}(t,1)+A_1^1\dot{x}(t,0)+B_0 u(t,1)\\ +B_1\dot{u}(t,1)+B_2 u(t,2).\end{aligned} \tag{6b}$$

Substituting (5) into (6b) we obtain

$$\begin{aligned}F(t,1)&=A_0^0 e^{A_2^0 t}x(0,1)+A_0^0\int_0^t e^{A_2^0(t-\tau)}A_2^1 x(\tau-d,1)d\tau+A_0^0\int_0^t e^{A_2^0(t-\tau)}F(\tau,0)d\tau\\ &+A_1^0\frac{d}{dt}\left(e^{A_2^0 t}x(0,1)+\int_0^t e^{A_2^0(t-\tau)}A_2^1 x(\tau-d,1)d\tau+\int_0^t e^{A_2^0(t-\tau)}F(\tau,0)d\tau\right)\\ &+A_1^1\dot{x}(t,0)+A_0^1(t-d,0)+B_0 u(t,1)+B_1\dot{u}(t,1)+B_2 u(t,2)\\ &=e^{A_2 t}(A_0^0+A_1^0A_2^0)\left(x(0,1)-F(0,0)-A_3 x(-d,1)\right)\\ &+A_0^0\left(A_2^1 x(t-d,1)+F(t,0)\right)+A_1^0\left(A_2^1\dot{x}(t-d,1)+\dot{F}(t,0)\right)+A_1^1\dot{x}(t,0)\\ &+A_0^1(t-d,0)+B_0 u(t,1)+B_1\dot{u}(t,1)+B_2 u(t,2).\end{aligned} \tag{6c}$$

For given nonnegative initial conditions (2), nonnegative inputs we obtain $F(t,1)\in\Re_+^n$, if and only if $A_0^0, A_0^1, A_1^0, A_1^1\in\Re_+^{n\times n}$, $A_0^0+A_1^0A_2^0\in\Re_+^{n\times n}$, $B_0, B_1, B_2\in\Re_+^{n\times m}$. The solution of the equation (6a) has the form

$$x(t,2)=e^{A_2^0 t}x(0,2)+\int_0^t e^{A_2^0(t-\tau)}\left(A_2^1 x(\tau-d,2)+F(\tau,1)\right)d\tau\in\Re_+^n \tag{7}$$

if the conditions (3) are met.

Continuing the procedure for $q = 1$, $i > 0$ with nonnegative initial conditions (2), non-negative inputs and conditions (3) we may show that

$$F(t,i) = A_0^0 x(t,i) + A_0^1 x(t-d,i-1) + A_1^0 \dot{x}(t,i) + A_1^1 \dot{x}(t,i-1) + B_0 u(t,i) + B_1 \dot{u}(t,i) + B_2 u(t,i+1) \in \Re_+^n \tag{8a}$$

and

$$x(t,i+1) = e^{A_2^0 t} x(0,i+1) + \int_0^t e^{A_2^0(t-\tau)} \left(A_2^1 x(\tau-d,i+1) + F(\tau,i)\right) d\tau \in \Re_+^n \tag{8b}$$

for $t \in \Re_+$ and $i \in Z_+$. In a similar way, for $q > 1$ we may show that the hybrid system with delays (1) is internally positive if the conditions (3) are satisfied. □

3. Stability of positive continuous-discrete time systems with delays

Definition 2 *The continuous-discrete linear system with delays* (1) *is called asymptotically stable if*

$$\lim_{t,i \to \infty} x(t,i) = 0 \tag{9}$$

for bounded initial conditions and for $u(t,i) = 0$, $t \geqslant 0$, $i \in Z_+$.

The matrix $A \in \Re^{n \times n}$ is called asymptotically stable (Hurwitz) if all its eigenvalues lie in the open left half of the complex plane.

Definition 3 . *The point* x_e *is called equilibrium point of the asymptotically stable system (1) for* $Bu = 1_n = [1 \ \dots \ 1]^T \in \Re_+^n$ *if the equation is satisfied*

$$0 = \bar{A}_0 x_e + \bar{A}_2 x_e + 1_n \tag{10}$$

where $\bar{A}_0 = \sum_{k=0}^{q} A_0^k$, $\bar{A}_2 = \sum_{k=0}^{q} A_2^k$. *Asymptotic stability implies* $\det[\bar{A}_0 + \bar{A}_2] \neq 0$ *and from* (10) *we have*

$$x_e = -[\bar{A}_0 + \bar{A}_2]^{-1} 1_n. \tag{11}$$

Theorem 2 [9] *The linear continuous-discrete time system* (1) *for* $q = 0$ *is asymptotically stable if and only if the zeros of the polynomial*

$$\det[I_n sz - A_0 - A_1 s - A_2 z] = s^n z^n + a_{n,n-1} s^n z^{n-1} + a_{n-1,n} s^{n-1} z^n + \dots + a_{10} s + a_{01} z + a_{00} \tag{12}$$

are located in the left half of the complex plane s and in the unit circle of the complex plane z.

