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TESTS FOR PRACTICAL STABILITY OF POSITIVE FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

The practical stability of positive fractional discrete-time linear systems is addressed. It is shown that: 1) the positive fractional systems are unstable if at least one diagonal entry of the system matrix is greater one, 2) checking of the practical stability of the systems can be reduced to checking of the asymptotic stability of corresponding positive linear systems. The considerations are illustrated by a numerical example.

METODY BADANIA PRAKTYCZNEJ STABILNOŚCI DODATNICH LINIOWYCH DYSKRETNYCH UKŁADÓW NIECAŁKOWITEGO RZĘDU

Zostaną podane metody badania stabilności praktycznej dodatnich dyskretnych liniowych układów niecałkowitego rzędu. Wykazane zostanie, że: 1) dodatnie układy niecałkowitego rzędu są niestabilne, jeżeli przynajmniej jeden element na głównej przekątnej macierzy stanu jest większy od jedności, 2) sprawdzanie stabilności praktycznej można zredukować do sprawdzania stabilności asymptotycznej odpowiadającego dodatniego układu liniowego. Rozważania zostaną zobrazowane przykładami numerycznymi.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [3, 5].

Mathematical fundamentals of fractional calculus are given in the monographs [17-19, 23]. The fractional positive linear continuous-time and discrete-time systems have been addressed in [7, 9, 20, 23, 25]. The first monograph on analysis and synthesis of control systems with delays was the monograph published by Gorecki in 1971 [28]. Stability of positive 1D and 2D systems has been addressed in [12-14, 26, 27] and the stability of positive and fractional linear systems has been investigated in [1, 2]. The reachability and controllability to zero of positive fractional linear systems have been considered in [6, 8, 16]. The fractional order controllers have been developed in [22]. A generalization of the Kalman filter for fractional order systems has been proposed in [24]. Fractional polynomials and n D systems have been investigated in [4]. The notion of standard and positive 2D fractional linear systems has been introduced in [10, 11].

In the paper [15] a new concept of the practical stability of positive fractional discrete-time linear systems was introduced and necessary and sufficient conditions for the practical stability were established.

In this paper it will be shown that the positive fractional systems are unstable if at least one diagonal entry of the system matrix is greater one and that checking of the practical stability

of the systems can be reduced to checking of the asymptotic stability of corresponding positive linear systems.

The paper is organized as follows. In section 2 the basic definitions and necessary and sufficient conditions for the positivity of the discrete-time linear systems are recalled. The positive fractional discrete-time linear systems are introduced in section 3. The main result of the paper are presented in section 4. Concluding remarks are given in section 5.

The following notation will be used in the paper. The set of real $n \times m$ matrices with nonnegative entries will be denoted by $\mathfrak{R}_+^{n \times m}$ and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$. A matrix $A = [a_{ij}] \in \mathfrak{R}_+^{n \times m}$ (a vector x) will be called strictly positive and denoted by $A > 0$ ($x > 0$) if $a_{ij} > 0$ for $i = 1, \dots, n, j = 1, \dots, m$. The set of nonnegative integers will be denoted by Z_+ .

2. PPOSITIVE DISCRETE-TIME LINEA SYSTEMS

Consider the linear discrete-time system:

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ \tag{1a}$$

$$y_i = Cx_i + Du_i \tag{1b}$$

where, $x_i \in \mathfrak{R}^n, u_i \in \mathfrak{R}^m, y_i \in \mathfrak{R}^p$ are the state, input and output vectors and,

$$A \in \mathfrak{R}^{n \times n}, \quad B \in \mathfrak{R}^{n \times m}, \quad C \in \mathfrak{R}^{p \times n}, \quad D \in \mathfrak{R}^{p \times m}.$$

Definition 1. The system (1) is called (internally) positive if $x_i \in \mathfrak{R}_+^n, y_i \in \mathfrak{R}_+^p, i \in Z_+$, for any initial conditions $x_0 \in \mathfrak{R}_+^n$ and every input sequence $u_i \in \mathfrak{R}_+^m, i \in Z_+$.

