

Positive minimal realizations of continuous-discrete linear systems with transfer function with separable denominator or numerator

Łukasz Sajewski

Faculty of Electrical Engineering, Białystok University of Technology, Poland

Abstract: The positive minimal realization problem for continuous-discrete linear single-input, single-output (SISO) systems is formulated. Two special case of the continuous-discrete systems are analyzed. Method based on the state variable diagram for finding positive minimal realizations of given proper transfer functions is proposed. Sufficient conditions for the existence of positive minimal realizations of given proper transfer functions with separable numerator or transfer functions with separable denominator are established. Two procedures for computation of positive minimal realizations are proposed and illustrated by numerical examples.

Keywords: continuous-discrete, minimal, positive, realization, existence, computation

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are: industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1, 2]. The realization problem for positive discrete-time and continuous-time systems without and with delays was considered in [3–7].

Continuous-discrete 2D linear system is a dynamic system that incorporates both continuous-time and discrete-time dynamics. It means that state, input and output vectors of 2D system depend on continuous and discrete variables. Examples of continuous-discrete systems include systems with relays, switches, and hysteresis, transmissions, and other motion controllers, constrained robotic systems, automated highway systems, flight control and management systems, analog/digital circuit. Wide variety of not only 2D system examples can be found e.g. in book [8]. The positive continuous-discrete 2D linear

systems have been introduced in [9], positive hybrid linear systems in [10] and the positive fractional 2D hybrid systems in [11]. Different methods of solvability of 2D hybrid linear systems have been discussed in [12] and the solution to singular 2D hybrids linear systems has been derived in [13]. The realization problem for positive 2D hybrid systems have been addressed in [2, 14–16] and the minimal realization problem for the transfer function with separable denominators and the transfer function with separable numerators of 2D systems has been addressed in [17, 18].

Positive minimal realization problem for 1D systems is well-known [2]. The same problem for 2D continuous-discrete systems is much more complicated and there is only a few publications concerning positive minimal realizations. The presented paper is focused on extending the state diagram method [2, 17], previously used to solve positive realization problem (non-minimal) [13, 19], on minimal realizations. In this paper it will be shown that the state variable diagram method can be used to compute the positive minimal realizations for special case of 2D transfer function – proper transfer functions with separable denominators or separable numerators. Also, the sufficient conditions for the existence of positive minimal realizations of the given proper transfer functions will be established, and procedures for computation of positive minimal realizations for the two cases of transfer functions will be proposed.

The paper is divided in 3 sections. In section 1 some preliminaries concerning the positive continuous-discrete 2D linear systems and minimal realization are recalled and the positive minimal realization problem is formulated. Two special cases of continuous-discrete systems are analyzed in section 2. In the same section the solution to the positive minimal realization problem for two cases of transfer functions are presented and the sufficient conditions for existence of positive minimal realization are established. Concluding remarks are given in section 3.

In the paper the following notation will be used: the set of $n \times m$ real matrices will be denoted by $\mathfrak{R}^{n \times m}$ and $\mathfrak{R}^n = \mathfrak{R}^{n \times 1}$. The set of $n \times m$ real matrices with nonnegative entries will be denoted by $\mathfrak{R}_+^{n \times m}$ and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$. M_n be the set of $n \times m$ Metzler matrices (real matrices with nonnegative off-diagonal entries). The $n \times n$ identity matrix will be denoted by I_n and the transpose will be denoted by T .

1. Preliminaries and problem formulation

Consider a continuous-discrete linear system described by the equations [2]:

$$\dot{x}_1(t, i) = A_{11}x_1(t, i) + A_{12}x_2(t, i) + B_1u(t, i), \tag{1a}$$

$$x_2(t, i + 1) = A_{21}x_1(t, i) + A_{22}x_2(t, i) + B_2u(t, i), \tag{1b}$$

$$y(t, i) = C_1x_1(t, i) + C_2x_2(t, i) + Du(t, i), \tag{1c}$$

where $t \in \mathfrak{R}_+ = [0, +\infty]$, $i \in Z_+$, $\dot{x}_1(t, i) = \frac{\partial x_1(t, i)}{\partial t}$,

$x_1(t, i) \in \mathfrak{R}^{n_1}$, $x_2(t, i) \in \mathfrak{R}^{n_2}$, $u(t, i) \in \mathfrak{R}^m$, $y(t, i) \in \mathfrak{R}^p$ and $A_{11} \in M_{n_1}$, $A_{12} \in \mathfrak{R}^{n_1 \times n_2}$, $A_{21} \in \mathfrak{R}^{n_2 \times n_1}$, $A_{22} \in \mathfrak{R}^{n_2 \times n_2}$, $B_1 \in \mathfrak{R}^{n_1 \times m}$, $B_2 \in \mathfrak{R}^{n_2 \times m}$, $C_1 \in \mathfrak{R}^{p \times n_1}$, $C_2 \in \mathfrak{R}^{p \times n_2}$, $D \in \mathfrak{R}^{p \times m}$ are real matrices.

Boundary conditions for (1a) and (1b) have the form:

$$x_1(0, i) = x_1(i), \quad i \in Z_+ \quad \text{and} \quad x_2(t, 0) = x_2(t), \quad t \in \mathfrak{R}_+. \tag{2}$$

Note that the continuous-discrete linear system (1) has a similar structure as the Roesser model [10, 20].

Definition 1. The continuous-discrete linear system (1) is called internally positive if $x_1(t, i) \in \mathfrak{R}_+^{n_1}$, $x_2(t, i) \in \mathfrak{R}_+^{n_2}$, and $y(t, i) \in \mathfrak{R}_+^p$, $t \in \mathfrak{R}_+$, $i \in Z_+$ for all arbitrary boundary conditions $x_1(i) \in \mathfrak{R}_+^{n_1}$, $i \in Z_+$, $x_2(t) \in \mathfrak{R}_+^{n_2}$, $t \in \mathfrak{R}_+$ and all inputs $u(t, i) \in \mathfrak{R}_+^m$, $t \in \mathfrak{R}_+$, $i \in Z_+$.

