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# ERGODIC THEORY APPROACH TO CHAOS: REMARKS AND COMPUTATIONAL ASPECTS

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We discuss basic notions of the ergodic theory approach to chaos. Based on simple examples we show some characteristic features of ergodic and mixing behaviour. Then we investigate an infinite dimensional model (delay differential equation) of erythropoiesis (red blood cell production process) formulated by Lasota. We show its computational analysis on the previously presented theory and examples. Our calculations suggest that the infinite dimensional model considered possesses an attractor of a nonsimple structure, supporting an invariant mixing measure. This observation verifies Lasota's conjecture concerning nontrivial ergodic properties of the model.

**Keywords:** ergodic theory, chaos, invariant measures, attractors, delay differential equations.

## 1. Introduction

In the literature concerning dynamical systems we can find many definitions of chaos in various approaches (Rudnicki, 2004; Devaney, 1987; Bronsztejn *et al.*, 2004). Our central issue here will be the ergodic theory approach. Ergodic theory in general has its origin in physical systems of a large number of particles, where microscopic chaos leads to macroscopic (statistical) regularity. As the beginning of ergodic theory, the moment when Boltzmann formulated his famous *ergodic hypothesis*, in 1868 (see, e.g., Nadzieja, 1996; Górnicki, 2001) or in 1871 (Lebowitz and Penrose, 1973), can probably be considered. For more information about the ergodic hypothesis, consult also the works of Birkhoff and Koopman (1932) as well as Dorfman (2001).

## 2. Ergodic theory and chaos: Basic facts

One of the most fundamental notions in ergodic theory is that of *invariant measure* (see Lasota and Mackey, 1994; Fomin *et al.*, 1987; Bronsztejn *et al.*, 2004; Rudnicki, 2004; Dawidowicz, 2007), which is a consequence of Liouville's theorem (see, e.g., Szlenk, 1982; Landau and Lifszyc, 2007; Arnold, 1989; Nadzieja, 1996; Dorfman, 2001). Transformations (or flows) with an invariant measure display three main levels of irregular behaviour, i.e., (ranging from the lowest to the highest) *ergodicity*, *mixing* and *exactness*. Between ergodicity and mixing we can also distinguish *light mixing*, *mild mixing* and *weak mixing* (Lasota and Mackey, 1994; Silva, 2010) and, on the level similar to exactness, the type of *K-flows* (or *K-property*, *K-automorphism*) (cf. Rudnicki 1985a; 1985b; 2004; Lasota and Mackey,1994). In this article we will consider only ergodicity and mixing. First we formalize these notions and show some simple examples of ergodic and mixing transformations. Then in Section 3. we analyze an infinite dimensional system which additionally has interesting medical (hematological) interpretations.

By $\{S_t\}_{t\geq 0}$ we denote a *semidynamical system* or a *semiflow* on the metric space $X$, i.e.,

(i) $S_0(x) = x$ for all $x \in X$;

(ii) $S_t(S_{t'}(x)) = S_{t+t'}(x)$ for all $x \in X$, and $t, t' \in \mathbb{R}^+$;

(iii) $S\colon X \times \mathbb{R}^+ \to X$ is a continuous function of $(t, x)$.

By a measure on $X$ we mean any probability measure defined on the $\sigma$-algebra $\mathcal{B}(X)$ of Borel subsets of $X$. A measure $\mu$ is called *invariant* under a semiflow $\{S_t\}_{t\geq 0}$, if $\mu(A) = \mu(S_t^{-1}(A))$ for each $t \geq 0$ and each $A \in \mathcal{B}$.

**2.1. Ergodicity.** A Borel set $A$ is called invariant with respect to the semiflow $\{S_t\}_{t\geq 0}$ if $S_t^{-1}(A) = A$ for all $t \geq 0$. We now denote by $(S, \mu)$ a semiflow $\{S_t\}_{t\geq 0}$ with an invariant measure $\mu$. The semiflow $(S, \mu)$ is *ergodic* (we say also that the measure is ergodic) if the measure $\mu(A)$ of any invariant set $A$ equals 0 or 1. Let us now consider two simple examples.