Theorem 1 [3, 5]. The system (1) is positive if and only if

$$A \in \mathfrak{R}_+^{n \times n}, B \in \mathfrak{R}_+^{n \times m}, C \in \mathfrak{R}_+^{p \times n}, D \in \mathfrak{R}_+^{p \times m} \tag{2}$$

The positive system (1) is called asymptotically stable if the solution

$$x_i = A^i x_0 \tag{3}$$

of the equation

$$x_{i+1} = Ax_i, \quad A \in \mathfrak{R}_+^{n \times n}, \quad i \in Z_+ \tag{4}$$

satisfies the condition

$$\lim_{i \rightarrow \infty} x_i = 0 \quad \text{for every } x_0 \in \mathfrak{R}_+^n \tag{5}$$

Theorem 2. [3, 5, 12] For the positive system (4) the following statements are equivalent:

- 1) The system is asymptotically stable
- 2) Eigenvalues z_1, z_2, \dots, z_n of the matrix A have moduli less than 1, i.e. $|z_k| < 1$ for $k = 1, \dots, n$
- 3) $\det[I_n z - A] \neq 0$ for $|z| \geq 1$
- 4) $\rho(A) < 1$

where $\rho(A)$ is the spectral radius of the matrix A defined by $\rho(A) = \max_{1 \leq k \leq n} \{|z_k|\}$

- 5) All coefficients $\hat{a}_i, i = 0, 1, \dots, n-1$ of the characteristic polynomial

$$p_{\hat{A}}(z) = \det[I_n z - \hat{A}] = z^n + \hat{a}_{n-1}z^{n-1} + \dots + \hat{a}_1z + \hat{a}_0 \tag{6}$$

of the matrix $\hat{A} = A - I_n$ are positive.

- 6) All principal minors of the matrix

$$\bar{A} = I_n - A = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots & \bar{a}_{nn} \end{bmatrix} \tag{7a}$$

are positive, i.e.

$$|\bar{a}_{11}| > 0, \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{vmatrix} > 0, \dots, \det \bar{A} > 0 \tag{7b}$$

7) There exists a strictly positive vector $\bar{x} > 0$ such that

$$8) [A - I_n]\bar{x} < 0 \tag{8}$$

Theorem 3. [5] The positive system (4) is unstable if at least one diagonal entry of the matrix A is greater than 1.

3. POSITIVE FRACTIONAL SYSTEMS

In this paper the following definition of the fractional difference

$$\Delta^\alpha x_k = \sum_{j=0}^k (-1)^j \binom{\alpha}{j} x_{k-j}, \quad 0 < \alpha < 1 \tag{9}$$

will be used, where $\alpha \in R$ is the order of the fractional difference, and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \dots \end{cases} \tag{10}$$

Consider the fractional discrete-time linear system, described by the state-space equations

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad u \in Z_+ \tag{11a}$$

$$y_k = Cx_k + Du_k \tag{11b}$$

where $x_k \in \mathfrak{R}^n$, $u_k \in \mathfrak{R}^m$, $y_k \in \mathfrak{R}^p$ are the state, input and output vectors and $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$, $C \in \mathfrak{R}^{p \times n}$, $D \in \mathfrak{R}^{p \times m}$.

Using the definition (9) we may write the equations (11) in the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{12a}$$

$$y_k = Cx_k + Du_k \tag{12b}$$

Definition 2. The system (12) is called the (internally) positive fractional system if and only if $x_k \in \mathfrak{R}_+^n$ and $y_k \in \mathfrak{R}_+^p$, $k \in Z_+$ for any initial conditions $x_0 \in \mathfrak{R}_+^n$ and all input sequences $u_k \in \mathfrak{R}_+^m$, $k \in Z_+$.