Theorem 1. [2, 10] The continuous-discrete linear system (1) is internally positive if and only if:

$$\begin{aligned} A_{11} &\in M_{n_1}, \quad A_{12} \in \mathfrak{R}_+^{n_1 \times n_2}, \quad A_{21} \in \mathfrak{R}_+^{n_2 \times n_1}, \quad A_{22} \in \mathfrak{R}_+^{n_2 \times n_2}, \\ B_1 &\in \mathfrak{R}_+^{n_1 \times m}, \quad B_2 \in \mathfrak{R}_+^{n_2 \times m}, \quad C_1 \in \mathfrak{R}_+^{p \times n_1}, \quad C_2 \in \mathfrak{R}_+^{p \times n_2}, \\ D &\in \mathfrak{R}_+^{p \times m}. \end{aligned} \tag{3}$$

The transfer matrix of the system (1) is given by the formula:

$$T(s, z) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} I_{n_1}s - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \in \mathfrak{R}^{p \times m}(s, z) \tag{4}$$

where $\mathfrak{R}^{p \times m}(s, z)$ is the set of $p \times m$ real matrices in s and z with real coefficient. For the m -inputs and p -outputs continuous-discrete linear system (1), the proper transfer matrix has the following form:

$$T(s, z) = \begin{bmatrix} T_{11}(s, z) & \dots & T_{1m}(s, z) \\ \vdots & \vdots & \vdots \\ T_{p1}(s, z) & \dots & T_{pm}(s, z) \end{bmatrix} \in R^{p \times m}(s, z) \tag{5a}$$

where:

$$T_{kl}(s, z) = \frac{Y_{kl}(s, z)}{U_{kl}(s, z)} = \frac{\sum_{i=0}^{n_{1,kl}} \sum_{j=0}^{n_{2,kl}} b_{i,j}^{kl} s^i z^j}{s^{n_{1,kl}} z^{n_{2,kl}} - \sum_{\substack{i=0 \\ i+j \neq n_{1,kl} + n_{2,kl}}}^{n_{1,kl}} \sum_{j=0}^{n_{2,kl}} a_{i,j}^{kl} s^i z^j} \tag{5b}$$

for $k = 1, 2, \dots, p$; $l = 1, 2, \dots, m$ where $U(s, z) = Z\{\mathcal{L}[u(t, i)]\}$, $Y(s, z) = Z\{\mathcal{L}[y(t, i)]\}$ and Z and \mathcal{L} are the Z-transform and Laplace operators.

Multiplying the numerator and denominator of transfer matrix (5b) by $s^{-n_{1,kl}} z^{-n_{2,kl}}$ we obtain the transfer matrix in the state space form, eg. form which is desired to draw the state space diagram [9, 15, 20]:

$$T_{kl}(s^{-1}, z^{-1}) = \frac{\sum_{i=0}^{n_{1,kl}} \sum_{j=0}^{n_{2,kl}} b_{i,j}^{kl} s^{-i} z^{-j}}{1 - \sum_{\substack{i=0 \\ i+j \neq n_{1,kl} + n_{2,kl}}}^{n_{1,kl}} \sum_{j=0}^{n_{2,kl}} a_{i,j}^{kl} s^{-i} z^{-j}} \tag{6}$$

for $k = 1, 2, \dots, p$; $l = 1, 2, \dots, m$.

Definition 2. The matrices (3) are called the positive realization of the transfer matrix $T(s, z)$ if they satisfy the equality (4). The realization is minimal if the matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ has the lowest possible dimension among

all realizations. The positive minimal realization problem can be stated as follows: given a proper rational matrix $T(s, z) \in R^{p \times m}(s, z)$, find its positive and minimal realization (3).

Remark 1. For 1D systems the minimal realization is the one with the matrix A of dimension $n \times n$ where n is the degree of the characteristic polynomial of the system [9]. This was implicated by controllability and observability of the 1D system. For 2D system in general case this relationship is not true [19] and the observability and controllability of the 2D system does not implicate the minimality of its realization.

Remark 2. The minimal realization for 2D system is the one with the matrix A of dimension $(n_1 + n_2) \times (n_1 + n_2)$ where n_1 and n_2 are the degrees of the characteristic polynomial in s and z of the system [19].

2. Problem solution for SISO systems

The solution to the minimal positive realization problem will be presented on two special cases of the 2D transfer functions. Proposed method will be based on the state variable diagram [2, 17, 15].

Two cases of the transfer functions of continuous-discrete linear system will be considered.

Case 1. The transfer function with separable denominators:

$$T(s^{-1}, z^{-1}) = \frac{\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} b_{i,j} s^{-i} z^{-j}}{\left(1 - \sum_{k=1}^{n_1} \beta_k s^{-k}\right) \left(1 - \sum_{l=1}^{n_2} \lambda_l z^{-l}\right)} = \frac{Y(s^{-1}, z^{-1})}{U(s^{-1}, z^{-1})} \quad (7)$$

Case 2. The transfer function with separable numerators:

$$T(s^{-1}, z^{-1}) = \frac{\left(\sum_{i=0}^{n_1} \beta_i s^{-i}\right) \left(\sum_{j=0}^{n_2} \lambda_j z^{-j}\right)}{1 - \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} a_{k,l} s^{-k} z^{-l}} = \frac{Y(s^{-1}, z^{-1})}{U(s^{-1}, z^{-1})} \quad (8)$$

2.1. Case 1

Defining:

$$E(s^{-1}, z^{-1}) = \frac{U(s^{-1}, z^{-1})}{1 - \beta_1 s^{-1} - \beta_2 s^{-2} - \dots - \beta_{n_1} s^{-n_1}},$$

$$Y(s^{-1}, z^{-1}) = \frac{(b_{00} + b_{01} z^{-1} + b_{10} s^{-1} + \dots + b_{n_1, n_2} s^{-n_1} z^{-n_2}) E(s^{-1}, z^{-1})}{1 - \lambda_1 z^{-1} - \lambda_2 z^{-2} - \dots - \lambda_{n_2} z^{-n_2}} \quad (9)$$

from (9) and (7) we obtain:

$$E(s^{-1}, z^{-1}) = U(s^{-1}, z^{-1}) + (\beta_1 s^{-1} + \beta_2 s^{-2} + \dots + \beta_{n_1} s^{-n_1}) E(s^{-1}, z^{-1}),$$

$$Y(s^{-1}, z^{-1}) = (b_{00} + b_{01} z^{-1} + b_{10} s^{-1} + \dots + b_{n_1, n_2} s^{-n_1} z^{-n_2}) E(s^{-1}, z^{-1}) + (\lambda_1 z^{-1} + \lambda_2 z^{-2} + \dots + \lambda_{n_2} z^{-n_2}) Y(s^{-1}, z^{-1}). \quad (10)$$

Using (10) we may draw the state variable diagram shown in fig. 1.

