An application of generalized least squares method to the conduction heat transfer problem

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Abstract: Article presents an application of generalized least squares method to heat transfer problem (the steady-state one-dimensional heat conduction). The theoretical basis of mathematical method was presented as well as general model of conduction heat transfer problem was introduced. During model creation boundary and internal (additional) measurements of temperature in the plate were used. In article the different locations of additional measuring points were checked and the local and global error of obtained models were determined.

Keywords: generalized least squares method, conduction heat transfer, error of measurement

1. Introduction

The mathematical modeling of heat transfers problems often use numerical methods which allows for finding approximate parameters describing real system. Those solutions usually are not precise – due to the lack of different types of errors. Especially in the case when physical measurements are used, it is important to find deviation of obtained results (which derives from measured data).The generalized least squares method lets for finding the most accurate results (models parameters) in the case when the number of measurements exceeds the amount of data necessary to define the model (there is additional information about system available, but all data are affected by measurement error).

Article presents the application of generalized least squares method to the steady-state one-dimensional heat conduction problem. To define the temperature profile in the plate only two measurements of temperature have to be made. But the measurements are affected by measurement instrument error so obtained profile is not precise. It is checked how additional temperature measurements affect the error distribution. Several cases with different error types (percent and constant error) and different number of additional measuring points are analyzed. Analyze of different conditions and possibilities of additional measurement lets for proper planning of experiment.

Presented methodology can be applied to many other problems, including more complicated heat transfer issues.

2. Mathematical method

The least squares method is used in finding approximate solution of overdetermined systems of equations. Overdetermined equations sets have more equations than unknowns. Alternative methods of finding solution in that type of problems are more complicated and less popular. The least squares method was introduced by Legendre (1805) and Gauss (1809). Initially it was used in geodetic calculations and then it found an application in astronomy and science. Moreover this method is the basis of modern error analysis. The least squares method finds the most likely value of variable which was measured with experimental errors in several different experiments. This method can be used also in more complicated problems with many unknown variables, defined as a functions of measured values. The generalized least squares method (GLS) with assumed normal error distribution lets for finding values of unknown variables with their errors.

Generalized least squares method (unified least squares method) was introduced in 1976 by Mikhail and Ackermann. Nowdays it is widely used in energy science problems, for example to model heat and mass transfer or prepare balances of energy systems [2, 3, 5].

2.1. Least squares method

The basic idea of least squares method is based on the Legendre postulate. Value of measurement y_j can be defined as sum of unknown value x and measurment error ε_j:

$$y_j = x + \varepsilon_j .$$

The goal is to find values of ε_j which gave minimal sum of squared measurement errors:

$$\sum_j \varepsilon_j^{\,2} = \sum_j \left(x - y_j\right)^2 = \min .$$

2.2. Generalized least squares method

Mathematical models for real problems consiss usually many complicated non-linear differential equations. Finding algebraic solution is often very laborious or even impossible. Those problems, on the other hand, can be solved by using numerical methods and linearization of the equations [1]

Assume, that problem is defined by the following set of J equations:

$$f_j\left(\mathbf{u}^*, \mathbf{x}^*\right) = 0, \qquad j = 1, 2, \ldots, J .$$

There are two different types of variables in equations: experimentally measured (vector $\mathbf{u}^*$) and unknowns variables which values is looked for (vector $\mathbf{x}^*$ represents best approximation of unknowns). The goal is to find vectors $\mathbf{u}^*$ and $\mathbf{x}^*$, which satisfied given equations set most precisely. For initial values of measured variables $\mathbf{u}$ and unknown approximations $\mathbf{x}$ the model equation set is not satisfied precisely:

$$f_j(\mathbf{u}, \mathbf{x}) = -\mathrm{w}_j, \qquad j = 1, 2, \ldots, J ,$$

where w_j is error (residuum) of j equation.

The corrections vector are defined:

$$\mathbf{u}^* = \mathbf{u} + \mathbf{v},$$
$$\mathbf{x}^* = \mathbf{x} + \mathbf{y},$$

where **v** is experimental measurements correction vector and **y** is unknowns correction vector.

Finally it can be written following equation:

$$f_j(\mathbf{u}+\mathbf{v}, \mathbf{x}+\mathbf{y}) = 0, \qquad j = 1, 2, \ldots, J\,.$$

In mathematical models f_j functions are unrestricted algebraic functions. Least squares method requires linear constraint equations. If the f_j functions are differentiable it is possible to expanse them into Taylor series in the neighborhood of the point P(**u**, **x**). Omission of higher derivatives lets for linearization of constraint equations.