**Example 1.** Let $S\colon [0, 2\pi) \to [0, 2\pi)$ be a transformation generating rotation through an angle $\phi$ on a circle with unit radius (see Lasota and Mackey, 1994; Bronsztejn *et al.*, 2004; Devaney, 1987; Dorfman, 2001):

$$S(x) = x + \phi \qquad (\mathrm{mod}\ 2\pi). \tag{1}$$

If $\phi/2\pi$ is rational, we can find invariant sets which have measure different from 0 or 1, and thus $S$ is not ergodic. However, if $\phi/2\pi$ is irrational, then $S$ is ergodic (for a proof, see the work of Lasota and Mackey (1994, p. 75) or Devaney (1987, p. 21)). If we take, e.g., $\phi = \sqrt{2}$ and pick an arbitrary point on the circle, we can observe that successive iterations of this point under the action of $S$ will densely fill the whole available space (circle) (see Fig. 1). ♦

**Example 2.** To understand better the typical features of ergodic behaviour, let us consider the following transformation (see Lasota and Mackey, 1994, p. 68):

$$S(x, y) = (\sqrt{2} + x, \sqrt{3} + y) \qquad (\mathrm{mod}\ 1). \tag{2}$$

This is an extension of the rotational transformation (1) on the space $[0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$. In Fig. 2 we can observe the result of the action of $S$ on the ensemble of $10^3$ points distributed randomly in the area $[0, 0.1] \times [0, 0.1]$. The transformation (2) shifts the initial area and does not spread the points over the space. When we measure the Euclidean distance during iterations between two arbitrarily chosen close points, we notice that it is constant (Fig. 2(d)). Thus the popular criterion of chaos, i.e., sensitivity to initial conditions, is not a property of ergodic transformations. Their property is the dense trajectory (we formalize this fact in the last paragraph of this section). ♦

One of the most important theorems in ergodic theory is the *Birkhoff individual ergodic theorem* (Birkhoff, 1931a; 1931b; Birkhoff and Koopman, 1932; Lasota and Mackey, 1994, Fomin *et al.*, 1987; Szlenk 1982; Dawidowicz, 2007; Nadzieja, 1996; Gornicki, 2001; Dorfman, 2001). Here we cite a popular extension of this theorem (see Lasota and Mackey, 1994, p. 64; Fomin *et al.*, 1987, p. 46). Recall that by $(S, \mu)$ we denote a semiflow $\{S_t\}_{t\geq 0}$ with an invariant measure $\mu$.

**Theorem 1.** (Extension of the Birkhoff theorem) *Let $(S, \mu)$ be ergodic. Then, for each $\mu$-integrable function $f\colon X \to \mathbb{R}$, the mean of $f$ along the trajectory of $S$ is equal almost everywhere to the mean of $f$ over the space $X$, that is,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(S_t(x))\,\mathrm{d}t = \frac{1}{\mu(X)} \int_X f(x)\,\mu(\mathrm{d}x), \tag{3}$$

*$\mu$-almost everywhere.*

If we substitute $f = \mathbf{1}_A$ in Eqn. (3) ($\mathbf{1}_A$ is the characteristic function of $A$) (see Lasota and Mackey, 1994; Rudnicki, 2004; Dawidowicz, 2007), then the left-hand

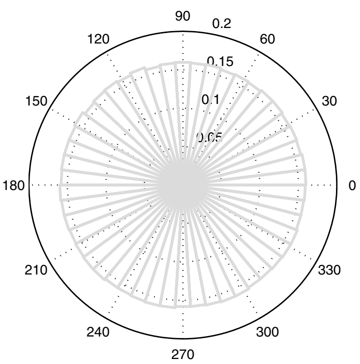



Fig. 1. Normalized (to the probability density function) round histogram (bars inside the circle) showing that a single point under the action of the ergodic transformation (1) with $\phi = \sqrt{2}$ fills densely the whole circle.