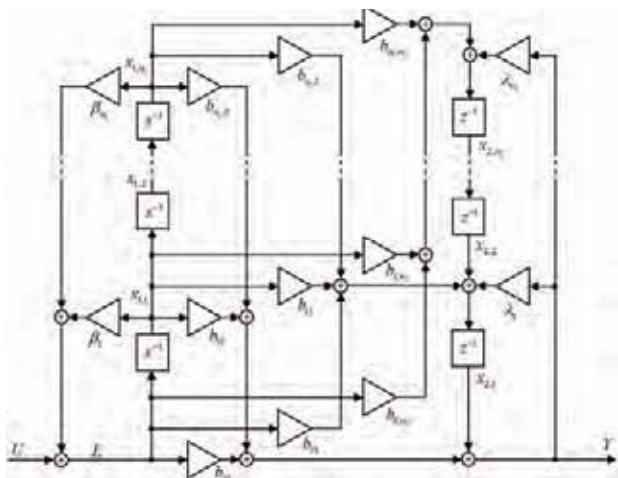


Fig. 1. State variable diagram for transfer function (7) with separable denominators

Rys. 1. Schemat zmiennych stanu dla transmitancji (7) z separowanym mianownikiem

As state variables we choose the outputs of integrators $(x_{1,1}(t, i), x_{1,2}(t, i), \dots, x_{1,n_1}(t, i))$ and of delay elements $(x_{2,1}(t, i), x_{2,2}(t, i), \dots, x_{2,n_2}(t, i))$. Using the state variable diagram (fig.1) we can write the following differential and difference equations:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= e(t, i), \\ \dot{x}_{1,2}(t, i) &= x_{1,1}(t, i), \\ &\vdots \\ \dot{x}_{1,n_1}(t, i) &= x_{1,n_1-1}(t, i), \end{aligned} \quad (11a)$$

$$\begin{aligned} x_{2,1}(t, i+1) &= b_{01} e(t, i) + b_{11} x_{1,1}(t, i) + b_{21} x_{1,2}(t, i) \\ &\quad + \dots + b_{n_1,1} x_{1,n_1}(t, i) + x_{2,2}(t, i) + \lambda_1 y(t, i), \\ &\vdots \\ x_{2,n_2-1}(t, i+1) &= b_{0,n_2-1} e(t, i) + b_{1,n_2-1} x_{1,1}(t, i) + b_{2,n_2-1} x_{1,2}(t, i) \\ &\quad + \dots + b_{n_1,n_2-1} x_{1,n_1}(t, i) + x_{2,n_2}(t, i) + \lambda_{n_2-1} y(t, i), \\ x_{2,n_2}(t, i+1) &= b_{0,n_2} e(t, i) + b_{1,n_2} x_{1,1}(t, i) + b_{2,n_2} x_{1,2}(t, i) \\ &\quad + \dots + b_{n_1,n_2} x_{1,n_1}(t, i) + \lambda_{n_2} y(t, i) \end{aligned}$$

where:

$$\begin{aligned} y(t, i) &= b_{00} e(t, i) + b_{10} x_{1,1}(t, i) + b_{20} x_{1,2}(t, i) \\ &\quad + \dots + b_{n_1,0} x_{1,n_1}(t, i) + x_{2,1}(t, i), \end{aligned} \quad (11b)$$

$$e(t, i) = \beta_1 x_{1,1}(t, i) + \beta_2 x_{1,2}(t, i) + \dots + \beta_{n_1} x_{1,n_1}(t, i) + u(t, i).$$

Substituting (11b) into (11a) we obtain:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= \beta_1 x_{1,1}(t, i) + \beta_2 x_{1,2}(t, i) + \dots + \beta_{n_1} x_{1,n_1}(t, i) + u(t, i), \\ \dot{x}_{1,2}(t, i) &= x_{1,1}(t, i), \\ &\vdots \\ \dot{x}_{1,n_1}(t, i) &= x_{1,n_1-1}(t, i), \\ x_{2,1}(t, i+1) &= \bar{b}_{11} x_{1,1}(t, i) + \bar{b}_{21} x_{1,2}(t, i) + \dots + \bar{b}_{n_1,1} x_{1,n_1}(t, i) \\ &\quad + \lambda_1 x_{2,1}(t, i) + x_{2,2}(t, i) + (b_{01} + \lambda_1 b_{00}) u(t, i), \\ &\vdots \\ x_{2,n_2-1}(t, i+1) &= \bar{b}_{1,n_2-1} x_{1,1}(t, i) + \bar{b}_{2,n_2-1} x_{1,2}(t, i) + \dots + \bar{b}_{n_1,n_2-1} x_{1,n_1}(t, i) \\ &\quad + \lambda_{n_2-1} x_{2,1}(t, i) + x_{2,2}(t, i) + (b_{0,n_2-1} + \lambda_{n_2-1} b_{00}) u(t, i), \\ x_{2,n_2}(t, i+1) &= \bar{b}_{1,n_2} x_{1,1}(t, i) + \bar{b}_{2,n_2} x_{1,2}(t, i) + \dots + \bar{b}_{n_1,n_2} x_{1,n_1}(t, i) \\ &\quad + \lambda_{n_2} x_{2,1}(t, i) + (b_{0,n_2} + \lambda_{n_2} b_{00}) u(t, i), \\ y(t, i) &= (b_{10} + \beta_1 b_{00}) x_{1,1}(t, i) + (b_{20} + \beta_2 b_{00}) x_{1,2}(t, i) \\ &\quad + \dots + (b_{n_1,0} + \beta_{n_1} b_{00}) x_{1,n_1}(t, i) + x_{2,1}(t, i) + b_{00} u(t, i) \end{aligned} \quad (12a)$$

where

$$\bar{b}_{k,l} = b_{k,l} + \beta_k b_{0,l} + \lambda_l b_{k,0} + \beta_k \lambda_l b_{00} \quad (12b)$$

for $k = 1, 2, \dots, n_1$; $l = 1, 2, \dots, n_2$.