Theorem 3 [9] *The positive linear system*

$$\dot{x} = Ax, \quad A \in M_n \tag{13}$$

is asymptotically stable if and only if the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{14}$$

has positive coefficients, i.e. $a_k > 0$ *for* $k = 0, 1, \ldots, n-1$.

Lemma 1 [9] *A nonnegative matrix* $A \in \Re_+^{n\times n}$ *is asymptotically stable (nonnegative Shur matrix) if and only if the Metzler matrix* $A - I_n$ *is asymptotically stable (Metzler Hurwitz matrix).*

Theorem 4 *The linear continuous-discrete positive system with delays* (1) *is asymptotically stable if and only if all coefficients of the polynomial*

$$\begin{aligned}&\det[I_n s(z+1) - \bar{A}_0 - \bar{A}_1 s - \bar{A}_2(z+1)]\\ &\qquad = s^n z^n + \bar{a}_{n,n-1}s^n z^{n-1} + \bar{a}_{n-1,n}s^{n-1}z^n + \ldots + \bar{a}_{10}s + \bar{a}_{01}z + \bar{a}_{00}\end{aligned} \tag{15}$$

are positive, i.e. $\bar{a}_{k,l} > 0$ *for* $k, l = 0, 1, \ldots, n$ $(\bar{a}_{n,n} = 1)$.

Proof From (9) follows that: 1) $\lim_{i\to\infty} x(\infty, i) = 0$ and 2) $\lim_{t\to\infty} x(t, \infty) = 0$. The conditions 1) implies that the coefficient of the polynomial (15) by Theorem 3 should be positive for $s = 0$. Condition 2) implies that the coefficient of the polynomial (15) by Theorem 3 should be positive for $z = -1$. Substitution $z = z + 1$ in (12) is equivalent to shifting the position of the zeros located in the unit circle located in the left half of the complex plane to the unit circle located in the center of the complex plane. Therefore, if the coefficients of the polynomial (15) are positive then the zeros of the polynomial (15) are located in the left half of the complex plane s and in the unit circle of the complex plane z and the positive continuous-discrete time system with delays (1) is asymptotically stable. □

Theorem 5 *Let the matrix* $\bar{A}_1 - I_n$ *be a Hurwitz Metzler matrix. The positive continuous-discrete linear system with delays (1) is asymptotically stable if and only if there exists a strictly positive vector* $\lambda \in \Re_+^n$ *(all components of the vectors are positive) such that*

$$(\bar{A}_0 + \bar{A}_2)\lambda < 0 \tag{16}$$

where $\bar{A}_0 = \sum_{k=0}^{q} A_0^k$, $\bar{A}_1 = \sum_{k=0}^{q} A_1^k$, $\bar{A}_2 = \sum_{k=0}^{q} A_2^k$.

Proof Integrating the equation (1a) with $B_0, B_1, B_2 = 0$ in the interval $(0, +\infty)$ for $i \to +\infty$ we obtain

$$x(+\infty, +\infty) - x(0, +\infty) = \sum_{k=0}^{q} A_0^k \int_0^{+\infty} x(t - kd, +\infty) + \sum_{k=0}^{q} A_1^k x(+\infty, +\infty)$$

$$
\begin{aligned}
&-\sum_{k=0}^{q} A_1^k x(0,+\infty) + \sum_{k=0}^{q} A_2^k \int_0^{+\infty} x(t-kd,+\infty) \\
&= A_0^0 \int_0^{+\infty} x(\tau,+\infty)d\tau + \sum_{k=1}^{q} A_0^k \int_0^{+\infty} x(t-kd,+\infty) + \sum_{k=0}^{q} A_1^k x(+\infty,+\infty) - \sum_{k=0}^{q} A_1^k x(0,+\infty) \\
&+A_2^0 \int_0^{+\infty} x(\tau,+\infty)d\tau + \sum_{k=1}^{q} A_2^k \int_0^{+\infty} x(t-kd,+\infty).
\end{aligned} \tag{17}
$$