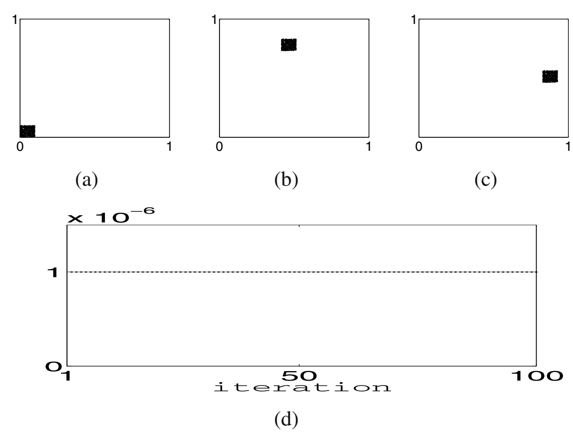



Fig. 2. Iterations of the ergodic transformation (2) acting on an ensemble of $10^3$ points randomly distributed in $[0, 0.1] \times [0, 0.1]$: 1st iteration (a), 2nd iteration (b), 3rd iteration (c), Euclidean metric between two arbitrarily chosen close points from the ensemble (d).

side of (3) is the mean time of visiting the set $A$ and the right-hand side is $\mu(A)$, and this corresponds to ergodicity in the sense of Boltzmann, which roughly speaking is the mean time that a particle of a physical system spends in some region and it is proportional to its natural probabilistic measure (Dawidowicz, 2007; Dorfman, 2001; Nadzieja, 1996; Górnicki, 2001; Birkhoff and Koopman, 1932; Lebowitz and Penrose, 1973)

We can see that ergodic behaviour in the "pure" form does not need to be very irregular and unpredictable. In fact, an invariant and ergodic measure should have some additional properties to be interesting from the point of view of dynamics. Briefly speaking, it should be nontrivial—for example, we intuitively understand that to have interesting dynamics the measure should not be concentrated on a single point. According to our knowledge, two approaches to this problem appear in the literature. In the main ideas, both seem to be similar, but in the literature exist separately. One is the theory of Prodi (1960) (and Foias (1973)), which says that stationary turbulence occurs when the flow admits nontrivial invariant ergodic measure. This theory was strongly developed by Lasota (1979; 1981) (see also Lasota and Yorke, 1977; Lasota and Myjak, 2002; Lasota and Szarek, 2004) and further by Rudnicki (1985a; 1988; 2009) (see also Myjak and Rudnicki, 2002) as well as Dawidowicz (1992a; 1992b) (see also Dawidowicz *et al.*, 2007). Another one uses the notion of *SRB* (*Sinai, Ruelle, Bowen*) *measures* (see, e.g., Bronsztejn *et al.*, 2004; Dorfman, 2001; Taylor, 2004; Tucker, 1999). Roughly speaking, both the approaches say that to have interesting dynamics the support of the measure should be possibly a large set.

Let us now assume that $X$ is a separable metric space and $\mu$ is a probability Borel measure on $X$ such that $\operatorname{supp} \mu = X$. We can state that (see Rudnicki, 2004, p. 727, Proposition 1), if a semiflow $(S, \mu)$ is ergodic, then for $\mu$-almost all $x$ the trajectory $S_t(x),\ t \geq 0$ is dense.

### 2.2. Mixing.

Now we will consider the notion of *mixing*, which exhibits a higher level of irregular behaviour than ergodicity. The literature says that the concept of a mixing system was introduced by J.W. Gibbs (see, e.g., Dorfman, 2001, p. 18, 65). A semiflow $(S, \mu)$ is mixing (see, e.g., Lasota and Mackey, 1994; Rudnicki, 2004; Bronsztejn *et al.*, 2004) if

$$\lim_{t\to\infty} \mu(A \cap S_t^{-1}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{B}. \tag{4}$$

This means that the fraction of points which at $t = 0$ are in $A$ and for large $t$ are in $B$ is given by the product of the measures of $A$ and $B$ in $X$. Mixing systems are also ergodic.