Defining state vectors in the form:

$$x_1(t, i) = \begin{bmatrix} x_{1,1}(t, i) \\ \vdots \\ x_{1,n_1}(t, i) \end{bmatrix}, \quad x_2(t, i) = \begin{bmatrix} x_{2,1}(t, i) \\ \vdots \\ x_{2,n_2}(t, i) \end{bmatrix} \quad (13)$$

we can write the equations (12) in the form:

$$\begin{bmatrix} \dot{x}_1(t, i) \\ x_2(t, i+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t, i) \\ x_2(t, i) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t, i), \quad (14)$$

$$y(t, i) = [C_1 \quad C_2] \begin{bmatrix} x_1(t, i) \\ x_2(t, i) \end{bmatrix} + Du(t, i)$$

where:

$$A_{11} = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_{n_1-1} & \beta_{n_1} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \mathfrak{R}^{n_1 \times n_1}, \quad A_{12} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in \mathfrak{R}^{n_1 \times n_2},$$

$$A_{21} = \begin{bmatrix} \bar{b}_{1,1} & \dots & \bar{b}_{1,n_1} \\ \vdots & \dots & \vdots \\ \bar{b}_{1,n_2} & \dots & \bar{b}_{1,n_2} \end{bmatrix} \in \mathfrak{R}^{n_2 \times n_1}, \quad A_{22} = \begin{bmatrix} \lambda_1 & 1 & 0 & \dots & 0 \\ \lambda_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{n_2-1} & 0 & 0 & \dots & 0 \\ \lambda_{n_2} & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathfrak{R}^{n_2 \times n_2},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathfrak{R}^{n_1 \times 1}, \quad B_2 = \begin{bmatrix} b_{01} + \lambda_1 b_{00} \\ \vdots \\ b_{0,n_2} + \lambda_{n_2} b_{00} \end{bmatrix} \in \mathfrak{R}^{n_2 \times 1},$$

$$C_1 = [b_{10} + \beta_1 b_{00} \quad \dots \quad b_{1,n_1} + \beta_{n_1} b_{00}] \in \mathfrak{R}^{1 \times n_1}, \quad C_2 = [1 \quad 0 \quad \dots \quad 0] \in \mathfrak{R}^{1 \times n_2},$$

$$D = [b_{00}] \in \mathfrak{R}^{1 \times 1}.$$

Therefore, the consequent theorem has been proved.

Theorem 2. There exists positive realization of dimension $(n_1 + n_2) \times (n_1 + n_2)$ of transfer function (7) if it has separable denominator with nonnegative coefficients λ_k, β_l for $k = 1, 2, \dots, n_1; l = 1, 2, \dots, n_2$ and arbitrary nonnegative numerator coefficients $b_{i,j}$ for $i = 0, 1, \dots, n_1, j = 0, 1, \dots, n_2$.

If the assumptions of Theorem 2 are satisfied, then positive realization (3) of (7) can be found by the use of the following procedure:

Procedure 1.

- Step 1. Using the transfer function (7) write (10).
- Step 2. Using (10) draw the state variable diagram shown in Fig.1.
- Step 3. Choose as the state variables the outputs of integrators and of delay elements and write equations (12).
- Step 4. Using (12) find the desired realization (15).

Example 1. Find positive realization (3) of the continuous-discrete system with proper transfer function:

$$T(s^{-1}, z^{-1}) = \frac{0.6 + 0.5z^{-1} + 0.4s^{-1} + 0.3s^{-1}z^{-1} + 0.2s^{-2} + 0.1s^{-2}z^{-1}}{1 - 4z^{-1} - 2s^{-1} + 8s^{-1}z^{-1} - 3s^{-2} + 12s^{-2}z^{-1}} \quad (16a)$$

In this case $n_1 = 2, n_2 = 1$ and transfer function has separable denominator, since:

$$\begin{aligned} d(s^{-1}, z^{-1}) &= 1 - 4z^{-1} - 2s^{-1} + 8s^{-1}z^{-1} - 3s^{-2} + 12s^{-2}z^{-1} \\ &= (1 - 2s^{-1} - 3s^{-2})(1 - 4z^{-1}) \end{aligned} \quad (16b)$$

Using Procedure 1 we obtain the following:

Step 1. Using transfer function (16a) we can write:

$$E = U + (2s^{-1} + 3s^{-2})E,$$

$$Y = (0.6 + 0.5z^{-1} + 0.4s^{-1} + 0.3s^{-1}z^{-1} + 0.2s^{-2} + 0.1s^{-2}z^{-1})E + (4z^{-1})Y. \quad (17)$$

Step 2. State variable diagram has the form shown in fig.2.