If the system is asymptotically stable then $x(+\infty,+\infty)=0$ and from (17) we obtain

$$
\left(\sum_{k=0}^{q} A_1^k - I_n\right) x(0,+\infty) - \sum_{k=1}^{q} \left(A_0^k + A_2^k\right) \int_{-kd}^{0} x(\tau,+\infty)d\tau = \sum_{k=0}^{q} \left(A_0^k + A_2^k\right) \int_0^{+\infty} x(\tau,+\infty)d\tau. \tag{18}
$$

If the matrix $\bar{A}_1 - I_n$ is Hurwitz Metzler matrix then for every $x(0,+\infty) \in \Re_+^n$ such that $(\bar{A}_1 - I_n)x(0,+\infty)$ is a strictly negative vector (and $\sum_{k=1}^{q} \left(A_0^k + A_2^k\right) \int_{-kd}^{0} x(\tau,+\infty)d\tau > 0$), $\lambda = \int_0^{+\infty} x(\tau,+\infty)d\tau$ is a strictly positive vector and (16). holds.

Now we shall show that if there exists a strictly positive vector λ such that (16) holds then the positive system with delays (1) is asymptotically stable. It is well-known that the positive system (1) with $B_0, B_1, B_2 = 0$ is asymptotically stable if and only if the correspond-ing transpose positive system

$$
\dot{x}(t,i+1) = \bar{A}_0^T x(t,i) + \bar{A}_1^T \dot{x}(t,i) + \bar{A}_2^T x(t,i+1), \ \ t \in \Re_+, \ \ i \in Z_+ \tag{19}
$$

is asymptotically stable. As a candidate for a Lyapunov function for the positive system (19) we choose

$$
V(t,x(i)) = x^T(t,i)\lambda \tag{20}
$$

which is positive for every nonzero $x(t,i) \in \Re_+^n$ and strictly positive vector $\lambda > 0$. Using (20) and (19) we obtain

$$
\begin{aligned}
\Delta\dot{V}(t,x(i)) &= \dot{V}(t,x(i+1)) - \dot{V}(t,x(i)) = \dot{x}^T(t,i+1)\lambda - \dot{x}^T(t,i)\lambda \\
&= \dot{x}^T(t,i)[\bar{A}_1 - I_n]\lambda + x^T(t,i)\bar{A}_0\lambda + x^T(t,i+1)\bar{A}_2\lambda \\
&\leqslant \begin{cases} x^T(t,i)(\bar{A}_0 + \bar{A}_2)\lambda & \text{for} \quad x(t,i) \geqslant x(t,i+1) \\ x^T(t,i+1)(\bar{A}_0 + \bar{A}_2)\lambda & \text{for} \quad x(t,i) < x(t,i+1) \end{cases}
\end{aligned} \tag{21}
$$

since by assumption $[\bar{A}_1 - I_n]\lambda < 0$. If (16) holds then from (21) we have $\Delta\dot{V}(t,x(i)) < 0$ and the positive system is asymptotically stable. □

Remark 1 As the strictly positive vector λ we may choose the equilibrium point (6) since for $\lambda = x_e$ we have

$$(\bar{A}_0 + \bar{A}_2)\lambda = -(\bar{A}_0 + \bar{A}_2)(\bar{A}_0 + \bar{A}_2)^{-1}1_n = -1_n. \tag{22}$$

Theorem 6 *The positive system* (1) *is asymptotically stable if and only if both matrices*

$$\bar{A}_1 - I_n, \quad \bar{A}_0 + \bar{A}_2 \tag{23}$$

are Hurwitz Metzler matrices, where $\bar{A}_0 = \sum_{k=0}^{q} A_0^k$, $\bar{A}_1 = \sum_{k=0}^{q} A_1^k$, $\bar{A}_2 = \sum_{k=0}^{q} A_2^k$.

Proof From Remark 1 it follows that the positive system (1) is asymptotically stable only if the matrix $\bar{A}_1 - I_n$ is Hurwitz Metzler matrix. By Theorem 5 the positive system is asymptotically stable if and only if there exists a strictly positive vector λ such that (16) holds but this is equivalent that the matrix $\bar{A}_0 + \bar{A}_2$ is Hurwitz Metzler matrix. □

To test if the matrices (23) are Hurwitz Metzler matrices the following theorem is recommended [9, 15].