**Example 3.** Let us consider the mixing transformation (see Lasota and Mackey, 1994, p. 57, pp. 65–68)

$$S(x, y) = (x + y, x + 2y) \qquad (\mathrm{mod}\ 1). \tag{5}$$

This is an example of the Anosov diffeomorphism (Anosov, 1963) (see also Bronsztejn *et al.*, 2004, p. 903). In Fig. 3 we can see the first the fifth and the tenth iteration of the mixing tranformation (5) acting on the ensemble of $10^3$ points distributed randomly in the area $[0, 0.1] \times [0, 0.1]$. The points are being spread over the space and afterwards that transformation is literally mixing these points in the whole space. The Euclidean distance between close points first grows quickly and then fluctuates irregularly (Fig. 3 (d)). The difference between the ergodic transformation (2) (cf. Fig. 2) is noticeable. Typical for mixing is the sensitivity to initial conditions (we will formalize this fact further on). ♦

We can say more about the chaoticity of mixing systems. First let us recall the following definition (Auslander and Yorke, 1980) (see also Rudnicki, 2004).

**Definition 1.** The flow is *chaotic in the sense of Auslander and Yorke* if

(i) there exists a dense trajectory, and

(ii) each trajectory is unstable.

Instability here means that there exists a constant $\eta > 0$ such that for each point $x \in X$ and for each $\epsilon > 0$ there exists a point $y \in B(x, \epsilon)$ and $t > 0$ such that $\rho(S_t(x), S_t(y)) > \eta$, where $\rho$ is the metric in $X$ and $B(x, r)$ is the open ball in $X$ with center $x$ and radius $r > 0$. Instability can be also described here as the sensitivity to initial conditions, which is a "popular" criterion of chaos. Now, with the assumption that $X$ is a separable metric space and $\mu$ is a probability Borel measure on $X$ such

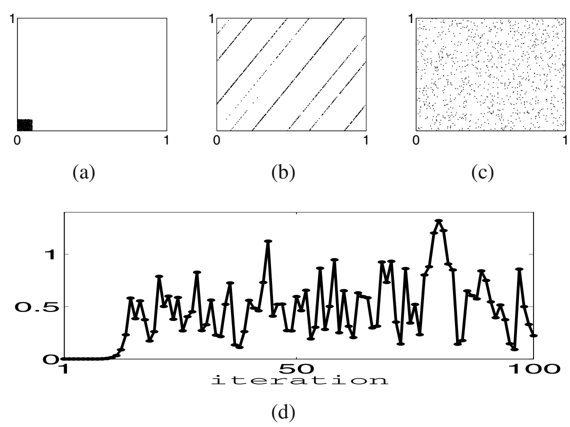



Fig. 3. Iterations of the mixing transformation (5) acting on an ensemble of $10^3$ points randomly distributed in $[0, 0.1] \times [0, 0.1]$: 1st iteration (a), 5th iteration (b), 10th iteration (c), Euclidean metric between two arbitrarily chosen close points from the ensemble (d).

that supp $\mu = X$, we can state that (see Rudnicki, 2004, p. 727, Proposition 1), if a semiflow $(S, \mu)$ is mixing, then the semiflow $\{S_t\}_{t\geq 0}$ is chaotic in the sense of Auslander and Yorke.

**Example 4.** Once again let us consider the mixing transformation (5) from Example 3. Let us consider a correlation coefficient in the form (see de Larminat and Thomas, 1983)

$$\gamma_{xy}(\tau) = \frac{c_{xy}(\tau)}{\sigma_x \sigma y}, \quad \tau = 0, 1, 2, \ldots, \tag{6}$$

where

$$c_{xy}(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - x_0)(y_{i+\tau} - y_0(\tau)), \tag{7}$$

$$x_0 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i,\ y_0(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} y_{i+\tau} \tag{8}$$

and

$$\sigma_x = \sqrt{\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - x_0)^2}, \tag{9}$$

$$\sigma_y = \sqrt{\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (y_{i+\tau} - y_0(\tau))^2}. \tag{10}$$

Once again the tranformation (5) is acting on the ensemble of points (this time $10^4$ for higher accuracy). After a few iterations it reaches the statistical equilibrium on the ensemble and with further iterations it is "mixing" the ensemble in the space. We take a sequence $x_i$ of the euclidean norms for the ensemble in the equilibrium, so we have a sequence of $10^4$ values. $y_{i+\tau}$ for $\tau = 0$ is the same as $x_i$ and for $\tau = 1, 2, \ldots$ it forms a sequence for further iterations. So using the formula (6) we obtain a correlation function where for $\tau = 0$ we have correlation $x_i$ with $x_i$ (Fig. 4(c)) and for $\tau = 1, 2, \ldots$ we have correlation between $x_i$ and $y_{i+\tau}$ which is moving away in time. The result is visible in Fig. 4(a).