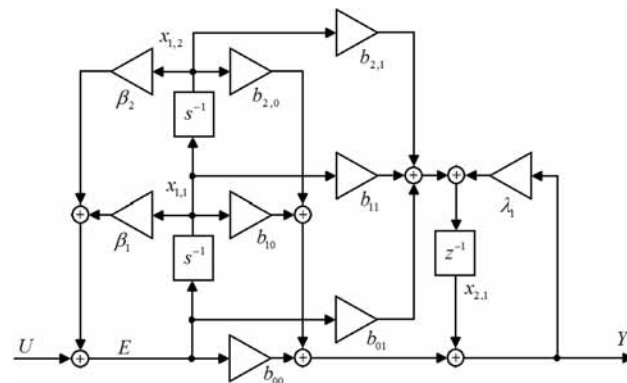


Fig. 2. State space diagram for transfer function (16) for: $\lambda_1 = 4, \beta_1 = 2, \beta_2 = 3, b_{00} = 0.6, b_{01} = 0.5, b_{10} = 0.4, b_{11} = 0.3, b_{20} = 0.2, b_{21} = 0.1$

rys. 2. Schemat zmiennych stanu dla transmitancji (16), przy czym: $\lambda_1 = 4, \beta_1 = 2, \beta_2 = 3, b_{00} = 0.6, b_{01} = 0.5, b_{10} = 0.4, b_{11} = 0.3, b_{20} = 0.2, b_{21} = 0.1$

Step 3. Using state variable diagram we can write the following equations:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= e(t, i), \\ \dot{x}_{1,2}(t, i) &= x_{1,1}(t, i), \\ x_{2,1}(t, i+1) &= 0.5e(t, i) + 0.3x_{1,1}(t, i) \\ &\quad + 0.1x_{1,2}(t, i) + 4y(t, i) \end{aligned} \quad (18a)$$

and

$$\begin{aligned} y(t, i) &= 0.6e(t, i) + 0.4x_{1,1}(t, i) \\ &\quad + 0.2x_{1,2}(t, i) + x_{2,1}(t, i), \\ e(t, i) &= 2x_{1,1}(t, i) + 3x_{1,2}(t, i) + u(t, i). \end{aligned} \quad (18b)$$

Substituting (18b) into (18a) we have:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= 2x_{1,1}(t, i) + 3x_{1,2}(t, i) + u(t, i), \\ \dot{x}_{1,2}(t, i) &= x_{1,1}(t, i), \\ x_{2,1}(t, i+1) &= 7.7x_{1,1}(t, i) + 9.6x_{1,2}(t, i) + 4x_{2,1}(t, i) + 2.9u(t, i), \\ y(t, i) &= 1.6x_{1,1}(t, i) + 2x_{1,2}(t, i) + x_{2,1}(t, i) + 0.6u(t, i) \end{aligned} \quad (19)$$

Step 4. The desired realization of (16) has the form:

$$A_{11} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{21} = [7.7 \quad 9.6], \quad A_{22} = [4],$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = [2.9], \quad C_1 = [1.6 \quad 2], \quad C_2 = [1], \quad D = [0.6]. \quad (20)$$

Obtained realization has only nonnegative entries and its dimension is minimal.

2.2. Case 2

Defining:

$$E(s^{-1}, z^{-1}) = \frac{(\lambda_0 + \lambda_1 z^{-1} + \lambda_2 z^{-2} + \dots + \lambda_{n_2} z^{-n_2})U(s^{-1}, z^{-1})}{1 - a_{01}z^{-1} - a_{10}s^{-1} - \dots - a_{n_1, n_2} s^{-n_1} z^{-n_2}} \quad (21)$$

from (8) and (21) we obtain:

$$\begin{aligned} E(s^{-1}, z^{-1}) &= (\lambda_0 + \lambda_1 z^{-1} + \lambda_2 z^{-2} + \dots + \lambda_{n_2} z^{-n_2})U(s^{-1}, z^{-1}) \\ &\quad + (a_{01}z^{-1} + a_{10}s^{-1} + \dots + a_{n_1, n_2} s^{-n_1} z^{-n_2})E(s^{-1}, z^{-1}), \\ Y(s^{-1}, z^{-1}) &= (\beta_0 + \beta_1 s^{-1} + \beta_2 s^{-2} + \dots + \beta_{n_1} s^{-n_1})E(s^{-1}, z^{-1}). \end{aligned} \quad (22)$$

Using (22) we may draw the state variable diagram shown in fig. 3.

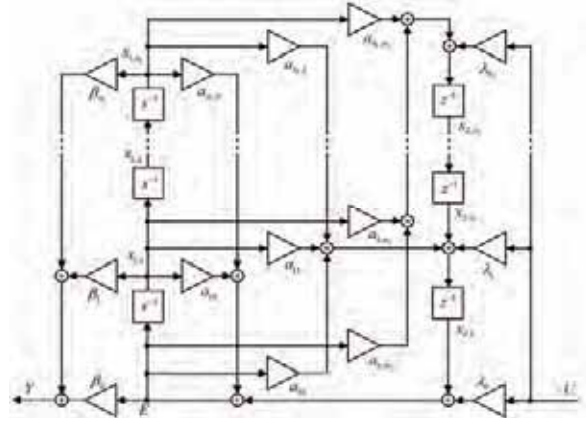


Fig. 3. State space diagram for transfer function (8) with separable numerators

Fig. 3. Schemat zmiennych stanu dla transmitancji (8) z separowanym licznikiem

Similarly as in section 2.1 as state variables we choose the outputs of integrators ($x_{1,1}(t, i), \dots, x_{1, n_1}(t, i)$) and of delay elements ($x_{2,1}(t, i), \dots, x_{2, n_2}(t, i)$). Using state variable diagram (fig. 3) we can write the following differential and difference equations:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= e(t, i), \\ \dot{x}_{1,2}(t, i) &= x_{1,1}(t, i), \\ &\vdots \\ \dot{x}_{1, n_1}(t, i) &= x_{1, n_1-1}(t, i), \\ x_{2,1}(t, i+1) &= a_{01}e(t, i) + a_{11}x_{1,1}(t, i) + a_{21}x_{1,2}(t, i) \\ &\quad + \dots + a_{n_1,1}x_{1, n_1}(t, i) + x_{2,2}(t, i) + \lambda_1 u(t, i) \\ &\vdots \\ x_{2, n_2-1}(t, i+1) &= a_{0, n_2-1}e(t, i) + a_{1, n_2-1}x_{1,1}(t, i) + a_{2, n_2-1}x_{1,2}(t, i) \\ &\quad + \dots + a_{n_1, n_2-1}x_{1, n_1}(t, i) + x_{2, n_2}(t, i) + \lambda_{n_2-1}u(t, i), \\ x_{2, n_2}(t, i+1) &= a_{0, n_2}e(t, i) + a_{1, n_2}x_{1,1}(t, i) + a_{2, n_2}x_{1,2}(t, i) \\ &\quad + \dots + a_{n_1, n_2}x_{1, n_1}(t, i) + \lambda_{n_2}u(t, i), \\ y(t, i) &= \beta_0 e(t, i) + \beta_1 x_{1,1}(t, i) + \beta_2 x_{1,2}(t, i) + \dots + \beta_{n_1} x_{1, n_1}(t, i) \end{aligned} \quad (23a)$$

where:

$$\begin{aligned} e(t, i) &= a_{10}x_{1,1}(t, i) + a_{20}x_{1,2}(t, i) + \dots + a_{n_1,0}x_{1, n_1}(t, i) \\ &\quad + x_{2,1}(t, i) + \lambda_0 u(t, i). \end{aligned} \quad (23b)$$

