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EXISTENCE AND DETERMINATION OF THE SET OF METZLER MATRICES FOR GIVEN STABLE POLYNOMIALS

TADEUSZ KACZOREK

Faculty of Electrical Engineering
Białystok Technical University, ul. Wiejska 45D, 15-351 Białystok, Poland
e-mail: kaczorek@isep.pw.edu.pl

The problem of the existence and determination of the set of Metzler matrices for given stable polynomials is formulated and solved. Necessary and sufficient conditions are established for the existence of the set of Metzler matrices for given stable polynomials. A procedure for finding the set of Metzler matrices for given stable polynomials is proposed and illustrated with numerical examples.

Keywords: determination, existence, Metzler matrix, polynomial, stability.

1. Introduction

Determination of the state space equations for a given transfer matrix is a classical problem, called the realization problem, which has been addressed in many papers and books (Farina and Rinaldi, 2000; Benvenuti and Farina, 2004; Kaczorek, 1992; 2009b; 2011d; 2012; Shaker and Dixon, 1977). An overview on the positive realization problem is given by Farina and Rinaldi (2000), Kaczorek (2002), as well as Benvenuti and Farina (2004). The realization problem for positive continuous-time and discrete-time linear systems was considered by Kaczorek (2006a; 2006b; 2011a; 2011b; 2006c; 2004; 2011c) along with the positive realization problem for discrete-time systems with delays (Kaczorek, 2006c; 2004; 2005). Fractional positive linear systems were addressed by Kaczorek (2008c; 2009a; 2011d), together with the realization problem for fractional linear systems (Kaczorek, 2008a) and for positive 2D hybrid systems (Kaczorek, 2008b). A method based on similarity transformation of the standard realization to the discrete positive one was proposed (Kaczorek, 2011c), and conditions for the existence of a positive stable realization with a system Metzler matrix for a transfer function were established (Kaczorek, 2011a). The problem of determination of the set of Metzler matrices for given stable polynomials was formulated and partly solved by Kaczorek (2012).

It is well known (Farina and Rinaldi, 2000; Kaczorek, 2002; 1992) that to find a realization for a given transfer function, first we have to find a state matrix for a given denominator of the transfer function.

In this paper the problem of the existence and determination of the set of Metzler matrices for a given stable polynomial will be established and solved. Necessary and sufficient conditions will be established for the existence of the set of Metzler matrices for a given stable polynomial and a procedure will be proposed for finding the desired set of Metzler matrices.

The paper is organized as follows. In Section 2 some preliminaries concerning positive stable continuous-time linear systems are recalled and the problem formulation is given. The problem solution is presented in Section 3, which consists of four subsections. In Section 3.1 the problem is solved for second-order stable polynomials, and in Section 3.2 and 3.3 for third- and fourth- order stable polynomials. The general case is addressed in Section 3.4. Concluding remarks are given in Section 4.

The following notation will be used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, $\mathbb{R}_+^{n\times m}$ is the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$, M_n is the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n is the $n \times n$ identity matrix.

2. Preliminaries and problem formulation

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 1. (Farina and Rinaldi, 2000; Kaczorek, 2002) The system (1) is called (*internally*) *positive* if $x(t) \in \mathbb{R}^n_+$, $y(t) \in \mathbb{R}^p_+$, $t \geq 0$ for any initial conditions $x(0) = x_0 \in \mathbb{R}^n_+$ and all inputs $u(t) \in \mathbb{R}^m_+$, $t \geq 0$.

Theorem 1. (Farina and Rinaldi, 2000; Kaczorek, 2002) *The system (1) is positive if and only if*

$$A \in M_n, \quad B \in \mathbb{R}^{n\times m}_+, \quad C \in \mathbb{R}^{p\times n}_+, \quad D \in \mathbb{R}^{p\times m}_+. \tag{2}$$

Definition 2. (Farina and Rinaldi, 2000; Kaczorek, 2002) The positive system (1) is called *asymptotically stable* if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for any } x_0 \in \mathbb{R}^n_+. \tag{3}$$

Theorem 2. (Farina and Rinaldi, 2000; Kaczorek, 2002) *The positive system (1) is asymptotically stable if and only if all coefficients of the polynomial*

$$\begin{aligned} p_n(s) &= \det[I_n s - A] \\ &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \end{aligned} \tag{4}$$

are positive, i.e., $a_i > 0$ *for* $i = 0, 1, \ldots, n-1$.

Definition 3. (Kaczorek, 2002) A matrix $P \in \mathbb{R}^{n\times n}_+$ is called the *monomial matrix* (or the generalized permutation matrix) if its every row and its every column contains only one positive entry and its remaining entries are zero.

Lemma 1. (Kaczorek, 2002) *The inverse matrix* A^{-1} *of the monomial matrix A is equal to the transpose matrix in which every nonzero entry is replaced by its inverse.*

Lemma 2. *If* $A_M \in M_n$*, then* $\bar{A}_M = PA_M P^{-1} \in M_n$ *for every monomial matrices* $P \in \mathbb{R}^{n\times n}_+$ *and*

$$\det[I_n s - \bar{A}_M] = \det[I_n s - A_M]. \tag{5}$$

Proof. By Lemma 1, if $P \in \mathbb{R}^{n\times n}_+$, then $P^{-1} \in \mathbb{R}^{n\times n}_+$ and $\bar{A}_M = PA_M P^{-1} \in M_n$ if $A_M \in M_n$. It is easy to check that

$$\begin{aligned} \det[I_n s - \bar{A}_M] &= \det[I_n s - PA_M P^{-1}] \\ &= \det\{P[I_n s - A_M]P^{-1}\} \\ &= \det P \det[I_n s - A_M] \det P^{-1} \\ &= \det[I_n s - A_M] \end{aligned} \tag{6}$$

since $\det P \det P^{-1} = 1$. ■

The problem under consideration can be stated as follows: Given a stable polynomial

$$p_n(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \quad a_i > 0 \tag{7}$$

for $i = 0, 1, \ldots, n-1$, find a class of Metzler matrices $A_M \in M_n$ (if it exists) such that

$$\det[I_n s - A_M] = p_n(s). \tag{8}$$

The following two subproblems will be analyzed.

Subproblem 1. Find a class of stable polynomials (7) for which there exists a class of Metzler matrices $A_M \in M_n$ satisfying the condition (8).