The model constraints after linearization can be written in the following form:

$$\mathbf{A_B}\mathbf{V_B} = \mathbf{W_B},$$

where $\mathbf{A_B}$ i $\mathbf{V_B}$ are defined simultaneously for experimental measurements and unknowns:

$$\mathbf{A_B} = [\mathbf{A}, \mathbf{B}],$$
$$\mathbf{V_B} = [\mathbf{v}, \mathbf{y}]^{\mathrm{T}}.$$

A and **B** are Jacobi's matrixes defined as:

$$\mathbf{A} = \frac{\partial f_j}{\partial u_k},$$

$$\mathbf{B} = \frac{\partial f_j}{\partial x_n}.$$

The least squares methods requires minimization of the following function:

$$\mathbf{\Phi}(\mathbf{v}, \mathbf{y}) = \sum_{i=1}^{K}\left(\frac{v_i}{s_i}\right)^2 + \sum_{j=1}^{N}\left(\frac{y_j}{s_j}\right)^2 = \min$$

where **v** – experimental measurements correction vector; **y** – unknowns correction vector; v_i – element i of vector **v**; y_j – element j of vector **y**; s_i, s_j – error of i measurement (j unknown).

Covariance matrix is defined as:

$$\mathbf{C_B} = \begin{bmatrix} \mathbf{C_S} & 0 \\ 0 & \mathbf{C_{SX}} \end{bmatrix}$$

Where $\mathbf{C_S}$ and $\mathbf{C_{SX}}$ are covariance matrixes for experimental measurements and unknowns with their variance values on diagonal: σ_i^2 and σ_j^2 (it is assumed that $\sigma_j^2 \gg \sigma_i^2$).

The solution of minimization problem is following:

$$\mathbf{V_B} = \mathbf{C_B}\mathbf{A_B}^{\mathrm{T}}\,\mathbf{F_B}^{-1}\,\mathbf{W_B}$$

where:

$$\mathbf{F_B} = \mathbf{A_B}\mathbf{C_B}\mathbf{A_B}^{\mathrm{T}}$$

Final covariance matrix after application of least squares method is:

$$\mathbf{C_{VB}} = \mathbf{C_B} - \mathbf{C_B}\mathbf{A_B}^{\mathrm{T}}\,\mathbf{F_B}^{-1}\,\mathbf{A_B}\,\mathbf{C_B}$$

or in the matrix form:

$$\mathbf{C_{VB}} = \begin{bmatrix} \mathbf{C_U} & \mathbf{C_{UX}} \\ \mathbf{C_{UX}}^{\mathrm{T}} & \mathbf{C_X} \end{bmatrix}.$$

Matrix $\mathbf{C_{VB}}$ represents the measurement errors and it is possible to obtain from it the measurement's standard deviation values.

3. Conduction heat transfer problem

Presented methodology was applied to the steady-state one-dimensional heat conduction problem.

Considered wall has 0.1 m width, what is significantly less than other dimensions – because of that the one-dimensional heat transfer model can be used.

Heat transfer by conduction can by described by the general Fourier-Kirchhoff equation:

$$\rho c_p \frac{\partial T}{\partial t} = \lambda \nabla^2 T + \dot{q}_V,$$

where:
T – temperature [K], t – time [s], λ – thermal conductivity [W/(m·K)], ρ – density [kg/m^3], cp – heat capacity [J/(kg·K)], $\dot{q}_V$ – heat source.

The investigated problem assumes that system is in the steady state:

$$\frac{\partial T}{\partial t} = 0,$$

and there is no additional heat source:

$$\dot{q}_V = 0.$$

Moreover, for one dimension problem it can be assumed that temperature is one variable function dependent from x coordinate

$$T = f(x).$$

On the basis of following assumptions the final form of Fourier-Kirchhoff equation for the one-dimensional steady-state heat transfer problem can be presented:

$$\frac{d^2T(x)}{dx^2} = 0.$$

The wall was divided into 10 modules (11 variable points; presented on fig. 1) and numerical modeling of differential equation was used:

$$\frac{d^2T(x)}{dx^2} \cong \frac{T(i+1) - 2T(i) + T(i-1)}{(\Delta x)^2} = 0.$$

Presented equation was used to prepare constraint equation set.