Theorem 7 *The matrix* $A \in \Re^{n\times n}$ *is a Hurwitz Metzler matrix if and only if one of the following equivalent conditions is satisfied:*

i) all coefficients $a_0, \dots, a_{n-1}$ *of the characteristic polynomial*

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \tag{24}$$

are positive, i.e. $a_i \geqslant 0$, $i = 0, 1, \dots, n-1$,

ii) the diagonal entries of the matrices

$$A_{n-k}^{(k)} \quad \text{for} \quad k = 1, \dots, n-1 \tag{25}$$

are negative, where

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,1}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{n,n}^{(0)} \end{bmatrix},$$

$$A_{n-1}^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n-1}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n-1,1}^{(0)} & \dots & a_{n-1,n-1}^{(0)} \end{bmatrix}$$

$$b_{n-1}^{(0)} = \begin{bmatrix} a_{1,n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix}, \quad c_{n-1}^{(0)} = \begin{bmatrix} a_{n,1}^{(0)} & \dots & a_{n,n-1}^{(0)} \end{bmatrix} \tag{26}$$

$$A_{n-k}^{(k)} = A_{n-k}^{(n-1)} - \frac{b_{n-k}^{(k-1)} c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} = \begin{bmatrix} a_{11}^{(k)} & \dots & a_{1,n-k}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k}^{(k)} \end{bmatrix}$$

$$= \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix},$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix}, \quad c_{n-k-1}^{(k)} = \begin{bmatrix} a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k-1}^{(k)} \end{bmatrix}.$$

To check the stability of the positive system (1) the following procedure can be used.

Procedure 1

Step 1. Check if at least one diagonal entry of the matrix $\sum_{k=0}^{q} A_1^k \in \Re_+^{n\times n}$ is equal or greater than 1. If this holds then the positive system with delays (1) is unstable [9].

Step 2. Using Theorem 7 check if the matrix $\sum_{k=0}^{q} A_1^k - I_n$ is Hurwitz Metzler matrix. If not the positive system with delays (1) is unstable.

Step 3. Using Theorem 7 check if the matrix $A_2^0 + \sum_{k=1}^{q} A_2^k + \sum_{k=0}^{q} A_0^k$ is Hurwitz Metzler matrix. If yes then the positive system with delays (1) is asymptotically stable.

Example 1 Consider the positive system (1) with the matrices

$$A_0^0 = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}, \quad A_1^0 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_2^0 = \begin{bmatrix} -0.6 & 0 \\ 0.05 & -0.95 \end{bmatrix},$$
$$A_0^1 = \begin{bmatrix} 0.01 & 0.02 \\ 0.01 & 0.01 \end{bmatrix}, \quad A_1^1 = \begin{bmatrix} 0.1 & 0.05 \\ 0.05 & 0.09 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 0.1 & 0.15 \\ 0.01 & 0.2 \end{bmatrix}. \tag{27}$$

By Theorem 1 the system is positive since $A_2^0 \in M_n$, $A_0^0, A_0^1, A_1^0, A_1^1, A_2^1 \in \Re_+^{2\times 2}$ and

$$A_0^0 + A_1^0 A_2^0 = \begin{bmatrix} 0.07 & 0.01 \\ 0.055 & 0.115 \end{bmatrix} \in \Re_+^{2\times 2}.$$

Using Procedure 1 we obtain the following:

Step 1. All diagonal entries of the matrix $A = A_1^0 + A_1^1$ are less than 1.

Step 2. The matrix

$$A_1^0 + A_1^1 - I_2 = \begin{bmatrix} -0.5 & 0.25 \\ 0.15 & -0.61 \end{bmatrix}$$

is Hurwitz Metzler matrix since the coefficients of the polynomial

$$\det[I_2 s - A_1^0 - A_1^1 + I_2] = \begin{vmatrix} s+0.5 & -0.25 \\ -0.15 & s+0.61 \end{vmatrix} = s^2 + 1.11s + 0.27 \tag{28}$$

are positive.

Step 3. The matrix

$$A_0^0 + A_0^1 + A_2^0 + A_2^1 = \begin{bmatrix} -0.19 & 0.37 \\ 0.17 & -0.34 \end{bmatrix}$$

is also Hurwitz since the coefficients of the polynomial

$$\det[I_2 s - A_0^0 - A_0^1 - A_2^0 - A_2^1] = \begin{vmatrix} s+0.19 & -0.37 \\ -0.17 & s+0.34 \end{vmatrix} = s^2 + 0.53s + 0.0017 \tag{29}$$

are positive.