Subproblem 2. Given a stable polynomial of the form (7) for which there exists a class of Metzler matrices $A_M \in M_n$, propose a procedure for computation of the desired class of Metzler matrices.

3. Problem solution

3.1. Second-degree polynomials. In the work of Kaczorek (2012) it was shown that the Metzler matrix

$$A_M = \begin{bmatrix} -a_{11} & a_{12} \\ a_{21} & -a_{22} \end{bmatrix}, \quad a_{i,j} \geq 0 \tag{9}$$

for $i, j = 1, 2$, has only real eigenvalues, and for a given stable polynomial

$$p_2(s) = s^2 + a_1 s + a_0 \tag{10}$$

there exists a set of Metzler matrices (9) with diagonal entries

$$\begin{aligned} a_{11} &= \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4(a_0 + a_{12}a_{21})}\right), \\ a_{22} &= \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4(a_0 + a_{12}a_{21})}\right) \end{aligned}$$

and off-diagonal entries $a_{12} \geq 0$, $a_{21} \geq 0$ satisfying the condition

$$a_1^2 - 4(a_0 + a_{12}a_{21}) \geq 0$$

if and only if

$$a_1^2 \geq 4a_0. \tag{11}$$

Theorem 3. *For a given stable polynomial (10) there exists a set of Metzler matrices* $\bar{A}_M = PA_M P^{-1}$*, where* $P \in \mathbb{R}^{2\times 2}_+$ *is a monomial matrix and matrix* A_M *has one of the following forms:*

$$\begin{aligned} A_{M1} &= \begin{bmatrix} -a & a_1 a - a^2 - a_0 \\ 1 & a - a_1 \end{bmatrix}, \\ A_{M2} &= \begin{bmatrix} -a & 1 \\ a_1 a - a^2 - a_0 & a - a_1 \end{bmatrix}, \\ &0 < a < a_1, \quad a_1 a - a^2 - a_0 \geq 0, \end{aligned} \tag{12}$$

if and only if the condition (11) is met.

Proof. If the matrix A_M has the form (9) for $a_{21} = 1$, then its characteristic polynomial is

$$\begin{aligned}\det[I_2 s - A_M] &= \begin{vmatrix} s + a_{11} & -a_{12} \\ -1 & s + a_{22} \end{vmatrix} \\ &= s^2 + (a_{11} + a_{22})s + a_{11}a_{22} - a_{12} \\ &= s^2 + a_1 s + a_0, \end{aligned} \tag{13a}$$

where

$$a_1 = a_{11} + a_{22}, \quad a_0 = a_{11}a_{22} - a_{12}. \tag{13b}$$

From (13b) for $a_{11} = a$ we have $a_{22} = a_1 - a$ and $a_{12} = a(a_1 - a) - a_0 = a_1 a - a^2 - a_0 \geq 0$. By Lemma 2 the condition (5) is satisfied for any monomial matrix $P \in \mathbb{R}_+^{2\times 2}$. The proof for the matrix A_{M2} is similar. ■

Example 1. Find the set of Metzler matrices (12) corresponding to the stable polynomial

$$p_2(s) = s^2 + 5s + 6. \tag{14}$$

The polynomial (14) satisfies the condition (11) since $a_1^2 = 25$, $4a_0 = 24$ and its zeros are $s_1 = -2$, $s_2 = -3$. The desired set of Metzler matrices corresponding to (14) has the form

$$\bar{A}_{M1} = PA_{M1}P^{-1} \quad \text{or} \quad \bar{A}_{M2} = PA_{M2}P^{-1}, \tag{15a}$$

where

$$A_{M1} = \begin{bmatrix} -a & 5a - a^2 - 6 \\ 1 & a - 5 \end{bmatrix}, \quad A_{M2} = \begin{bmatrix} -a & 1 \\ 5a - a^2 - 6 & a - 5 \end{bmatrix} \tag{15b}$$

for $2 \leq a \leq 3$ and any monomial matrix $P \in \mathbb{R}_+^{2\times 2}$. Choosing the monomial matrix

$$P = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \tag{16}$$

and using (15), we obtain

$$\begin{aligned}\bar{A}_{M1} &= PA_{M1}P^{-1} \\ &= \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} -a & 5a - a^2 - 6 \\ 1 & a - 5 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} a - 5 & \frac{2}{3} \\ \frac{3}{2}(5a - a^2 - 6) & -a \end{bmatrix}, \end{aligned} \tag{17a}$$

$$\begin{aligned}\bar{A}_{M2} &= PA_{M2}P^{-1} \\ &= \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} -a & 1 \\ 5a - a^2 - 6 & a - 5 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} a - 5 & \frac{2}{3}(5a - a^2 - 6) \\ \frac{3}{2} & -a \end{bmatrix} \end{aligned} \tag{17b}$$

for $2 \leq a \leq 3$. Note that the set of diagonal entries of both matrices (17) is the same and

$$\begin{aligned}\operatorname{trace} \bar{A}_{M1} &= \operatorname{trace} \bar{A}_{M2} = \operatorname{trace} A_{M1} \\ &= \operatorname{trace} A_{M2} = -5. \end{aligned}$$

♦

3.2. Third-degree polynomials. In the work of Kaczorek (2012) it was shown that if the stable polynomial

$$\begin{aligned} p_3(s) &= (s + \alpha_1)(s + \alpha_2)(s + \alpha_3) \\ &= s^3 + a_2 s^2 + a_1 s + a_0 \end{aligned}$$

has only real negative zeros $s_1 = -\alpha_1$, $s_2 = -\alpha_2$, $s_3 = -\alpha_3$, then the desired set of Metzler matrices is given by the set of lower or upper triangular matrices with diagonal entries $-a_{i,i}$, $i = 1, 2, 3$ equal to the negative zeros $-\alpha_1, -\alpha_2, -\alpha_3$ and any nonnegative off-diagonal entries.

In what follows it will be assumed that the polynomial $p_3(s)$ has one real zero and a pair of complex conjugate zeros.