Theorem 4. [7] Let $0 < \alpha < 1$. Then the fractional system (12) is positive if and only if

$$A + I_n \alpha \in \mathfrak{R}_+^{n \times n}, \quad B \in \mathfrak{R}_+^{n \times m}, \quad C \in \mathfrak{R}_+^{p \times n}, \quad D \in \mathfrak{R}_+^{p \times m} \tag{13}$$

4. PRACTICAL STABILITY OF POSITIVE FRACTIONAL SYSTEMS

From (10) it follows that the coefficients

$$c_j = c_j(\alpha) = (-1)^j \binom{\alpha}{j+1}, \quad j = 1, 2, \dots \tag{14}$$

strongly decrease for increasing j and they are positive for $0 < \alpha < 1$ [7]. In practical problems it is assumed that j is bounded by some natural number h .

In this case the equation (12a) takes the form

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^h c_j x_{k-j} + Bu_k, \quad k \in Z_+ \tag{15}$$

where

$$A_\alpha = A + I_n \alpha \tag{16}$$

Note that the equations (15) and (12b) describe a linear discrete-time system with h delays in state.

Definition 3. The positive fractional system (12) is called practically stable if and only if the system (15) is asymptotically stable.

Defining the new state vector

$$\tilde{x}_k = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-h} \end{bmatrix} \tag{17}$$

we may write the equations (15) and (12b) in the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k, \quad k \in Z_+ \tag{18a}$$

$$y_k = \tilde{C}\tilde{x}_k + \tilde{D}u_k \tag{18b}$$

where

$$\tilde{A} = \begin{bmatrix} A_\alpha & c_1 I_n & c_2 I_n & \dots & c_{h-1} I_n & c_h I_n \\ I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_n & 0 \end{bmatrix} \in \mathfrak{R}_+^{\tilde{n} \times \tilde{n}}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathfrak{R}_+^{\tilde{n} \times m} \tag{18c}$$

$$\tilde{C} = [C \ 0 \ \dots \ 0] \in \mathfrak{R}_+^{p \times \tilde{n}}, \quad \tilde{D} = D \in \mathfrak{R}_+^{p \times m}, \quad \tilde{n} = (1+h)n$$

To test the practical stability of the positive fractional system (12) the conditions of Theorem 2 can be applied to the system (18).

Theorem 5. The positive fractional system (12) is practically stable if and only if one of the following condition is satisfied

- 1) Eigenvalues $\tilde{z}_k, \quad k = 1, \dots, \tilde{n}$ of the matrix \tilde{A} have moduli less than 1, i.e.

$$|\tilde{z}_k| < 1 \text{ for } k = 1, \dots, \tilde{n} \tag{19}$$
- 2) $\det[I_{\tilde{n}}z - \tilde{A}] \neq 0$ for $|z| \geq 1$

3) $\rho(\tilde{A}) < 1$, where $\rho(\tilde{A})$ is the spectral radius defined by $\rho(\tilde{A}) = \max_{1 \leq k \leq \tilde{n}} \{|\tilde{z}_k|\}$ of the matrix \tilde{A}

4) All coefficients $\tilde{a}_i, i = 0, 1, \dots, \tilde{n} - 1$ of the characteristic polynomial

$$p_{\tilde{A}}(z) = \det[I_{\tilde{n}}(z+1) - \tilde{A}] = z^{\tilde{n}} + \tilde{a}_{\tilde{n}-1}z^{\tilde{n}-1} + \dots + \tilde{a}_1z + \tilde{a}_0 \tag{20}$$

of the matrix $[\tilde{A} - I_{\tilde{n}}]$ are positive

5) All principal minors of the matrix

$$[I_{\tilde{n}} - \tilde{A}] = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1\tilde{n}} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2\tilde{n}} \\ \vdots & \vdots & \dots & \vdots \\ \tilde{a}_{\tilde{n}1} & \tilde{a}_{\tilde{n}2} & \dots & \tilde{a}_{\tilde{n}\tilde{n}} \end{bmatrix} \tag{21a}$$

are positive, i.e.

$$|\tilde{a}_{11}| > 0, \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix} > 0, \dots, \det[I_{\tilde{n}} - \tilde{A}] > 0 \tag{21b}$$

6) There exist strictly positive vectors $\bar{x}_i \in \mathfrak{R}_+^n, i = 0, 1, \dots, h$ satisfying

$$\bar{x}_0 < \bar{x}_1, \bar{x}_1 < \bar{x}_2, \dots, \bar{x}_{h-1} < \bar{x}_h \tag{22a}$$

such that

$$A_\alpha \bar{x}_0 + c_1 \bar{x}_1 + \dots + c_h \bar{x}_h < \bar{x}_0 \tag{22b}$$

Proof is given in [15].