Substituting (23b) into (23a) we obtain:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= a_{10}x_{1,1}(t, i) + a_{20}x_{1,2}(t, i) + \dots + a_{n_1,0}x_{1, n_1}(t, i) \\ &\quad + x_{2,1}(t, i) + \lambda_0 u(t, i), \\ \dot{x}_{1,2}(t, i) &= x_{1,1}(t, i), \\ &\vdots \\ \dot{x}_{1, n_1}(t, i) &= x_{1, n_1-1}(t, i), \\ x_{2,1}(t, i+1) &= (a_{11} + a_{10}a_{01})x_{1,1}(t, i) + (a_{21} + a_{20}a_{01})x_{1,2}(t, i) + \dots \\ &\quad + (a_{n_1,1} + a_{n_1,0}a_{01})x_{1, n_1}(t, i) + a_{01}x_{2,1}(t, i) + x_{2,2}(t, i) + (\lambda_1 + a_{01}\lambda_0)u(t, i), \\ &\vdots \\ x_{2, n_2-1}(t, i+1) &= (a_{1, n_2-1} + a_{10}a_{0, n_2-1})x_{1,1}(t, i) + (a_{2, n_2-1} + a_{20}a_{0, n_2-1})x_{1,2}(t, i) \\ &\quad + \dots + (a_{n_1, n_2-1} + a_{n_1,0}a_{0, n_2-1})x_{1, n_1}(t, i) + a_{0, n_2-1}x_{2,1}(t, i) + x_{2,2}(t, i) \\ &\quad + (\lambda_{n_2-1} + a_{0, n_2-1}\lambda_0)u(t, i), \\ x_{2, n_2}(t, i+1) &= (a_{1, n_2} + a_{10}a_{0, n_2})x_{1,1}(t, i) + (a_{2, n_2} + a_{20}a_{0, n_2})x_{1,2}(t, i) \\ &\quad + \dots + (a_{n_1, n_2} + a_{n_1,0}a_{0, n_2})x_{1, n_1}(t, i) + a_{0, n_2}x_{2,1}(t, i) + (\lambda_{n_2} + a_{0, n_2}\lambda_0)u(t, i), \\ y(t, i) &= (\beta_1 + a_{10}\beta_0)x_{1,1}(t, i) + (\beta_2 + a_{20}\beta_0)x_{1,2}(t, i) + \dots \\ &\quad + (\beta_{n_1} + a_{n_1,0}\beta_0)x_{1, n_1}(t, i) + \beta_0 x_{2,1}(t, i) + \lambda_0 \beta_0 u(t, i). \end{aligned} \quad (24)$$

Defining state vectors in the form (13) we can write the equations (24) in the matrix form (14) where:

$$\begin{aligned} A_{11} &= \begin{bmatrix} a_{10} & a_{20} & \dots & a_{n_1-1,0} & a_{n_1,0} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \mathfrak{R}^{n_1 \times n_1}, \\ A_{12} &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathfrak{R}^{n_1 \times n_2}, \end{aligned} \quad (25)$$

$$\begin{aligned} A_{21} &= \begin{bmatrix} a_{11} + a_{10}a_{01} & \dots & a_{n_1,1} + a_{n_1,0}a_{01} \\ \vdots & \dots & \vdots \\ a_{1, n_2} + a_{10}a_{0, n_2} & \dots & a_{n_1, n_2} + a_{n_1,0}a_{0, n_2} \end{bmatrix} \in \mathfrak{R}^{n_2 \times n_1}, \\ A_{22} &= \begin{bmatrix} a_{01} & 1 & 0 & \dots & 0 \\ a_{02} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0, n_2-1} & 0 & 0 & \dots & 0 \\ a_{0, n_2} & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathfrak{R}^{n_2 \times n_2}, \\ B_1 &= \begin{bmatrix} \lambda_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathfrak{R}^{n_1 \times 1}, \quad B_2 = \begin{bmatrix} \lambda_1 + a_{01}\lambda_0 \\ \vdots \\ \lambda_{n_2} + a_{0, n_2}\lambda_0 \end{bmatrix} \in \mathfrak{R}^{n_2 \times 1}, \\ C_1 &= [\beta_1 + a_{10}\beta_0 \quad \dots \quad \beta_{n_1} + a_{n_1,0}\beta_0] \in \mathfrak{R}^{1 \times n_1}, \\ C_2 &= [\beta_0 \quad 0 \quad \dots \quad 0] \in \mathfrak{R}^{1 \times n_2}, \\ D &= [\lambda_0 \beta_0] \in \mathfrak{R}^{1 \times 1}. \end{aligned}$$

Therefore, the consequent theorem has been proved.

Theorem 3. There exists a positive realization of dimension $(n_1 + n_2) \times (n_1 + n_2)$ of transfer function (8) if it has separable numerator with nonnegative coefficients λ_i, β_j for $i = 0, 1, \dots, n_1, j = 0, 1, \dots, n_2$ and arbitrary nonnegative denominator coefficients $a_{k,l}$ for $k = 0, 1, \dots, n_1, l = 0, 1, \dots, n_2$ and $k + l \neq 0$.

If the assumptions of Theorem 3 are satisfied, then positive realization (3) of (8) can be found by the use of the following procedure:

Procedure 2.

- Step 1. Using the transfer function (8) write (22).
- Step 2. Using (22) draw the state variable diagram shown in fig. 3.
- Step 3. Choose as the state variables the outputs of integrators and of delay elements and write equations (24).
- Step 4. Using (24) find the desired realization (25).