Theorem 4. *For the given stable polynomial*

$$p_3(s) = s^3 + a_2 s^2 + a_1 s + a_0, \quad a_k > 0, \quad k = 0, 1, 2, \tag{18}$$

there exists the set of Metzler matrices

$$\bar{A}_{Mk} = PA_{Mk}P^{-1}, \quad k = 1, 2, \dots, 6, \tag{19}$$

if and only if

$$a_2^2 - 3a_1 \geq 0, \tag{20a}$$

$$-2a_2^3 + 9a_1 a_2 - 27a_0 \geq 0, \tag{20b}$$

where $P \in \mathbb{R}_+^{3\times 3}$ is a monomial matrix and matrix A_M has one of the following forms:

$$\begin{aligned} A_{M1} &= \begin{bmatrix} -a_{11} & 1 & a_{13} \\ 0 & -a_{22} & a_{23} \\ 1 & 0 & -a_{33} \end{bmatrix}, \\ A_{M2} &= \begin{bmatrix} -a_{11} & 0 & 1 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 1 & -a_{33} \end{bmatrix}, \\ A_{M3} &= \begin{bmatrix} -a_{11} & a_{12} & 0 \\ 0 & -a_{22} & 1 \\ 1 & a_{32} & -a_{33} \end{bmatrix}, \\ A_{M4} = A_{M1}^T &= \begin{bmatrix} -a_{11} & 0 & 1 \\ 1 & -a_{22} & 0 \\ a_{13} & a_{23} & -a_{33} \end{bmatrix}, \\ A_{M5} = A_{M2}^T &= \begin{bmatrix} -a_{11} & a_{21} & a_{31} \\ 0 & -a_{22} & 1 \\ 1 & 0 & -a_{33} \end{bmatrix}, \\ A_{M6} = A_{M3}^T &= \begin{bmatrix} -a_{11} & 0 & 1 \\ a_{12} & -a_{22} & a_{32} \\ 0 & 1 & -a_{33} \end{bmatrix}, \end{aligned} \tag{21}$$

and T denotes the transpose.

Proof. The characteristic polynomial of A_{M1} has the form

$$\det[I_3 s - A_{M1}] = \begin{vmatrix} s+a_{11} & -1 & -a_{13} \\ 0 & s+a_{22} & -a_{23} \\ -1 & 0 & s+a_{33} \end{vmatrix}$$
$$= (s+a_{11})(s+a_{22})(s+a_{33}) - \begin{vmatrix} -1 & -a_{13} \\ s+a_{22} & -a_{23} \end{vmatrix}$$
$$= s^3 + a_2 s^2 + a_1 s + a_0, \tag{22a}$$

where

$$a_2 = a_{11} + a_{22} + a_{33},$$
$$a_1 = a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_{13},$$
$$a_0 = a_{11}a_{22}a_{33} - a_{22}a_{13} - a_{23}. \tag{22b}$$

From (22b) and (18) we have

$$a_{13} = a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_1 \geq 0, \tag{23a}$$
$$a_{23} = a_{11}a_{22}a_{33} - a_{22}a_{13} - a_0 \geq 0. \tag{23b}$$

By Lemma 3 the functions $a_{11}(a_{22} + a_{33}) + a_{22}a_{33}$ and $a_{11}a_{22}a_{33}$ for $a_{11} + a_{22} + a_{33} = a_2$ reach their maximal values if

$$a_{11} = a_{22} = a_{33} = \frac{a_2}{3} \quad (a_2 \text{ is given}). \tag{24}$$

Substitution of (24) into (23) yields

$$a_{13} = \frac{a_2^2}{3} - a_1 \geq 0, \tag{25a}$$
$$a_{23} = \left(\frac{a_2}{3}\right)^3 - \left(\frac{a_2}{3}\right)\left(\frac{a_2^2}{3} - a_1\right) - a_0$$
$$= -2\left(\frac{a_2}{3}\right)^3 + \frac{a_1 a_2}{3} - a_0 \geq 0, \tag{25b}$$

and these conditions are equivalent to the conditions (20). The proof for the remaining matrices A_{M2} and A_{M3} is similar and proofs for the matrices A_{M4}, A_{M5} and A_{M6} follow immediately from the equations

$$\det[I_3 s - A_{M4}] = \det[I_3 s - A_{M1}],$$
$$\det[I_3 s - A_{M5}] = \det[I_3 s - A_{M2}], \tag{26}$$
$$\det[I_3 s - A_{M6}] = \det[I_3 s - A_{M3}].$$

■

Theorem 5. *Let $s_1 = -\alpha$ and $s_2 = -\alpha_1 + j\beta_1$, $s_2' = -\alpha_1 - j\beta_1$ be the zeros of the polynomial (18). Then the conditions (20) are satisfied if and only if*

$$(\alpha - \alpha_1)^2 \geq 3\beta_1^2 \tag{27}$$

and

$$\alpha_1 \geq \alpha. \tag{28}$$

Proof. Taking into account that

$$p_3(s) = (s+\alpha)(s+\alpha_1+j\beta_1)(s+\alpha_1-j\beta_1)$$
$$= s^3 + a_2 s^2 + a_1 s + a_0, \tag{29}$$

where

$$a_2 = \alpha + 2\alpha_1, \quad a_1 = 2\alpha\alpha_1 + \alpha_1^2 + \beta_1^2,$$
$$a_0 = \alpha(\alpha_1^2 + \beta_1^2), \tag{30}$$

and using (20), we obtain ■

$$a_2^2 - 3a_1 = (\alpha + 2\alpha_1)^2 - 3(2\alpha\alpha_1 + \alpha_1^2 + \beta_1^2)$$
$$= \alpha^2 - 2\alpha\alpha_1 + \alpha_1^2 - 3\beta_1^2$$
$$= (\alpha - \alpha_1)^2 - 3\beta_1^2 \geq 0 \tag{31a}$$

and

$$-2a_2^3 + 9a_1a_2 - 27a_0$$
$$= -2(\alpha + 2\alpha_1)^3$$
$$+ 9(2\alpha\alpha_1 + \alpha_1^2 + \beta_1^2)(\alpha + 2\alpha_1)$$
$$- 27\alpha(\alpha_1^2 + \beta_1^2)$$
$$= 2(\alpha_1 - \alpha)^3 + 18\beta_1^2(\alpha_1 - \alpha) \geq 0. \tag{31b}$$

Therefore, the inequalities (27) and (31a) are equivalent, and the condition (20b) is satisfied if and only if (28) holds.

If the conditions (20) are satisfied, then to find the entries of the matrix A_{M1} of the form given in (21) the following procedure can be used.

Procedure 1.