We can see that the correlation function (6) for the ensemble decreases to a value near 0 very quickly (already in the 2nd iteration). When we draw the spread of the ensembles on the space for $\tau > 0$, e.g., $\tau = 5$, we can see that points are correlated neither linearly nor in any other way (Fig. 4(d)). Since the mixing transformation is also ergodic, we can change averages over the ensemble to averages along a single trajectory. So instead of calculating a correlation function for the whole ensembles, we can calculate it for a single trajectory and its time shifts. The result is presented in Fig. 4(b); we can see that the correlation functions in both cases (ensemble and single trajectory) are almost the same. Such a rapid decrease in correlation is typical for mixing systems (see Bronsztejn *et al.*, 2004; Rudnicki, 2004; 1988). ♦

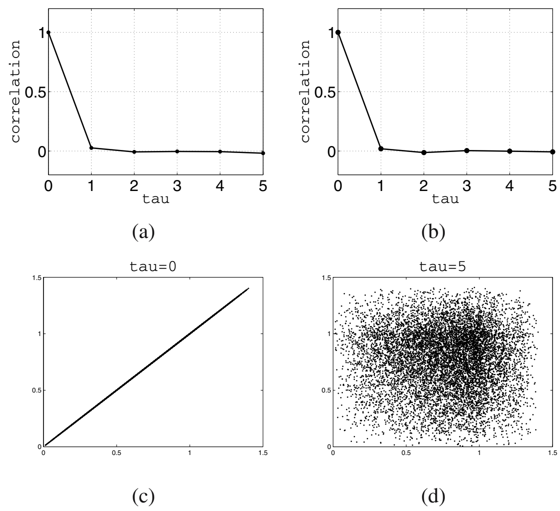



Fig. 4. Rapid decrease in the correlation for the mixing transformation (5) for an ensemble of $10^4$ points (a), correlation for a single trajectory and its time shift (b), spread of points of the ensemble for $\tau = 0$, i.e., correlation of the "initial" ensemble with itself (c), spread of points of the ensemble for $\tau = 5$ (d).

## 3. Infinite dimensional case

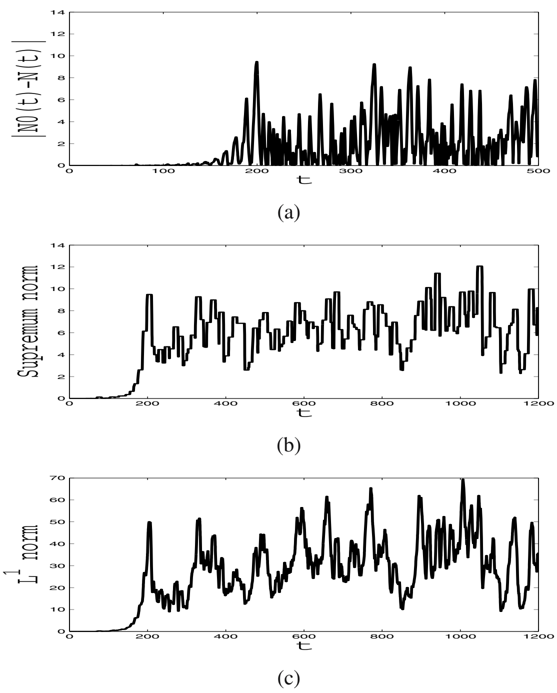



Fig. 5. Two trajectories of Eqn. (11) for constant initial functions different by 0.0001 of the absolute value of the distance between the values $N(t)$ (a), distance in the supremum norm (b), distance in the $L^1$ norm (c).