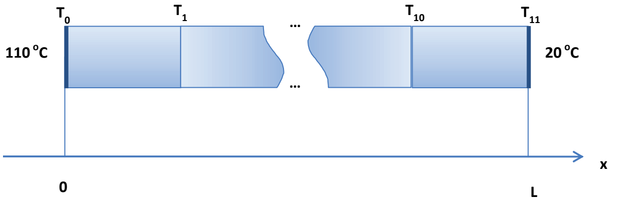


Fig. 1. Plate divided for modules used in the model

Rys. 1. Podział płyty wykorzystany w modelu

It is assumed that temperatures on the ends of the plate are equal 110 °C and 20 °C.

Measured variables **u** are temperatures on the boundaries of the plate and additional temperatures inside (one or two variables). Unknowns **x** are temperatures in the rest measuring points.

In the next chapter the several possibilities of choosing additional measuring points are presented.

4. Results

Figure 2 presents analysis of the case with one additional point inside the plate. The absolute error was constant for all points and equal 0.5 K. On the graphs measured points are pointed by black dot. Lines represents local error distribution for different analyzed cases – different placement of additional measuring point.

Figure 3 shows the sum of diagonal elements of covariance matrix C_{VB} for the presented case. It can be noticed that

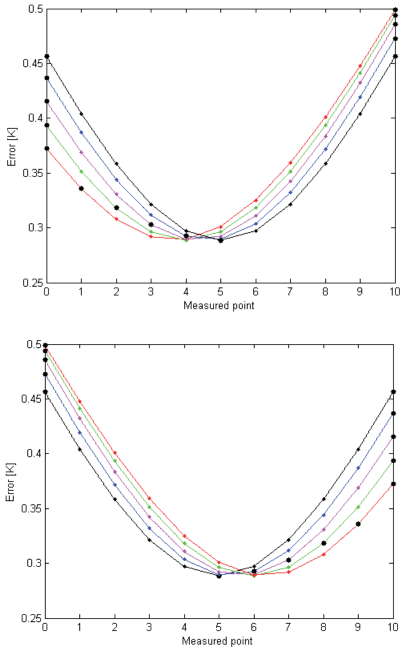


Fig. 2. Error distribution of temperatures determined on the basis of 3 measurement (pointed with black dots) with error 0.5 K. Different lines represent error based on different measurement points distribution

Rys. 2. Rozkład błędów temperatury wyznaczonej na podstawie 3 pomiarów (oznaczonych czarnymi punktami) z błędem 0,5 K. Poszczególne linie pokazują rozkład w przypadku różnego doboru punktów pomiarowych

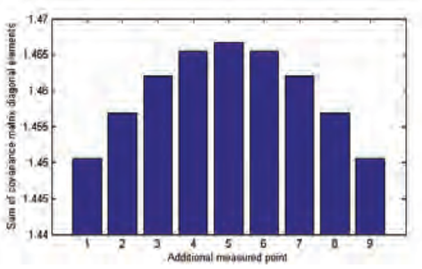


Fig. 3. Sum of diagonal elements of covariance matrix $\mathbf{C}_{VB}$ for case with 3 measurements with standard deviation 0.5 K

Rys. 3. Suma elementów diagonalnych macierzy kowariancji $\mathbf{C}_{VB}$ dla przypadku z 3 pomiarami z odchyleniem 0,5 K

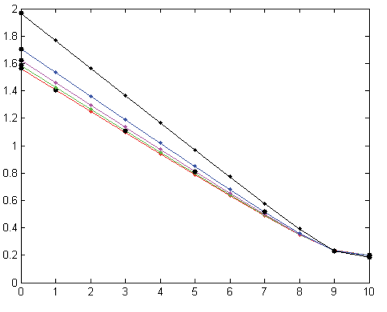


Fig. 4. Error distribution of temperatures determined on the basis of 3 measurement (pointed with black dots) with error 2 %. Different lines represent error based on different measurement points distribution

Rys. 4. Rozkład błędów temperatury wyznaczonej na podstawie 3 pomiarów (oznaczonych czarnymi punktami) z błędem 2 %. Poszczególne linie pokazują rozkład w przypadku różnego doboru punktów pomiarowych

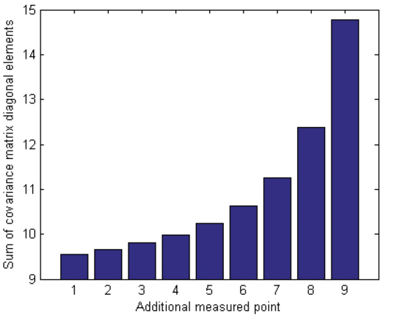


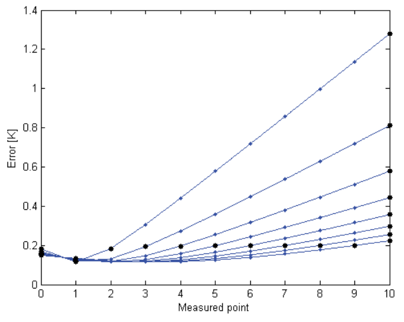
Fig. 5. Sum of diagonal elements of covariance matrix $\mathbf{C}_{VB}$ for case with 3 measurements with standard deviation 2 %

