# Monitoring of fatigue life of mechatronic elements using spectral method for fatigue life assessment including the mean stress value

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**Abstract:** The paper presents a proposal of accounting the mean stress value in the process of fatigue life assessment using spectral method in terms of monitoring the fatigue life of mechatronic elements. The existing approaches are being discussed, and some chosen stress models used to take into account the influence of the mean stress value in the process of the determination of fatigue life are being introduced. The authors refer to a broad range of widely used models proposed by Soderberg, Goodman, Morrow, Gerber, and Kwofie. Those models can be used to determine the Power Spectral Density Function (PSDF) of the stress after transformation due to the mean value. Such a transformation is of great importance in fatigue life assessment with spectral method since PSDF is the quantity which defines loading and should also include information about mean stress. Determination of power spectral density of transformed stress allows the use of well-known models used in the spectral method, which in principle does not include the effect of the mean stress on fatigue life.

**Keywords:** mean stress; fatigue life assessment; random loading; power spectral density function

## 1. Introduction

Machines, as well as mechatronic components being subjected to variable loads, require constant supervision during operation due to the emerging phenomenon of material fatigue. Also, when designing new constructions or modification of nodes of machine elements, it is required to check their load capacity and fatigue life before finally being put into operation. Such kind of verifications are performed in laboratories carrying out fatigue tests or, if it is not possible because of e.g. the size of element or cost of the tests, calculations are made with a view to the best possible estimate of fatigue life. Method of calculations depends of the character of the load. In the case of load-amplitude with no significant mean value, the expected number of cycles to fatigue crack initiation can be read out directly from S-N curve, for example, from well known Wöhler curve. If there are significant mean values in the stress history, then their effect must be taken into account while assessing fatigue life. For this purpose you can use the charts to take account of the impact of the mean load, for example, Smith diagram or Wöhler curves drawn up for various cycle asymmetry coefficients $R = \sigma_{min}/\sigma_{max}$. If the diagrams or curves of this type are not available, then appropriate mean stress effect models should be used while calculations.

## 2. Mean value in random loading

Determination of fatigue life under variable amplitude or random loading is generally done in the time domain using a cycle counting algorithm determining the cycles from the loading history, using a chosen model to describe the influence of the mean load on fatigue and the hypothesis of summation of fatigue damage. Łagoda et al. [1] presents fatigue tests under uniaxial random tension-compression with and without mean value performed on samples made of 10HNAP steel. They proposed an algorithm for calculating the fatigue life using rainflow cycle counting method and the linear hypothesis of fatigue damage summation by Palmgren-Miner. The authors of this work have analyzed three ways to take into account the influence of the mean value, see fig. 1, which are:

a) method I, not taking into account the mean value,
b) method II, taking into account the influence of the mean value by transforming each of the cycle amplitude obtained from rainflow algorithm on the basis of their local mean value (rainflow cycle mean value),
c) method III, taking into account the influence of the mean value by transforming the whole load course on the basis of its global mean value before the cycle counting.

In the work by Łagoda et al. [1] the $K$ coefficient has been introduced, which allows to calculate the transformed amplitude according to the method II

$$\sigma_{aTi} = \sigma_{ai} \cdot K_i(\sigma_{mi}), \tag{1}$$

for the $i$-th cycle with amplitude $\sigma_{ai}$ and the mean value $\sigma_{mi}$ specified by the rainflow algorithm from a registered part of the random course. Method III is based on the principle of

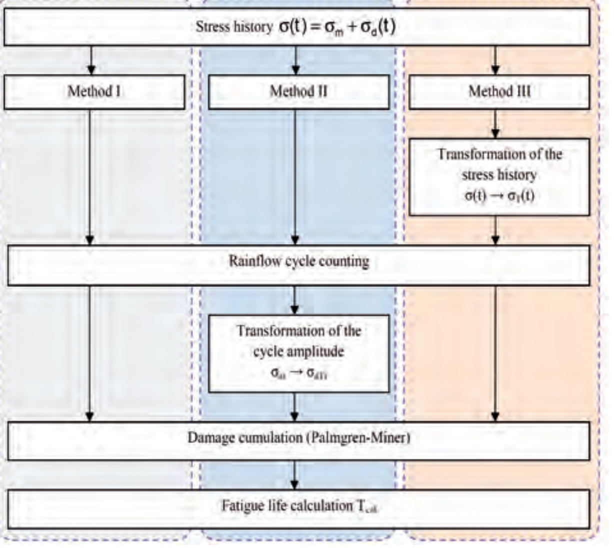


**Fig. 1.** Three methods for calculating the fatigue life $T_{cal}$ according to Łagoda et al. [1]

**Rys. 1.** Trzy metody obliczenia trwałości zmęczeniowej $T_{cal}$ (według Łagody i in. [1])

the transformation of the whole random stress course using the global mean value $\sigma_m$