Step 1. Given a_2, choose a_{11}, a_{22} and a_{33} so that

$$a_{11} + a_{22} + a_{33} = a_2. \tag{32a}$$

In a particular case,

$$a_{11} = a_{22} = a_{33} = \frac{a_2}{3}. \tag{32b}$$

Step 2. Knowing a_1 and using (23a), find

$$a_{13} = a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_1. \tag{33a}$$

In a particular case for (32b), we obtain

$$a_{13} = \frac{a_2^2}{3} - a_1. \tag{33b}$$

Step 3. Knowing a_0 and using (23b), find

$$a_{23} = a_{11}a_{22}a_{33} - a_{22}a_{13} - a_0. \tag{34a}$$

In a particular case for (32b), we obtain

$$a_{23} = \left(\frac{a_2}{3}\right)^3 - \left(\frac{a_2}{3}\right)a_{13} - a_0 \tag{34b}$$

and the desired set of Metzler matrices (19).

Example 2. Find the set of Metzler matrices (19) for the stable polynomial

$$p_3(s) = s^3 + 9s^2 + 25s + 17. \tag{35}$$

The polynomial (35) satisfies the conditions (20) since

$$a_2^2 - 3a_1 = 81 - 75 = 6 > 0,$$

and

$$-2a_2^3 + 9a_1a_2 - 27a_0 = -1458 + 2025 - 459 = 108 > 0$$

and its zeros are $s_1 = -1$, $s_2 = -4 + j$, $s_2' = -4 - j$.

Using Procedure 1, we obtain the following.

Step 1. We choose $a_{11} = 2$, $a_{22} = 3$, $a_{33} = 4$.

Step 2. Using (33a), we obtain

$$a_{13} = a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_1 = 2 \cdot 7 + 12 - 25 = 1. \tag{36}$$

Step 3. Using (34a) and 36, we obtain

$$a_{23} = a_{11}a_{22}a_{33} - a_{22}a_{13} - a_0 = 24 - 3 - 17 = 4. \tag{37}$$

The desired set of Metzler matrices corresponding to the polynomial (35) has the form

$$\begin{aligned} \bar{A}_{M1} &= PA_{M1}P^{-1} \\ &= \begin{bmatrix} 0 & 0 & p_1 \\ p_2 & 0 & 0 \\ 0 & p_3 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 4 \\ 1 & 0 & -4 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & \frac{1}{p_2} & 0 \\ 0 & 0 & \frac{1}{p_3} \\ \frac{1}{p_1} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -4 & \frac{p_1}{p_2} & 0 \\ \frac{p_2}{p_1} & -2 & \frac{p_2}{p_3} \\ \frac{4p_3}{p_1} & 0 & -3 \end{bmatrix} \end{aligned} \tag{38}$$

for any positive p_1, p_2, p_3. ♦

Example 3. Find the set of Metzler matrices (19) for the stable polynomial

$$p_3(s) = s^3 + 10s^2 + 33s + 34. \tag{39}$$

The polynomial satisfies the conditions (20) since $a_2^2 - 3a_1 = 100 - 99 = 1 > 0$ and

$$\begin{aligned} &-2a_2^3 + 9a_1a_2 - 27a_0 \\ &\qquad = -2000 + 2970 - 918 = 52 > 0, \end{aligned}$$

and its zeros are $s_1 = -2$, $s_2 = -4 + j$, $s_2' = -4 - j$. Using Procedure 1 and the particular choice (32b), we obtain the following.

Step 1. From (32b), we have $a_{11} = a_{22} = a_{33} = \frac{10}{3}$.
Step 2. Using (33b), we obtain

$$a_{13} = \frac{a_2^2}{3} - a_1 = \frac{100}{3} - 33 = \frac{1}{3}. \tag{40}$$

Step 3. Using (34b), we have

$$\begin{aligned} a_{23} &= \left(\frac{a_2}{3}\right)^3 - \left(\frac{a_2}{3}\right) a_{13} - a_0 \\ &= \left(\frac{10}{3}\right)^3 - \left(\frac{10}{3}\right)\frac{1}{3} - 34 = \frac{52}{27}. \end{aligned} \tag{41}$$

The desired set of Metzler matrices corresponding to the polynomial (39) has the form

$$\begin{aligned} \bar{A}_{M1} &= PA_{M1}P^{-1} \\ &= \begin{bmatrix} 0 & p_1 & 0 \\ p_2 & 0 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \begin{bmatrix} -\frac{10}{3} & 1 & \frac{1}{3} \\ 0 & -\frac{10}{3} & \frac{52}{27} \\ 1 & 0 & -\frac{10}{3} \end{bmatrix} \\ &\times \begin{bmatrix} 0 & \frac{1}{p_2} & 0 \\ \frac{1}{p_1} & 0 & 0 \\ 0 & 0 & \frac{1}{p_3} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{10}{3} & 0 & \frac{52p_1}{27p_3} \\ \frac{p_2}{p_1} & -\frac{10}{3} & \frac{p_2}{3p_3} \\ 0 & \frac{p_3}{p_2} & -\frac{10}{3} \end{bmatrix} \end{aligned} \tag{42}$$

for any positive p_1, p_2, p_3.

In the above method the set of Metzler matrices (42) depends on three arbitrary positive parameters p_1, p_2, p_3. In the following method, also based on Procedure 1, the set of Metzler matrices corresponding to the polynomial (39) will depends on five parameters.

Using Procedure 1, we obtain the following.

Step 1. We choose

$$a_{11} = p_4, \quad a_{22} = p_5$$

and

$$a_{33} = a_2 - p_4 - p_5.$$

Step 2. From (33a) for $a_1 = 33$, we have

$$\begin{aligned} a_{13} &= a_{11}(a_{22}+a_{33}) + a_{22}a_{33} - a_1 \\ &= p_4(a_2 - p_4) + p_5(a_2 - p_4 - p_5) - a_1 \\ &= 10(p_4+p_5) - p_4p_5 - p_4^2 - p_5^2 - 33. \end{aligned} \tag{43}$$

Step 3. Using (34a) and (43) for $a_0 = 34$, we obtain

$$\begin{aligned} a_{23} &= a_{11}a_{22}a_{33} - a_{22}a_{13} - a_0 \\ &= p_4p_5(10 - p_4 - p_5) - p_5[10(p_4+p_5) \\ &\quad - p_4p_5 - p_4^2 - p_5^2 - 33] - 34 \\ &= p_5^3 - 10p_5^2 + 33p_5 - 34. \end{aligned} \tag{44}$$