Rys. 5. Suma elementów diagonalnych macierzy kowariancji $\mathbf{C}_{VB}$ dla przypadku z 3 pomiarami z odchyleniem 2 %

for constant measurement error the best choice for additional measuring point is the place close to the boundary of a plate (point 1 or point 9) and the results for both plate's sides are symmetrical.

The next analyzed case was analogical to the previous one, but instead of constant measurement absolute error the percentage relative error value equals to the 2 % of measured value was assumed. The results of analysis are presented in the fig. 4 and fig. 5. It can be noticed that simi-

A.



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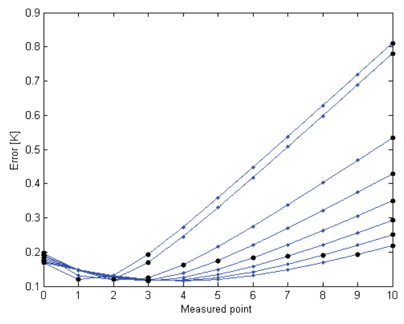

Fig. 6. Error distribution of temperatures determined on the basis of 3 measurement (pointed with black dots) with error 0.2 K. Different lines represent error based on different measurement points distribution

Rys. 6. Rozkład błędów temperatury wyznaczonej na podstawie 3 pomiarów (oznaczonych czarnymi punktami) z błędem 0,2 K. Poszczególne linie pokazują rozkład w przypadku różnego doboru punktów pomiarowych

larly to the previous condition the additional point should be located next to the plate's boundary, but in this case it should be end with higher temperature (we gain additional information in the part, where the biggest measurement error occurs).

Analogical analysis can be prepared for the bigger number of additional points. Figure 6 presents results obtained for 2 additional points inside the plate (4 measuring points total) and constant measurement error. It is seen that the best results are obtained when the points are equally distributed (for example, when one additional point is situated next to the end of plate, the second additional point should be placed next to the opposite end).

5. Conclusions

Article presents generalized least squares method and its application to the conduction heat transfer problem. All computations were conducted in MATLAB. Presented methodology and program implementation can be easy applied to many different configuration of the measuring system as well as to another heat transfer problems. What is significant least squares method lets for taking into account the measurement errors and their influence for final results. Presented results can be helpful in designing experimental measurements – they provide a method of minimalizing final error of obtained results.

Bibliography

1. Brandt S., *Statistical and Computational Methods in Data Analysis*, North-Holland Publishing Company, Amsterdam, Netherlands 1970.
2. Szargut J., *Rachunek wyrównawczy w technice cieplnej*, PAN, Komisja Energetyki, Wrocław 1984.
3. Szmyd J., Suzuki K., Kolenda Z., *An interactive computational – experimental methodology approach to convection and conduction heat transfer problems with uncertain specification*, Proc. of EUROTHERM Sem. 36 on Advanced Concepts and Techniques in Thermal Modeling, Poitiers, France, 1994, 277–283.
4. Taylor J., *An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurments*, University Science Books, USA, 1997.
5. Wroński P., *Wyznaczenie rozkładów pól temperatury i strumieni ciepła w procesach wymiany ciepła przez przewodzenie, przy nadmiarze danych obarczonych błędami*, Praca magisterska, AGH, Kraków 1997.

Wykorzystanie uogólnionej metody najmniejszych kwadratów w analizie przewodzenia ciepła

Streszczenie: Artykuł prezentuje zastosowanie uogólnionej metody najmniejszych kwadratów w analizie problemu transportu ciepła (stacjonarne, jednowymiarowe przewodzenie ciepła). Zaprezentowano teoretyczne podstawy metody matematycznej oraz wprowadzono ogólny model przewodzenia ciepła. Do stworzenia modelu wykorzystano pomiary temperatury na brzegach płyty oraz dodatkowe, wewnętrzne punkty pomiaru. Sprawdzono wpływ wyboru różnych punktów pomiarowych na sumaryczny oraz lokalny błąd uzyskanych modeli matematycznych.

Słowa kluczowe: uogólniona metoda najmniejszych kwadratów, przewodzenie ciepła, błąd pomiarowy

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