Theorem 6. If the sum of entries of every row of the adjoint matrix $\text{Adj}[I_{\tilde{n}} - \tilde{A}]$ is strictly positive i.e.

$$\text{Adj}[I_{\tilde{n}} - \tilde{A}]1_{\tilde{n}} > 0 \tag{23}$$

then the positive fractional system (12) is practically stable,

where $1_{\tilde{n}} = [1 \ 1 \ \dots \ 1]^T \in \mathfrak{R}_+^{\tilde{n}}, T$ denotes the transpose.

Proof is given in [15].

Theorem 7. The positive fractional system (12) is practically stable only if the positive system

$$x_{k+1} = A_\alpha x_k, \quad k \in Z_+ \tag{24}$$

is asymptotically stable.

Proof is given in [15].

From Theorem 7 we have the following important corollary.

Corollary. The positive fractional system (12) is practically unstable for any finite h if the positive system (24) is asymptotically unstable.

Theorem 8. The positive fractional system (12) is practically unstable if at least one diagonal entry of the matrix A_α is greater than 1.

Proof. The proof follows immediately from Theorems 7 and 3. \square

Theorem 9. The positive discrete-time system with delays

$$x_{i+1} = \sum_{k=0}^h A_k x_{i-k}, \quad i \in Z_+ \tag{25}$$

is asymptotically stable if and only if the positive system with delays

$$x_{i+1} = Ax_i, \quad A = \sum_{k=0}^h A_k \in \mathfrak{R}_+^{n \times n}, \quad i \in Z_+ \quad (26)$$

is asymptotically stable.

The proof is given in [2].

Applying Theorem 9 to the positive fractional system (15) we obtain the following theorem.

Theorem 10. The positive fractional system (15) is practically stable if and only if the positive system

$$x_{i+1} = \bar{A}x_i, \quad \bar{A} = A_\alpha + \sum_{j=1}^h c_j I_n, \quad i \in Z_+ \quad (27)$$

is asymptotically stable.

Example. Check the practical stability of the positive fractional system [15]

$$\Delta^\alpha x_{k+1} = 0.1x_k, \quad k \in Z_+ \quad (28)$$

for $\alpha = 0.5$ and $h = 2$.

Using (14), (16) and (27) we obtain

$$c_1 = \frac{\alpha(1-\alpha)}{2} = \frac{1}{8}, \quad c_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{3!} = \frac{1}{16}, \quad A_\alpha = 0.6$$

and

$$\bar{A} = A_\alpha + \sum_{j=1}^2 c_j I_n = 0.7875 \quad (29)$$

From (29) and Theorem 10 it follows that the positive fractional system (28) is practically stable.

5. CONCLUDING REMARKS

The notion of practical stability has been proposed. It has been shown that: 1) the positive fractional system (15) is unstable for any h if at least one diagonal entry of the matrix A_α is greater than 1; 2) that the positive fractional system (15) is practically stable if and only if the positive system (27) is asymptotically stable. The considerations can be easily extended to two-dimensional positive fractional linear systems.

This work was supported by Ministry of Science and Higher Education in Poland under work No NN514 1939 33.

REFERENCES

1. Busłowicz M., Robust stability of positive discrete-time linear systems with multiple delays with unity rank uncertainty structure or non-negative perturbation matrices, *Bull. Pol. Acad. Techn. Sci.* Vol. 55, No. 1, (2007) 1-5.
2. Busłowicz M., Simple stability conditions for linear positive discrete-time systems with delays, *Bull. Pol. Acad. Techn. Sci.*, Vol. 56, No. 4, (2008) 325-328.
3. Farina L., Rinaldi S., Positive Linear Systems; Theory and Applications, J. Wiley, New York, 2000.
4. Gałkowski K., Kummert A., Fractional polynomials and nD systems. *Proc IEEE Int. Symp. Circuits and Systems*, ISCAS'2005, Kobe, Japan, CD-ROM.
5. Kaczorek T., Positive 1D and 2D Systems, Springer-Verlag, London, (2002).