In this case the desired set of Metzler matrices corresponding to the polynomial (39) and the same monomial matrix P has the form

$$\begin{aligned} \bar{A}'_{M1} &= PA'_{M1}P^{-1} \\ &= \begin{bmatrix} 0 & p_1 & 0 \\ p_2 & 0 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \begin{bmatrix} -p_4 & 1 & a_{13} \\ 0 & -p_5 & a_{23} \\ 1 & 0 & -10+p_4+p_5 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & 1/p_2 & 0 \\ 1/p_1 & 0 & 0 \\ 0 & 0 & 1/p_3 \end{bmatrix} \\ &= \begin{bmatrix} -p_5 & 0 & a_{23}p_1/p_3 \\ p_2/p_1 & -p_4 & a_{13}p_2/p_3 \\ 0 & p_3/p_2 & -10+p_4+p_5 \end{bmatrix}, \end{aligned} \tag{45}$$

where a_{13} and a_{23} are given by (43) and (44), respectively, and p_1, p_2, p_3 are arbitrary positive parameters and $0 < p_4 + p_5 < 10$.

3.3. Fourth-degree polynomials.

It will be shown that there exists a set of Metzler matrices corresponding to the stable polynomial

$$p_4(s) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0,\ a_k > 0, \quad k = 0,1,2,3, \tag{46}$$

only if the polynomial has at least two real negative zeros. If the polynomial (46) has only real nonnegative zeros, then the desired set of Metzler matrices is given by the set of lower or upper triangular matrices with diagonal entries equal to the negative zeros and any nonnegative off-diagonal entries (Kaczorek, 2012). In what follows it will be assumed that the polynomial (46) has a pair of complex conjugate zeros.

Theorem 6. *For a given stable polynomial (46), the set of Metzler matrices exists and*

$$\bar{A}_{Mk} = PA_{Mk}P^{-1}, \quad k = 1,2,\dots,8, \tag{47}$$

if and only if

$$3a_3^2 - 8a_2 \ge 0, \tag{48a}$$

$$-a_3^3 + 4a_2a_3 - 8a_1 \ge 0, \tag{48b}$$

$$3a_3^4 - 16a_2a_3^2 + 64a_1a_3 - 256a_0 \ge 0, \tag{48c}$$

where $P \in \mathbb{R}_+^{4\times4}$ is a monomial matrix and matrix A_M has one of the following forms:

$$\begin{aligned} A_{M1} &= \begin{bmatrix} -a_{11} & 1 & 0 & a_{14} \\ 0 & -a_{22} & 1 & a_{24} \\ 0 & 0 & -a_{33} & a_{34} \\ 1 & 0 & 0 & -a_{44} \end{bmatrix}, \\ A_{M2} &= \begin{bmatrix} -a_{11} & 0 & 0 & 1 \\ a_{21} & -a_{22} & 0 & 0 \\ a_{31} & 1 & -a_{33} & 0 \\ a_{41} & 0 & 1 & -a_{44} \end{bmatrix}, \\ A_{M3} &= \begin{bmatrix} -a_{11} & a_{12} & 1 & 0 \\ 1 & -a_{22} & 0 & 0 \\ 0 & a_{32} & -a_{33} & 1 \\ 0 & a_{42} & 0 & -a_{44} \end{bmatrix}, \\ A_{M4} &= \begin{bmatrix} -a_{11} & 1 & a_{13} & 0 \\ 0 & -a_{22} & a_{23} & 1 \\ 1 & 0 & -a_{33} & 0 \\ 0 & 0 & a_{43} & -a_{44} \end{bmatrix}, \end{aligned} \tag{49}$$

$$A_{M5} = A_{M1}^T, \quad A_{M6} = A_{M2}^T, \quad A_{M7} = A_{M3}^T, \quad A_{M8} = A_{M4}^T.$$

Proof. The characteristic polynomial of A_{M1} has the form

$$\begin{aligned} &\det[I_4s - A_{M1}] \\ &= \begin{vmatrix} s+a_{11} & -1 & 0 & -a_{14} \\ 0 & s+a_{22} & -1 & -a_{24} \\ 0 & 0 & s+a_{33} & -a_{34} \\ -1 & 0 & 0 & s+a_{44} \end{vmatrix} \\ &= (s+a_{11})(s+a_{22})(s+a_{33})(s+a_{44}) \\ &\quad + \begin{vmatrix} -1 & 0 & -a_{14} \\ s+a_{22} & -1 & -a_{24} \\ 0 & s+a_{33} & -a_{34} \end{vmatrix} \\ &= s^4 + a_3s^3 + a_2s^2 + a_1s + a_0, \end{aligned} \tag{50a}$$

where

$$\begin{aligned} a_3 &= a_{11} + a_{22} + a_{33} + a_{44}, \\ a_2 &= a_{11}(a_{22}+a_{33}+a_{44}) + a_{22}(a_{33}+a_{44}) \\ &\quad + a_{33}a_{44} - a_{14}, \\ a_1 &= (a_{11}+a_{22})a_{33}a_{44} + (a_{33}+a_{44})a_{11}a_{22} \\ &\quad - a_{14}(a_{22}+a_{33}) - a_{24}, \\ a_0 &= a_{11}a_{22}a_{33}a_{44} - a_{14}a_{22}a_{33} - a_{24}a_{33} - a_{34}. \end{aligned} \tag{50b}$$

From (50b) and (47) we have

$$\begin{aligned} a_{14} &= a_{11}(a_{22}+a_{33}+a_{44}) + a_{22}(a_{33}+a_{44}) \\ &\quad + a_{33}a_{44} - a_2 \ge 0, \qquad (51a) \\ a_{24} &= (a_{11}+a_{22})a_{33}a_{44} + (a_{33}+a_{44})a_{11}a_{22} \\ &\quad - a_{14}(a_{22}+a_{33}) - a_1 \ge 0, \qquad (51b) \\ a_{34} &= a_{11}a_{22}a_{33}a_{44} - a_{14}a_{22}a_{33} - a_{24}a_{33} \\ &\quad - a_0 \ge 0. \qquad (51c) \end{aligned}$$