Example 2. Find positive realization (3) of the continuous-discrete system with proper transfer function:

$$T(s^{-1}, z^{-1}) = \frac{3 + 4z^{-1} + 6s^{-1} + 8s^{-1}z^{-1} + 5z^{-2} + 10s^{-1}z^{-2}}{1 - 0.5z^{-1} - 0.4s^{-1} - 0.3s^{-1}z^{-1} - 0.2z^{-2} - 0.1s^{-1}z^{-2}} \quad (26a)$$

In this case $n_1 = 1, n_2 = 2$ and transfer function has separable numerator, since:

$$n(s^{-1}, z^{-1}) = 3 + 4z^{-1} + 6s^{-1} + 8s^{-1}z^{-1} + 5z^{-2} + 10s^{-1}z^{-2} = (1 + 2s^{-1})(3 + 4z^{-1} + 5z^{-2}). \quad (26b)$$

Using Procedure 2 we obtain the following:

Step 1. Using the transfer function (26) we can write:

$$E = (3 + 4z^{-1} + 5z^{-2})U + (0.5z^{-1} + 0.4s^{-1} + 0.3s^{-1}z^{-1} + 0.2s^{-2} + 0.1s^{-2}z^{-1})E, \quad (27)$$

$$Y = (1 + 2s^{-1})E.$$

Step 2. State variable diagram has the form shown in fig.4.

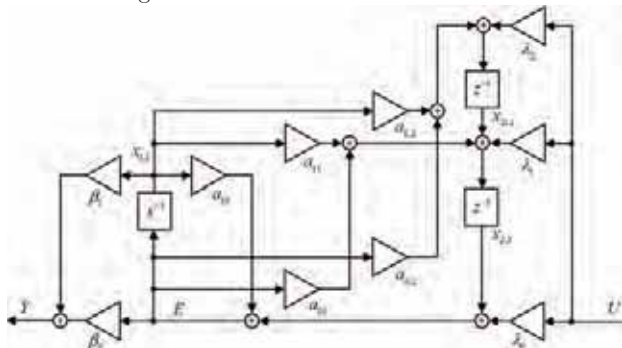


Fig. 4. State space diagram for transfer function (26) for $\lambda_0 = 3, \lambda_1 = 4, \lambda_2 = 5, \beta_0 = 1, \beta_1 = 2, a_{01} = 0.5, a_{10} = 0.4, a_{11} = 0.3, a_{02} = 0.2, a_{12} = 0.1$

Rys. 4. Schemat zmiennych stanu dla transmitancji (26), przy czym: $\lambda_0 = 3, \lambda_1 = 4, \lambda_2 = 5, \beta_0 = 1, \beta_1 = 2, a_{01} = 0.5, a_{10} = 0.4, a_{11} = 0.3, a_{02} = 0.2, a_{12} = 0.1$

Step 3. Using state variable diagram we can write the following equations:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= e(t, i), \\ x_{2,1}(t, i + 1) &= 0.5e(t, i) + 0.3x_{1,1}(t, i) + x_{2,2}(t, i) + 4u(t, i), \\ x_{2,1}(t, i + 1) &= 0.2e(t, i) + 0.1x_{1,1}(t, i) + 4u(t, i) \end{aligned} \quad (28a)$$

and

$$\begin{aligned} y(t, i) &= e(t, i) + 2x_{1,1}(t, i), \\ e(t, i) &= 0.4x_{1,1}(t, i) + x_{2,1}(t, i) + 3u(t, i). \end{aligned} \quad (28b)$$

Substituting (28b) into (28a) we have:

$$\begin{aligned} \dot{x}_{1,1}(t, i) &= 0.4x_{1,1}(t, i) + x_{2,1}(t, i) + 3u(t, i), \\ x_{2,1}(t, i + 1) &= 0.5x_{1,1}(t, i) + 0.5x_{2,1}(t, i) + x_{2,2}(t, i) + 5.5u(t, i), \\ x_{2,2}(t, i + 1) &= 0.18x_{1,1}(t, i) + 0.2x_{2,1}(t, i) + 5.6u(t, i), \\ y(t, i) &= 2.4x_{1,1}(t, i) + x_{2,1}(t, i) + 3u(t, i). \end{aligned} \quad (29)$$

Step 4. The desired realization of (26) has the form:

$$\begin{aligned} A_{11} &= [0.4], \quad A_{12} = [1 \quad 0], \quad A_{21} = \begin{bmatrix} 0.5 \\ 0.18 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.5 & 1 \\ 0.2 & 0 \end{bmatrix}, \\ B_1 &= [3], \quad B_2 = \begin{bmatrix} 5.5 \\ 5.6 \end{bmatrix}, \quad C_1 = [2.4], \quad C_2 = [1 \quad 0], \quad D = [3]. \end{aligned} \quad (30)$$

Obtained realization has only nonnegative entries and its dimension is minimal.

Presented approach can be used for transfer matrices (5a) of multi-input and multi-output systems. In MIMO case the transfer matrix (5a) should have the form:

$$T(s, z) = D^{-1}(s, z)N(s, z) \quad (\text{or } T(s, z) = \bar{N}(s, z)\bar{D}^{-1}(s, z)) \quad (31)$$

where $D(s, z)$ is diagonal matrix of common denominators for each row (column) of transfer matrix $T(s, z)$ [4].

3. Concluding remarks

A method for computation of positive minimal realizations of given proper transfer functions with separable numerator and with separable denominator of continuous-discrete linear systems has been proposed. Sufficient conditions for the existence of positive minimal realizations of given proper transfer function have been established. Two procedures for computation of positive minimal realizations have been proposed. The effectiveness of the procedures have been illustrated by numerical examples. Extension of these considerations for 2D continuous-discrete linear systems described by second Fornasini-Marchesini model [15] is possible.

An open problem is formulation of the necessary and sufficient conditions for the existence of solution of the positive minimal realization problem for 2D continuous-discrete linear systems in the general form [21].