6. Kaczorek T., Reachability and controllability to zero tests for standard and positive fractional discrete-time systems, *Journal of Automation and System Engineering*, Vol. 42, No. 6-8, (2008) 770-787.
7. Kaczorek T., Reachability and controllability to zero of positive fractional discrete-time systems. *Machine Intelligence and Robotic Control*, Vol. 6, No. 4, (2007) 139-143.
8. Kaczorek T., Reachability and controllability to zero of cone fractional linear systems, *Archives of Control Sciences*, Vol. 17, No. 3, (2007) 357-367.
9. Kaczorek T., Fractional positive continuous-time linear systems and their reachability, *Int. J. Appl. Math. Comput. Sci.*, Vol. 18, No. 2, (2008) 223-228.
10. Kaczorek T., Fractional 2D linear systems. *Journal of Automation, Mobile Robotics and Intelligent Systems*, Vol.2, No.2, (2008) 5-9.
11. Kaczorek T., Positive different orders fractional 2D linear systems. *Acta Mechanica et Automatica*, Vol. 2, No. 2, pp.51-58.
12. Kaczorek T., Choice of the forms of Lyapunov functions for positive 2D Roesser model, *Intern. J. Applied Math. and Comp. Sciences*, Vol. 17, No. 4, (2007) 471-475.
13. Kaczorek T., Asymptotic stability of positive 1D and 2D linear systems, *Recent Advances in Control and Automation*, Academ. Publ. House EXIT 2008, pp.41-52.
14. Kaczorek T., LMI approach to stability of 2D positive systems with delays. *Multidimensional Systems and Signal Processing*, Vol. 20 (2009), 39-54.
15. Kaczorek T., Practical stability of positive fractional discrete-time linear systems, *Bull. Pol. Acad. Techn. Sci.*, Vol. 56, No. 4, 313-318.
16. Klamka J., Positive controllability of positive systems, *Proc. of American Control Conference*, ACC-2002, Anchorage, (CD-ROM).
17. Miller K.S., Ross B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York 1993.
18. Nishimoto K., *Fractional Calculus*. Koriama: Decartess Press, 1984.
19. Oldham K. B., Spanier J., *The Fractional Calculus*. New York: Academic Press, 1974.
20. Ortigueira M. D., Fractional discrete-time linear systems, *Proc. of the IEE-ICASSP 97*, Munich, Germany, IEEE, New York, Vol. 3, (1997), 2241-2244.
21. Ostalczyk P., The non-integer difference of the discrete-time function and its application to the control system synthesis. *Int. J. Syst. Sci.* Vol. 31, No. 12, (2000) 1551-1561.
22. Oustalup A., *Commande CRONE*. Paris, Hermès, 1993.
23. Podlubny I., *Fractional Differential Equations*. San Diego: Academic Press, 1999.
24. Sierociuk D., Dzieliński D., Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation. *Int. J. Appl. Math. Comp. Sci.*, Vol. 16, No. 1, (2006) 129-140.
25. Vinagre M., Feliu V., Modeling and control of dynamic system using fractional calculus: Application to electrochemical processes and flexible structures. *Proc. 41st IEEE Conf. Decision and Control*, Las Vegas, NV, (2002), 214-239.
26. Twardy M., An LMI approach to checking stability of 2D positive systems, *Bull. Pol. Acad. Techn. Sci.* Vol. 55, No. 4, (2007) 379-383.
27. Twardy M., On the alternative stability criteria for positive systems, *Bull. Pol. Acad. Techn. Sci.* Vol. 55, No. 4, (2007) 385-393.
28. Gorecki H. *Analysis and Synthesis of Control Systems with Delay*. WNT Warszawa, 1971 (in Polish)