The functions $a_{11}(a_{22}+a_{33}+a_{44}) + a_{22}(a_{33}+a_{44}) + a_{33}a_{44}$, $(a_{11}+a_{22})a_{33}a_{44} + (a_{33}+a_{44})a_{11}a_{22}$ and $a_{11}a_{22}a_{33}a_{44}$ for $a_{11}+a_{22}+a_{33}+a_{44} = a_3$ reach their maximal values if

$$a_{11} = a_{22} = a_{33} = a_{44} = \frac{a_3}{4} \quad (a_3 \text{ is given}). \qquad (52)$$

Substitution of (52) into (51) yields

$$\begin{aligned} a_{14} &= 6\left(\frac{a_3}{4}\right)^2 - a_2 = \frac{3}{8}a_3^2 - a_2 \ge 0, \qquad (53a) \\ a_{24} &= 4\left(\frac{a_3}{4}\right)^3 - 2\left(\frac{a_3}{4}\right)\left(\frac{3}{8}a_3^2 - a_2\right) - a_1 \\ &= -\frac{a_3^3}{8} + \frac{a_2a_3}{2} - a_1 \ge 0, \qquad (53b) \\ a_{34} &= \left(\frac{a_3}{4}\right)^4 - \left(\frac{a_3}{4}\right)^2\left(\frac{3}{8}a_3^2 - a_2\right) \\ &\quad - \left(\frac{a_3}{4}\right)\left(-\frac{a_3^3}{8} + \frac{a_2a_3}{2} - a_1\right) - a_0 \\ &= \frac{3a_3^4}{256} - \frac{a_2a_3^2}{16} + \frac{a_1a_3}{4} - a_0 \ge 0. \qquad (53c) \end{aligned}$$

The conditions (53) are equivalent to the conditions (48). The remaining part of the proof is similar to the proof of Theorem 4. ■

Theorem 7. *For a given stable polynomial (46) the set of Metzler matrices exists only if the polynomial has at least two real nonnegative zeros.*

Proof. Let us assume that the polynomial has two pairs of complex zeros. Then

$$\begin{aligned} p_4(s) &= (s+\alpha_1+j\beta_1)(s+\alpha_1-j\beta_1) \\ &\quad \times (s+\alpha_2+j\beta_2)(s+\alpha_2-j\beta_2) \\ &= s^4 + a_3s^3 + a_2s^2 + a_1s + a_0, \qquad (54) \end{aligned}$$

where

$$\begin{aligned} a_3 &= 2(\alpha_1+\alpha_2), \\ a_2 &= 4\alpha_1\alpha_2 + \alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2, \\ a_1 &= 2\alpha_2(\alpha_1^2+\beta_1^2) + 2\alpha_1(\alpha_2^2+\beta_2^2), \\ a_0 &= (\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2). \qquad (55) \end{aligned}$$

In this case, using (48) and (55), we obtain

$$\begin{aligned} &3a_3^2 - 8a_2 \\ &\quad = 12(\alpha_1+\alpha_2)^2 - 8(\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2+4\alpha_1\alpha_2) \\ &\quad = 4(\alpha_1+\alpha_2)^2 - 8(\beta_1^2+\beta_2^2) \\ &\quad = 4[(\alpha_1+\alpha_2)^2 - 2(\beta_1^2+\beta_2^2)] \ge 0 \qquad (56a) \\ &\quad\quad - a_3^3 - 4_2a_3 - 8a_1 \\ &\quad = 8[\alpha_1(\beta_1^2-\beta_2^2) + \alpha_2(\beta_2^2-\beta_1^2)] \\ &\quad = 8(\alpha_1-\alpha_2)(\beta_1^2-\beta_2^2) \ge 0, \qquad (56b) \\ &3a_3^4 - 16a_2a_3^2 + 64a_1a_3 - 256a_0 \\ &\quad = -16(\alpha_1+\alpha_2)^4 - 64(\alpha_1+\alpha_2)^2(\beta_1^2+\beta_2^2) \\ &\quad\quad - 256\beta_1^2\beta_2^2 \\ &\quad = -16[(\alpha_1+\alpha_2)^4 + 4(\alpha_1+\alpha_2)^2(\beta_1^2+\beta_2^2) \\ &\quad\quad + 16\beta_1^2\beta_2^2] \ge 0. \qquad (56c) \end{aligned}$$

From (56c) it follows that the condition cannot be satisfied for two pairs of complex conjugate zeros, and by Theorem 6 there is no Metzler matrix corresponding to the stable polynomial (54). ■

The following example shows that a set of Metzler matrices (47) for a given stable polynomial does not exist, but there may exist a set of Metzler matrices of forms different from (47) corresponding to the stable polynomial.

Example 4. Find the set of Metzler matrices for the stable polynomial

$$p_4(s) = s^4 + 10s^3 + 34s^2 + 42s + 17. \qquad (57)$$

The polynomial does not satisfy the conditions (48) since $3a_3^2 - 8a_2 = 28 > 0$ and $-a_3^3 + 4a_2a_3 - 8a_1 = 24 > 0$ and $3a_3^4 - 16a_2a_3^2 + 64a_1a_3 - 256a_0 = -1872 < 0$. By Theorem 6 there is no set of Metzler matrices of the form (47). It will be shown that there exists another set of Metzler matrices corresponding to the polynomial (57). Note that the polynomial (57) can be decomposed into the following stable polynomials:

$$p_1(s) = s + 1, \quad p_3(s) = s^3 + 9s^2 + 25s + 17 \qquad (58)$$

since $p_4(s) = p_1(s)p_3(s)$. To the first polynomial $p_1(s)$ corresponds the matrix $A_{M1} = [-1]$ and to the second polynomial $p_3(s)$ the Metzler matrix (Kaczorek, 2012),

$$A_{M3} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 4 \\ 1 & 0 & -4 \end{bmatrix}. \qquad (59)$$

Therefore, the desired Metzler matrix corresponding to the polynomial (57) has the form

$$A_M = \begin{bmatrix} A_{M1} & 0 \\ 0 & A_{M3} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 4 \\ 0 & 1 & 0 & -4 \end{bmatrix}, \qquad (60)$$

and the desired Metzler matrices is given by

$$\bar{A}_M = PA_MP^{-1} \tag{61}$$

for any monomial matrix $P \in \mathbb{R}_+^{4\times 4}$. ♦

Therefore, we have the following important corollary.