$$\sigma_T(t) = \left[\sigma(t) - \sigma_m\right] \cdot K(\sigma_m)\,, \tag{2}$$

Amplitude of the transformed cycle $\sigma_{aTi}$ in this case is obtained directly by counting cycles of the course $\sigma_T(t)$ using rainflow cycle counting algorithm. Summation of fatigue damage is done according to the formula

$$D = \sum_{i=1}^{n} \frac{n_i}{N(\sigma_{aTi})}\,, \tag{3}$$

where: $D$ – fatigue damage parameter, $n_i$ – the number of cycles of amplitude $\sigma_{aTi}$, $N(\sigma_{aTi})$ – the number of cycles determined from S-N curve ($R = -1$) for the transformed amplitude $\sigma_{aTi}$. Fatigue life $N_{cal}$ expressed in cycles is determined from the formula

$$N_{cal} = \frac{N_{blok}}{D}\,, \tag{4}$$

where $N_{blok}$ is the number of counted cycles of the analyzed block of the stress course. The study carried out in [1] showed, that for the case of a stationary, random and symmetrically distributed relative to the mean value stress course the methods II and III are equivalent and can be used interchangeably in the calculations. In special cases, the $K$ coefficient is determined from the formulas derived on the basis of the adopted model to take account of the mean stress. In the literature you will find a significant number of models of this type [1, 2] for which the $K$ coefficient takes the form presented in tab. 1.

Fatigue life can be determined also in the frequency domain using a stochastic analysis of random processes. This method is known in the field of fatigue life assessment under the name spectral method and a lot of approaches including uniaxial and multiaxial cases were elaborated using this method [3, 6]. Taking into account the mean stress in this method is rather a difficult task, because the stress is represented by a power spectral density function, which contains information about the occurring locally and globally mean value in a way that is difficult to use in practice. In literature, however, we can find only a few suggestions on this issue. Kihl and Sarkani [3] and Sarkani et al. [4] show the effect of the mean value on fatigue life of welded steel joints. The tests were set to be run under both cyclic and random loadings with non-zero and zero mean stress value. The authors derived a formula to compute the expected number of cycles to fatigue failure in the case of random loads with extremes of Rayleigh distribution with a nonzero mean value of stress

$$N_{cal} = \left(1 - \frac{\sigma_m}{R_m}\right)^{-B} \frac{2^{\frac{B}{A}}\sigma_x^B A}{\Gamma\left(1 - \frac{B}{2}\right)} \tag{10}$$

where: $N_{cal}$ – number of cycles to fatigue failure, $A$ and $B$ – constant and slope of the Wöhler curve $\log(\sigma_a) = A + B \times \log(N)$, $\sigma_x$ – is the RMS stress value of the narrow-band random loading, $\Gamma(\cdot)$ – is the gamma function, $\sigma_m$ – global mean value of the random load, $R_m$ – tensile strength. It is easy to notice that in the eq. (10), the part being responsible for taking into account the mean value is $(1 - \sigma_m/R_m)^{-B}$, which modifies the expected cycle number till the fatigue failure determined by the narrow-band Miles formula [5].