Let us now consider the delay blood cell production model formulated by Lasota (1977):

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\sigma \cdot N(t) + (\rho \cdot N(t-h))^s \cdot e^{-\gamma \cdot N(t-h)}. \quad (11)$$

Biological interpretations of this equation have their origin in the famous research of Ważewska-Czyżewska and Lasota (1976) into mathematical modelling of the dynamics of erythropoiesis, which is a process of red blood cells (erythrocytes) formation in the bone marrow. For further insight into this research, consult the works of Ważewska-Czyżewska (1983) and Lasota *et al.* (1981). $N(t) \in \mathbb{R}$ is a global number of erythrocytes in blood circulation, $\sigma$ denotes the destruction rate of cells, $\rho$ is oxygen demand, $\gamma$ is the coefficient describing system excitation and $h$ is the delay time representing the time of maturation of erythrocytes.

The contribution of parameter $s$ to a biomedical interpretation can be found in the work of Mitkowski (2011). According to the authors' knowledge, the biomedical meaning of this parameter has not been explained in the literature yet. The production function of blood cells in Eqn. (11) (which can be interpreted as a feedback) has the form of the so-called unimodal function. Briefly speaking, it is a function with one smooth maximum. Because of such a form of the feedback, Eqn. (11) may display very complicated dynamics including chaos (see Ważewska-Czyżewska, 1983; Mackey, 2007; Liz and Rost, 2009; Mitkowski, 2011). Biological delay models with unimodal nonlinearities were considered also by Mackey and Glass (1977) as well as Gurney *et al.* (1980), who described experimental data of Nicholson (1954). However, the nonlinearity in Eqn. (11) is more "flexible" and gives stronger possibilities for applications (for a detailed discussion of this problem, see Mitkowski (2011).

**3.1. Conjecture of Lasota.** Lasota (1977, p. 248) formulated a conjecture concerning ergodic properties of Eqn. (11), i.e., let $C_h$ be the space of continuous functions $v : [-h, 0] \to \mathbb{R}$ with the supremum norm topology. For some positive values of parameters $\rho, h, s$ and $\sigma$, there exists a continuous measure on $C_h$ which is ergodic and invariant with respect to Eqn. (11). By a continuous measure we understand here a measure which vanishes at points (see Lasota, 1977; Lasota and Yorke, 1977) and in this sense the measure is nontrivial. Thus, according to our previous discussion, the conjecture concerns the chaotic behaviour of Eqn. (11). It might be very difficult to solve this problem using only mathematical tools. In general, according to the authors' knowledge, there are very few results where chaos for delay differential equations was proved using only mathematical tools. One of such results was given by Walther (1981). Our aim is to investigate Eqn. (11) numerically in order to check if it exhibits nontrivial ergodic properties.

There is also an interesting historical context of Lasota's hypothesis. Ulam (1960, p. 74) (see also Myjak, 2008) posed the problem of the existence of nontrivial invariant measures for transformations of the unit interval into itself defined by a sufficiently "simple" function (e.g., a piecewise linear function or a polynomial) whose graph does not cross the line $y = x$ with a slope in an absolute value less than 1. Later Lasota and Yorke (1973) solved the problem. The conjecture of Lasota for Eqn. (11) looks like a generalization of Ulam's conjecture to first order differential delay equations. This association comes up during numerical investigations of Lasota's delay equation, where we search for a proper "shape" of unimodal feedback to find nontrivial ergodic properties (see Fig. 7).

**3.2. Calculations.** Numerical investigations show that Eqn. (11) exhibits nontrivial ergodic properties for $\rho \in$

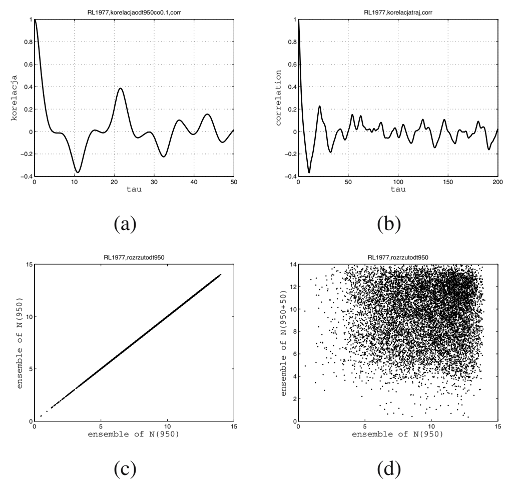



Fig. 6. Rapid decrease in the correlation for Eqn. (11) for an ensemble of $10^4$ trajectories (a), correlation for a single trajectory and its time shift (b), spread of points of the ensemble for $\tau = 0$, i.e., correlation of the "initial" ensemble with itself (c), spread of points of the ensemble for $\tau = 50$ (d).