Corollary 1. *If there does not exist a set of Metzler matrices of the form (47), there may exist a set of Metzler matrices of other forms corresponding to the given stable polynomials.*

3.4. General case: *n*-th degree polynomials.

If the polynomial

$$p_n(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,$$
$$a_k > 0, \quad k = 0, 1, \ldots, n-1 \tag{62}$$

has only negative zeros, then the desired set of Metzler matrices is given by the set of lower or upper triangular matrices with diagonal entries equal to the negative zeros and any nonnegative off-diagonal entries (Kaczorek, 2012). It will be assumed that the polynomial (62) has at least one pair of complex conjugate zeros.

Theorem 8. *For the given stable polynomial (62) there exists the set of Metzler matrices*

$$\bar{A}_{Mk} = PA_{Mk}P^{-1}, \quad k = 1, 2, \ldots, 2n, \tag{63}$$

if and only if

$$C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2} \geq 0,$$

$$C_3^n\left(\frac{a_{n-1}}{n}\right)^3 - \left[C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right] \times C_1^{n-2}\left(\frac{a_{n-1}}{n}\right) - a_{n-3} \geq 0,$$

$$\vdots$$

$$C_n^n\left(\frac{a_{n-1}}{n}\right)^n - \left[C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right] \times C_1^{n-2}\left(\frac{a_{n-1}}{n}\right)^{n-2} - \cdots - C_1^1\left(\frac{a_{n-1}}{n}\right) - a_0 \geq 0, \tag{64}$$

where $C_k^n = \binom{n}{k}$, $P \in \mathbb{R}_+^{n\times n}$ is a monomial matrix and matrix A_M has one of the following forms:

$$A_{M1} = \begin{bmatrix} -a_{11} & 1 & 0 & \ldots & 0 & a_{1,n} \\ 0 & -a_{22} & 1 & \ldots & 0 & a_{2,n} \\ 0 & 0 & -a_{33} & \ldots & 0 & a_{3,n} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_{n-2,n} \\ 0 & 0 & 0 & \ldots & -a_{n-1,n-1} & a_{n-1,n} \\ 1 & 0 & 0 & \ldots & 0 & -a_{n,n} \end{bmatrix},$$

$\ldots,$

$$A_{Mn} = \begin{bmatrix} -a_{11} & 0 & 0 & \ldots & 0 & 1 \\ a_{21} & -a_{22} & 0 & \ldots & 0 & 0 \\ a_{31} & 1 & -a_{33} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{n-2,1} & 0 & 0 & \ldots & 0 & 0 \\ a_{n-1,1} & 0 & 0 & \ldots & -a_{n-1,n-1} & 0 \\ a_{n,1} & 0 & 0 & \ldots & 1 & -a_{n,n} \end{bmatrix},$$

$$A_{Mn+1} = A_{M1}^T, \ldots, A_{M2n} = A_{Mn}^T. \tag{65}$$

Proof. The characteristic polynomial of A_{M1} has the form (66).

From (66b) and (62) we have

$$\begin{aligned} a_{1,n} &= a_{11}(a_{22} + a_{33} + \cdots + a_{n,n}) \\ &\quad + a_{22}(a_{33} + a_{44} + \cdots + a_{n,n}) + \ldots \\ &\quad + a_{n-2,n-2}(a_{n-1,n-1} + a_{n,n}) \\ &\quad + a_{n-1,n-1}a_{n,n} - a_{n-2} \geq 0, \\ &\vdots \\ a_{n-2,n} &= a_{11}a_{22}a_{33}\ldots a_{n-1,n-1} \\ &\quad + a_{11}a_{22}\ldots a_{n-2,n-2}a_{n,n} \\ &\quad + a_{22}a_{33}\ldots a_{n,n} - a_{1,n}(a_{22}a_{33}\ldots a_{n-2,n-2} \\ &\quad + \cdots + a_{33}a_{44}\ldots a_{n-1,n-1}) \\ &\quad - a_{2,n}(a_{33}a_{44}\ldots a_{n-2,n-2} + \ldots \\ &\quad + a_{44}a_{55}\ldots a_{n-1,n-1}) - \cdots - a_{n-3,n}a_{n-2,n-2} \\ &\quad - a_1 \geq 0, \\ a_{n-1,n} &= a_{11}a_{22}\ldots a_{n,n} - a_{1,n}a_{22}\ldots a_{n-1,n-1} \\ &\quad - a_{2,n}a_{33}\ldots a_{n-1,n-1} - a_{n-2,n}a_{n-1,n-1} \\ &\quad - a_0 \geq 0. \end{aligned} \tag{67}$$

The functions $a_{11}(a_{22}+a_{33}+\cdots+a_{n,n})+a_{22}(a_{33}+a_{44}+\cdots+a_{n,n})+\cdots+a_{n-2,n-2}(a_{n-1,n-1}+a_{n,n})$, $\ldots$, $a_{11}a_{22}\ldots a_{n,n}$ for $a_{11}+a_{22}+\cdots+a_{n,n} = a_{n-1}$ (given) reach their maximal values if (cf. Appendix)

$$a_{11} = a_{22} = \cdots = a_{n,n} = \frac{a_{n-1}}{n}. \tag{68}$$

$$
\begin{aligned}
&\det[I_n s - A_{M1}] \\
&= \begin{vmatrix} s+a_{11} & -1 & 0 & \dots & 0 & -a_{1,n} \\ 0 & s+a_{22} & -1 & \dots & 0 & -a_{2,n} \\ 0 & 0 & s+a_{33} & \dots & 0 & -a_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & -a_{n-2,n} \\ 0 & 0 & 0 & \dots & s+a_{n-1,n-1} & -a_{n-1,n} \\ -1 & 0 & 0 & \dots & 0 & s+a_{n,n} \end{vmatrix} \\
&= (s+a_{11})(s+a_{22})\dots(s+a_{n,n}) + (-1)^{n+2} \begin{vmatrix} -1 & 0 & \dots & 0 & -a_{1,n} \\ s+a_{22} & -1 & \dots & 0 & -a_{2,n} \\ 0 & s+a_{33} & \dots & 0 & -a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & -a_{n-2,n} \\ 0 & 0 & \dots & s+a_{n-1,n-1} & -a_{n-1,n} \end{vmatrix} \\
&= (s+a_{11})(s+a_{22})\dots(s+a_{n,n}) - a_{1,n}(s+a_{22})(s+a_{33})\dots(s+a_{n-1,n-1}) \\
&\quad - a_{2,n}(s+a_{33})(s+a_{44})\dots(s+a_{n-1,n-1}) - \dots - a_{n-2,n}(s+a_{n-1,n-1}) - a_{n-1,n} \\
&= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0,
\end{aligned}
\tag{66a}
$$