Acknowledgment

This work was supported by European Social Fund and Polish Government under scholarship No. WIEM/POKL/MD/III/2011/2 of Human Capital Programme.

References

1. Farina L., Rinaldi S., *Positive Linear Systems*, [in:] *Theory and Applications*, J. Wiley, New York 2000.
2. Kaczorek T., *Positive 1D and 2D Systems*, Springer-Verlag, London 2002.
3. Kaczorek T., Busłowicz M., *Minimal realization problem for positive multivariable linear systems with delay*, "Int. J. Appl. Math. Comput. Sci.", Vol. 14, No. 2, 2004, 181–187.
4. Kaczorek T., *A realization problem for positive continuous-time linear systems with reduced numbers of delay*, "Int. J. Appl. Math. Comp. Sci.", Vol. 16, No. 3, 2006, 325–331.
5. Kaczorek T., *Positive minimal realizations for singular discrete-time systems with delays in state and delays in control*, "Bull. Pol. Acad. Sci. Tech.", Vol. 53, No. 3, 2005, 293–298.
6. Kaczorek T., *Realization problem for positive discrete-time systems with delay*, "System Science", Vol. 30, No. 4, 2004, 117–130.
7. Kaczorek T., *Realization problem for positive multivariable discrete-time linear systems with delays in the state vector and inputs*, "Int. J. Appl. Math. Comp. Sci.", Vol. 16, No. 2, 2006, 101–106.
8. Gałkowski K., Wood J. (Eds), *Multidimensional Signals, Circuits and Systems*, Taylor and Francis, London 2001.
9. Kaczorek T., *Linear control systems*, Vol. 1, Research Studies Press J. Wiley, New York 1992.
10. Kaczorek T., *Positive 2D hybrid linear systems*, "Bull. Pol. Acad. Sci. Tech.", Vol. 55, No. 4, 2007, 351–358.
11. Kaczorek T., *Positive fractional 2D hybrid linear systems*, "Bull. Pol. Acad. Sci. Tech.", Vol. 56, No. 3, 2008, 273–277.
12. Kaczorek T., Marchenko V., Sajewski Ł., *Solvability of 2D hybrid linear systems - comparison of the different methods*, "Acta Mechanica et Automatica", Vol. 2, No. 2, 2008, 59–66.
13. Sajewski Ł., *Positive minimal realization of continuous-discrete linear systems with all-pole and all-zero transfer function*, "Acta Mechanica et Automatica", 2012 (in Press).
14. Kaczorek T., *Realization problem for positive 2D hybrid systems*, "COMPEL", Vol. 27, No. 3, 2008, 613–623.
15. Sajewski Ł., Kaczorek T., *Computation of positive realizations of MIMO hybrid linear systems in the form of second Fornasini-Marchesini model*, "Archives of Control Sciences", Vol. 20, No. 3, 2010, 253–271.
16. Sajewski Ł., Kaczorek T., *Computation of positive realizations of singular SISO hybrid linear systems*, "JAMRIS", Vol. 3, No. 4, 2009, 8–14.
17. Antoniou G.E., Paraskevopoulos P.N., Varoufakis S.J., *Minimal state space realization of factorable 2D transfer functions*, "IEEE Trans. on Circ. and Sys.", Vol. 35, No. 8, 1988, 1055–1058.
18. Sun-Yuan Kung, Levy B.C., Morf M., Kailath T., *New results in 2-D systems theory, Part II: 2-D state-space models-realization and the notions of controllability, observability and minimality*, "Proc. of the IEEE", Vol. 65, No. 6, 1977, 945–961.
19. Sajewski Ł., *Solution of 2D singular hybrid linear systems*, "Kybernetes", Vol. 38, No. 7/8, 2009, 1079–1092.
20. Roesser R.B., *A discrete state-space model for linear image processing*, "IEEE Trans. on Autom. Contr.", AC-20, 1975, 1–10.
21. Kurek J., *The general state-space model for a two-dimensional linear digital system*, "IEEE Trans. on Autom. Contr.", AC-30, 1985, 600–602.
22. Benvenuti L., Farina L., *A tutorial on the positive realization problem*, "IEEE Trans. on Autom. Control", Vol. 49, No. 5, 2004, 651–664.
23. Dymkov M., Gaishun I., Rogers E., Gałkowski K., Owens D.H., *Control theory for a class of 2D continuous-discrete linear systems*, "Int. J. Control", Vol. 77, No. 9, 2004, 847–860.
24. Varoufakis S.J., Paraskevopoulos P.N., Antoniou G.E., *On the minimal state-space realizations of all-pole and all-zero 2-D systems*, "IEEE Trans. on Circ. and Sys.", Vol. 34, No. 3, 1987, 289–292. ■

Wyznaczanie dodatnich realizacji minimalnych układów ciągoty-dyskretnych o transmitancji z separowanym licznikiem lub mianownikiem

Streszczenie: Sformułowany został problem wyznaczania dodatniej realizacji minimalnej dla klasy liniowych układów ciągoty-dyskretnych. Przeanalizowane zostały dwa przypadki szczególne układów ciągoty-dyskretnych. Zaproponowana została metoda, bazująca na schemacie zmiennych stanu, wyznaczania dodatniej realizacji minimalnej na podstawie znanej transmitancji operatorowej układu. Określono warunki wystarczające istnienia dodatniej realizacji minimalnej dla transmitancji operatorowej z separowanym licznikiem lub mianownikiem. Podano dwie procedury wyznaczania dodatniej realizacji minimalnej, których efektywność zobrazowano przykładami numerycznymi.

Słowa kluczowe: ciągoty-dyskretny, dodatni, minimalna, realizacja, wyznaczanie

Łukasz Sajewski, PhD Eng.

Born on 8th December 1981 in Białystok. MSc title in Electrical Engineering received in July 2006 on Białystok University of Technology. At the same University in June 2009 he defended his PhD thesis and obtained the PhD degree in Electrical Engineering. Currently he is with Faculty of Electrical Engineering of Białystok University of Technology. His main scientific interests are control theory especially positive, continuous-discrete and fractional systems as well as automatic control and microprocessor techniques.

e-mail: l.sajewski@pb.edu.pl