**Tab. 1.** Formulas for the $K$ coefficient according to the chosen models [1, 2]

**Tab. 1.** Wzory na współczynnik $K$ według wybranych modeli [1, 2]

| Eq. No. | According to: | Formula |
|---|---|---|
| Eq. (5) | Soderberg | $K_S = \dfrac{1}{1 - \dfrac{\sigma_m}{R_e}}$ |
| Eq. (6) | Goodman | $K_{Go} = \dfrac{1}{1 - \dfrac{\sigma_m}{R_m}}$ |
| Eq. (7) | Morrow | $K_M = \dfrac{1}{1 - \dfrac{\sigma_m}{\sigma'_f}}$ |
| Eq. (8) | Gerber | $K_{Ge} = \dfrac{1}{1 - \left(\dfrac{\sigma_m}{R_m}\right)^2}$ |
| Eq. (9) | Kwofie | $K_K = \dfrac{1}{\exp\left(-\alpha \cdot \dfrac{\sigma_m}{R_m}\right)}$ |

$K_s$, $K_{Go}$, $K_M$, $K_{Ge}$, $K_K$ – coefficients determined on the basis of appropriate models of Soderberg, Goodman, Morrow, Gerber and Kwofies, respectively,

$R_e$ – plasticity limit,

$R_m$ – tensile strength,

$\beta_f$ – fatigue strength coefficient,

$\alpha$ – mean stress sensitivity of the material [2].

## 3. PSD function of a random process with mean value

Let us consider an example of one-dimensional stationary random process $x(t)$ showing the property of ergodicity. Assuming that $x(t)$ represents the physical signal is often convenient to present as the sum of static component $x_m$ and dynamic or fluctuant component $x_d(t)$ [7, 8]

$$x(t) = x_m + x_d(t)\,. \tag{11}$$

Static component can be described by the expected value (mean value in deterministic case) given by the formula

$$x_m = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)dx\,. \tag{12}$$

And the dynamic component by the signals variance

$$\mu_x = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[x(t) - x_m\right]^2 dt\,. \tag{13}$$

The variance, however, does not describe the spectral structure of a random process, and this information is essential for the proper estimation of the number of cycles and the amplitude distribution of the load during the fatigue calculations. Therefore for this purpose the power spectral density function is being used. PSD of the signal describes the overall structure of a random process using the spectral density of root mean square of the physical signal in question. This value can be determined for the interval from $f$ to $f+\Delta f$ using a central-pass filter and averaging the square on the output of the filter [7]

$$\Psi_x(f,\Delta f) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x^2(t,f,\Delta f)dt, \tag{14}$$

where: $\Psi_x$ – the mean square value of the process $x(t)$, $T$ – time of the observation, $x(t, f, \Delta f)$ – component of $x(t)$ in the frequency range from $f$ to $f + \Delta f$. For small values of $\Delta f$ the eq. (14) shows the one-sided PSD function

$$G_x(f) = \lim_{\Delta f\to 0}\frac{\Psi_x(f,\Delta f)}{\Delta f} = = \lim_{\Delta f\to 0}\frac{1}{\Delta f}\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T x^2(t,f,\Delta f)dt\right] \tag{15}$$

A characteristic feature of the $G_x(f)$ function is the relation to the autocorrelation function. In particular, for stationary signals, these functions are closely related by the Fourier transformation

$$G_x(f) = 2\int_{-\infty}^{\infty} R_x(\tau)e^{-j2\pi f\tau}d\tau, \tag{16}$$

where

$$R_x(\tau) = \lim_{\tau\to\infty}\frac{1}{T}\int_0^T x(t)x(t+\tau)dt, \tag{17}$$

is the autocorrelation function of the signal $x(t)$. Mean value $x_m$ of the random process can be determined from the autocorrelation function

$$x_m = \sqrt{R_x(\infty)}, \tag{18}$$

and the mean value of $x(t)$ is also a function of the PSD presented as a Dirac function at zero frequency

$$x_m = \sqrt{\int_{0^-}^{0^+}\delta(0)G_x(f)df}\ . \tag{19}$$

The eq. (19) shows, that the mean value is equal to the positive square root of the 'field' underlying the Dirac function. This is an abstract field, as Dirac function takes the value $+\infty$ for an infinite small interval. For this reason, the direct use of eq. (19) to determine the expected value on the basis of a PSD function of a random function in practical cases is impossible. Numerical algorithms to estimate the PSD functions are subjected to some restrictions coming from the basic frequency resolution. Also, the value of PSD function for $f = 0$, i.e. $G_x(0)$, results from the mean value $x(t)$ and from the mean square value of a random process from the interval $(0, \Delta f)$. Proper separation of these two values is impossible without additional information such as of the static value of the processes $x_m$. Therefore, in practice, we analyze those two values separately, the dynamic and static component of the random process according to eq. (11).