By Theorem 6 the positive system (1) with (27) is asymptotically stable. The polynomial (15) for the positive system has the form

$$\begin{aligned} &\det[I_2 s(z+1) - A_0^0 - A_0^1 - (A_1^0 + A_1^1)s - (A_2^0 + A_2^1)(z+1)] \\ &= s^2z^2 + 1.11s^2z + 1.25sz^2 + 1.17sz + 0.27s^2 + 0.37z^2 + 0.19s + 0.26z + 0.0017 \end{aligned} \tag{30}$$

and its coefficients are positive. Therefore, by Theorem 2 the positive system (1) with (27) is asymptotically stable.

4. Concluding remarks

The new class of positive continuous-discrete linear systems with delays in state vector described by the model, similar to the first Fornaisni-Marchesini model, has been introduced. Necessary and sufficient conditions for the positivity of this class of continuous-discrete linear systems with delays in state vector has been established. Necessary and sufficient conditions for the asymptotic stability of this class of continuous-discrete linear systems with delays have been also established (Theorem 5 and 6). A procedure for checking the stability has been proposed and its effectiveness has been demonstrated on numerical example. The considerations can be also extended for fractional positive 2D continuous-discrete linear systems.

References

[1] Y. BISTRITZ: A stability test for continuous-discrete bivariate polynomials. *Proc. Int. Symp. on Circuits and Systems*, **3** (2003), 682-685.

[2] M. BUSLOWICZ: Improved stability and robust stability conditions for a general model of scalar continuous-discrete linear systems. *Measurement Automation and Monitoring*, (submitted for publication).

[3] M. BUSLOWICZ: Stability and robust stability conditions for a general model of scalar continuous-discrete linear systems. *Measurement Automation and Monitoring*, **56**(2), (2010), 133-135.

[4] M. BUSLOWICZ: Robust stability of the new general 2D model of a class of continuous-discrete linear systems. *Bull. Pol. Acad. Sci. Techn.*, **57**(4), (2010).

[5] M. DYMKOV, I. GAISHUN, E. ROGERS, K. GAŁKOWSKI and D. H. OWENS: Control theory for a class of 2D continuous-discrete linear systems. *Int. J. Control*, **77**(9), (2004), 847-860.

[6] L. FARINA and S. RINALDI: Positive linear systems; Theory and applications. J. Wiley, New York, 2000.

[7] K. GAŁKOWSKI, E. ROGERS, W. PASZKE and D.H. OWENS: Linear repetitive process control theory applied to a physical example. *Int. J. Appl. Math. Comput. Sci.*, **13**(1), (2003), 87-99.

[8] T. KACZOREK: Reachability and minimum energy control of positive 2D continuous-discrete systems. *Bull. Pol. Acad. Sci. Techn.*, **46**(1), (1998), 85-93.

[9] T. KACZOREK: Positive 1D and 2D Systems. Springer-Verlag, London, 2002.

[10] T. KACZOREK: Positive 2D hybrid linear systems. *Bull. Pol. Acad. Sci. Tech.*, **55**(4), (2007), 351-358.

[11] T. KACZOREK: Positive fractional 2D hybrid linear systems. *Bull. Pol. Acad. Tech.*, **56**(3), (2008), 273-277.

[12] T. KACZOREK: Realization problem for positive 2D hybrid systems. *COMPEL*, **27**(3), (2008), 613-623.

[13] T. KACZOREK: Stability of positive continuous-time linear systems with delays. *Bul. Pol. Acad. Sci. Techn.*, **57**(4), (2009), 395-398.

[14] T. KACZOREK, V. MARCHENKO and L. SAJEWSKI: Solvability of 2D hybrid linear systems - comparison of the different methods. *Acta Mechanica et Automatica*, **2**(2), (2008), 59-66.

[15] K.S. NARENDRA and R. SHORTEN: Hurwitz stability of Metzler matrices. *IEEE Trans. Autom. Contr.*, **55**(6), (2010), 1484-1487.

[16] L. SAJEWSKI: Solution of 2D singular hybrid linear systems. *Kybernetes*, **38**(7/8), (2009), 1079-1092.

[17] Y. XIAO: Stability test for 2-D continuous-discrete systems. *Proc. 40th IEEE Conf. on Decision and Control*, **4** (2001), 3649-3654.

[18] Y. XIAO: Stability, controllability and observability of 2-D continuous-discrete systems. *Proc. Int. Symp. on Circuits and Systems*, **4** (2003), 468-471.

[19] Y. XIAO: Robust Hurwitz-Schur stability conditions of polytopes of 2-D polynomials. *Proc. 40th IEEE Conf. on Decision and Control*, **4** (2001), 3643-3648.