where

$$
\begin{aligned}
a_{n-1} &= a_{11} + a_{22} + \dots + a_{n,n}, \\
a_{n-2} &= a_{11}(a_{22} + a_{33} + \dots + a_{n,n}) + a_{22}(a_{33} + a_{44} + \dots + a_{n,n}) + \dots + a_{n-2,n-2}(a_{n-1,n-1} + a_{n,n}) \\
&\quad + a_{n-1,n-1}a_{n,n} - a_{1,n}, \\
&\vdots \\
a_1 &= a_{11}a_{22}a_{33}\dots a_{n-1,n-1} + a_{11}a_{22}\dots a_{n-2,n-2}a_{n,n} + a_{22}a_{33}\dots a_{n,n} - a_{1,n}(a_{22}a_{33}\dots a_{n-2,n-2} \\
&\quad + \dots + a_{33}a_{44}\dots a_{n-1,n-1}) - a_{2,n}(a_{33}a_{44}\dots a_{n-2,n-2} + \dots + a_{44}a_{55}\dots a_{n-1,n-1}) \\
&\quad - \dots - a_{n-3,n}a_{n-2,n-2} - a_{n-2,n}, \\
a_0 &= a_{11}a_{22}\dots a_{n,n} - a_{1,n}a_{22}\dots a_{n-1,n-1} - a_{2,n}a_{33}\dots a_{n-1,n-1} - a_{n-2,n}a_{n-1,n-1} - a_{n-1,n}.
\end{aligned}
\tag{66b}
$$

Substitution of (68) into (67) yields

$$
\begin{aligned}
a_{1,n} &= C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2} \geq 0, \\
a_{2,n} &= C_3^n\left(\frac{a_{n-1}}{n}\right)^3 - \left[C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right] \\
&\quad \times C_1^{n-2}\left(\frac{a_{n-1}}{n}\right) - a_{n-3} \geq 0, \\
&\vdots \\
a_{n-1,n} &= C_n^n\left(\frac{a_{n-1}}{n}\right)^n - \left[C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right] \\
&\quad \times C_1^{n-2}\left(\frac{a_{n-1}}{n}\right)^{n-2} - \dots - C_1^1\left(\frac{a_{n-1}}{n}\right) \\
&\quad - a_0 \geq 0.
\end{aligned}
\tag{69}
$$

The conditions (69) are equivalent to the conditions (64). The remaining part of the proof is similar to the proof of Theorem 4. ■

4. Concluding remarks

The problem of the existence and determination of the set of Metzler matrices for given stable polynomials has been formulated and solved. Necessary and sufficient conditions for the existence of the set of Metzler matrices for a given second, third, fourth and *n*-th-order stable polynomial have been established. A procedure for finding the set of Metzler matrices for given stable polynomials has been proposed and illustrated with numerical examples. It has been shown that if there does not exist a set of Metzler matrices of the form (47), then there may exist a set of Metzler matrices of another form corresponding to a given polynomial (Example 4). The presented approach for positive continuous-time linear systems can be extended to positive discrete-time linear systems. The results of the paper will be used for the computation of positive realizations of a given transfer matrix.

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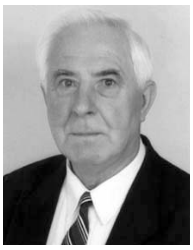

Tadeusz Kaczorek received the M.Sc., Ph.D. and D.Sc. degrees in electrical engineering from the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the years 1968–69 he was the dean of the Electrical Engineering Faculty, and in the period of 1970–73 he was a deputy rector of the Warsaw University of Technology. In 1971 he became a professor and in 1974 a full professor at the same university. Since 2003 he has been a professor at Białystok Technical University. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In the years 1988–1991 he was the director of the Research Centre of the Polish Academy of Sciences in Rome. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He has been granted honorary doctorates by nine universities. His research interests cover systems theory, especially singular multidimensional systems, positive multidimensional systems, singular positive 1D and 2D systems as well as positive fractional 1D and 2D systems. He initiated research in the field of singular 2D and positive 2D systems. He has published 24 books (six in English) and over 1000 scientific papers. He has also supervised 69 Ph.D. theses. He is the editor-in-chief of the *Bulletin of the Polish Academy of Sciences: Technical Sciences* and a member of editorial boards of ten international journals.

Appendix

Lemma 3. *Let*

$$x_1 + x_2 + x_3 = c(c - -a \textit{ given constant}). \tag{70}$$

Then the functions

$$f_1 = f_1(x_1, x_2, x_3) = x_1(x_2 + x_3) + x_2x_3, \tag{71a}$$

$$f_2 = f_2(x_1, x_2, x_3) = x_1x_2x_3 \tag{71b}$$

reach their maximal values for

$$x_1 = x_2 = x_3 = \frac{c}{3}. \tag{72}$$

Proof. From (70) we have

$$x_3 = c - x_1 - x_2. \tag{73}$$

Substitution of (73) into (71) yields

$$\begin{aligned} f_1 &= x_1x_2 + (x_1 + x_2)(c - x_1 - x_2) \\ &= c(x_1 + x_2) - x_1^2 - x_1x_2 - x_2^2, \end{aligned} \tag{74a}$$

$$f_2 = x_1x_2(c - x_1 - x_2) = cx_1x_2 - x_1^2x_2 - x_1x_2^2. \tag{74b}$$

The necessary conditions for the extremum of (74) are

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= c - 2x_1 - x_2 = 0, \\ \frac{\partial f_1}{\partial x_2} &= c - x_1 - 2x_2 = 0 \end{aligned} \tag{75a}$$

and

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} &= cx_2 - 2x_1x_2 - x_2^2 = 0, \\ \frac{\partial f_2}{\partial x_2} &= cx_1 - x_1^2 - 2x_1x_2 = 0. \end{aligned} \tag{75b}$$

From (75) we have

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c \\ c \end{bmatrix} \tag{76}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} c \\ c \end{bmatrix} = \frac{c}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{77}$$

Substitution of (77) into (73) yields $x_3 = c/3$. The proof of sufficiency is trivial. ∎

Lemma 3 can be easily extended for $n > 3$.

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