## 4. PSD function of a transformed stress course

The transition of the signal $x(t)$ by an linear system with constant parameters determined by the impulse response $h(\tau)$ and the transfer function $H(f)$ describes the following relationships [7]:

$$y(t) = \int_0^\infty h(\tau)x(t-\tau)d\tau, \tag{20}$$

$$G_y(f) = \left|H(f)\right|^2 G_x(f), \tag{21}$$

where $y(t)$ – output signal of the system, $G_x(f)$ and $G_y(f)$ – PSD's of input and output, respectively. From the eq. (21) we can notice that the power spectral density function of the output signal can be calculated knowing the gain factor $|H(f)|$ of the system. Fig. 2(a) shows schematically the transition of the signal $x(t)$ through a linear system. Spectral method for fatigue life determination uses PSD function to describe the stress state directly in the frequency domain. If the stress course includes a static and a fluctuant component then the transformed course should be computed according to the eq. (2). Treating the fluctuant component of the course $[\sigma(\tau)-\sigma_m]$ as an input signal of an linear system with constant gain factor $|H(f)| = K(\sigma_m)$ we can determine the PSD of a transformed stress

$$G_{\sigma T}(f) = \left[K(\sigma_m)\right]^2 G_\sigma(f), \tag{22}$$

where $G_\sigma(f)$ – power spectral density of a fluctuant component of the stress course. Fig. 2b presents the interpretation of the linear process of strain transformation due to the mean value, which can be compared to transition of a signal through a linear system, fig. 2(a). Eq. (22) allows the use of different forms of $K(\sigma_\mu)$ factors, for example, described by equations (5)–(9), in the process of determining the fatigue life by means of spectral method taking into account the static stress component.

If we consider a multiaxial loading case, then the transformation due to the mean stress has to be performed directly after crossing from the multiaxial stress state to the uniaxial, using appropriate multiaxial fatigue criteria's defined in the frequency domain. As an example we can use the criterion proposed by Macha [5] or Preumont and Pierford [5, 6, 9]. In this case the hydrostatic pressure value is used instead of the mean stress. It is a common and at the same time the simplest treatment used in the spatial stress state [6].

The main advantage of the proposed solution is that the transformation is subjected to power spectral density function before using known spectral models to determine fatigue life.

This gives the possibility of applying fatigue formulas in the spectral method developed for narrow-band frequency and the more universal solutions correctly describing most of the random loadings used in the fatigue life assessment [5].

Such a method is proposed by Dirlik [10] which is developed by using the empirical formula describing the probability density distribution of amplitudes ranges

$$p(\Delta\sigma) = \frac{1}{2\sqrt{m_0}}\left[\frac{K_1}{K_4}e^{\frac{-Z}{K_4}} + \frac{K_2 Z}{R^2}e^{\frac{-Z^2}{2R^2}} + K_3 Z e^{\frac{-Z^2}{2}}\right] \tag{23}$$

where: $Z$, $K_1$, $K_2$, $K_3$, $K_4$, $R$ – factors which are functions of the first five moments $m_k$ ($k = 0, ..., 4$) of the PSD function of transformed stress

$$m_k = \int_0^\infty G_{\sigma T}(f) f^k df \, . \quad (24)$$

Fatigue life is calculated using the selected hypothesis of fatigue damage accumulation, e.g. for a linear Palmgren-Miner hypothesis having regard to the amplitudes below the fatigue limit we obtain

$$N_{cal} = \frac{1}{\int_0^\infty \frac{p(\Delta\sigma)}{N(\Delta\sigma)} d\Delta\sigma} \quad (25)$$

where the number of cycles $N(\Delta\sigma)$ for stress range $\Delta\sigma$ is calculated on the basis of S-N curve

$$N(\Delta\sigma) = \sigma_{af}^m N_0 \left(\frac{\Delta\sigma}{2}\right)^{-m} . \quad (26)$$

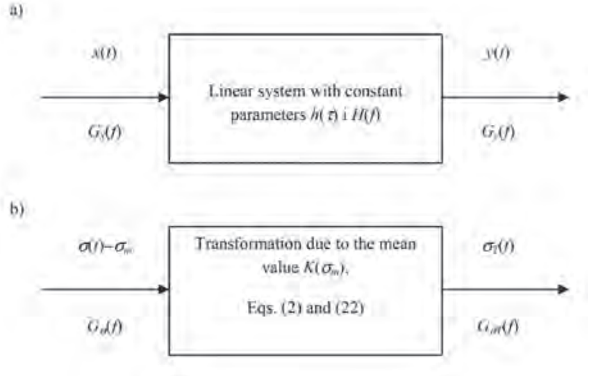


**Fig. 2.** One-input linear system (a) and interpretation of the linear process of strain course transformation due to the mean value (b)

**Rys. 2.** Jednowejściowy układ liniowy (a) oraz interpretacja transformacji liniowej przebiegu naprężenia ze względu na wartość średnią (b)

## 5. Computation algorithm

In order to calculate the fatigue life using the spectral method and taking into account the influence of the mean stress on fatigue life you should follow these steps:

a) designate or define PSD function of the fluctuant component of the stress course $G_\sigma(f)$ and establish its static part $\sigma_m$,
b) calculate the coefficient $K(\sigma_m)$ according to the right model, eq. (5)–(9). The choice of model depends of the mean stress value sensitivity of the material,
c) calculate PSD of then transformed stress $G_{\sigma T}(f)$ according to the eq. (22),
d) calculate the fatigue life using spectral method formulas, i.e. eq. (23) and (25) [5, 6].

## 6. Conclusions and observations

Based on the literature research it can be stated, that there are no papers that would propose the transformation of the power spectral density function of the stress, taking into account the influence of the mean stress value on the fatigue life. The presented equation (22) allows the calculation of the PSDF of the transformed stress, using models that are well known and widely verified in experimental researches. The proposal of Kihl and Sarkani [3] and Sarkani et al. [4] uses a Rayleigh amplitude distribution approximation, which reduces the area of application of the eq. (10) only to narrowband processes. The method proposed by the authors doesn't have this limitation and therefore allows a wide usage of many formulas used to predict the fatigue life by means of the spectral method. Compared with the time domain fatigue life prediction methods, the spectral method shows greater efficiency and it can be used there, where a multiplicand fatigue calculation is required (constructions optimization, fatigue damage maps etc.).

The experimental verification should be performed to verify the correctness of the fatigue calculations evaluated according to the proposed method, nevertheless the transformation of the PSD function in the spectral method is equivalent to the eq. (2) in the time domain.

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**Monitorowanie trwałości zmęczeniowej elementów mechatronicznych przy wykorzystaniu metody spektralnej wyznaczania trwałości zmęczeniowej z uwzględnieniem wartości średniej naprężenia**

**Streszczenie:** Praca przedstawia propozycję uwzględniania wartości średniej naprężenia w procesie wyznaczania trwałości zmęczeniowej przy wykorzystaniu metody spektralnej w odniesieniu do monitorowania trwałości zmęczeniowej elementów mechatronicznych. Opisano obecne podejścia oraz przedstawiono część wybranych modeli uwzględnienia wartości średniej naprężenia w procesie obliczania trwałości zmęczeniowej. Autorzy odnoszą się do szerokiej gamy stosowanych modeli zaproponowanych m.in. przez Soderberga, Goodmana, Morrowa, Gerbera oraz Kwofie'go. Te modele mogą zostać wykorzystane w celu wyznaczenia Gęstości Widmowej Mocy (GWM) naprężenia po transformacji ze względu na wartość średnią. Taka transformacja jest bardzo ważna w wyznaczaniu trwałości zmęczeniowej przy użyciu metody spektralnej, ponieważ Funkcja Gęstości Widmowej Mocy (FGWM) jest wielkością, która definiuje obciążenie (przy czym powinna również uwzględniać informacje o wartości średniej). Wyznaczanie Gęstości Widmowej Mocy transformowanego naprężenia pozwala na wykorzystanie znanych modeli uwzględnienia wartości średniej w metodzie spektralnej, która zasadniczo nie obejmuje wpływu wartości średniej naprężenia na trwałość zmęczeniową.

**Słowa kluczowe:** wyznaczanie trwałości zmęczeniowej; obciążenia losowe; naprężenie średnie; funkcja gęstości widmowej mocy

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