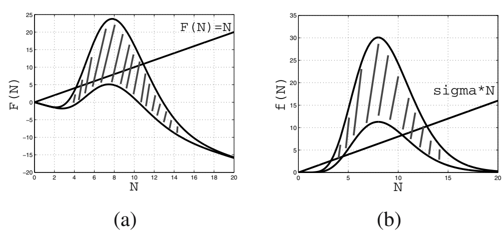



Fig. 7. Range of parameters (shaded area) which generate the nontrivial ergodic behaviour of the right-hand side of (11) (a), unimodal feedback function in reference to the linear destruction rate of the red blood cells (b). In both the cases the lower bound corresponds to $\rho = 0.46$ and the upper to $\rho = 0.52$.

$[0.46,\ 0.52]$, $\sigma = 0.8$, $s = 8$, $\gamma = 1$ and a delay of $h > 9$. In Fig. 7(a), we can see the range of the right-hand side $F(N)$ of (11), with the line $F(N) = N$. The lower bound of the shaded area corresponds to $\rho = 0.46$ and the upper bound to $\rho = 0.52$. In Fig. 7(b) there is the same range of parameters but presented in the form of the unimodal feedback function in reference to the linear destruction rate of red blood cells. Ergodic properties sustain for large values of $h$ (like $h = 50$); however, the more $h$ increases, the more trajectory is attracted to 0 and ergodic properties decay.

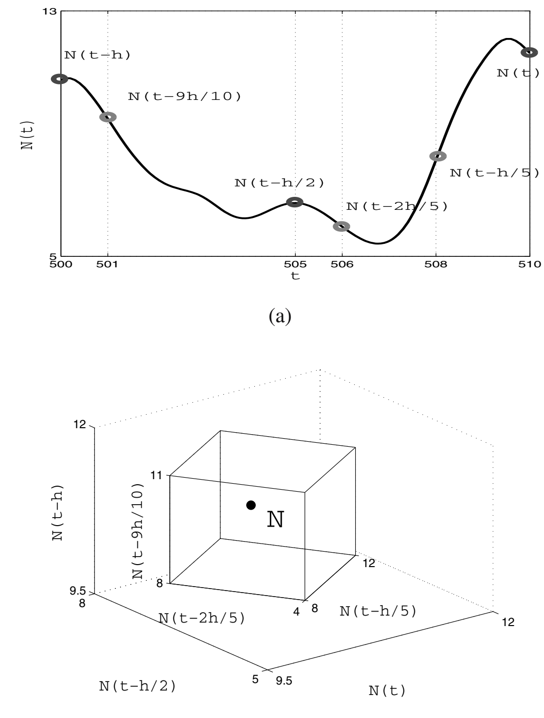
We will show now some numerical experiments indicating ergodic properties of Eqn. (11). We choose $\rho = 0.46$, $\sigma = 0.8$, $s = 8$, $\gamma = 1$, i.e., the lower bounds from Fig. 7(a) and (b). Equation (11) is solved using the MATLAB solver `de23` (see Shampine *et al.*, 2002).

Many important aspects concerning numerical investigations of probabilistic properties of delay differential equations were presented by Taylor (2004). Useful directions for computational analysis of ergodic properties were presented by Lasota and Mackey (1994), Kudrewicz (1991; 1993, 2007) as well as Ott (1993).

It is obvious that numerically we cannot show ergodic properties on the whole infinite dimensional space. We want to show that on some subspaces, Eqn. (11) has a smooth invariant density, which for a large ensemble (see Fig. 9) of trajectories is equal to the average along all single trajectories. That would indicate that the system exhibits basic ergodic properties. After that, using correlation techniques and examining the unstability of trajectories, we want to investigate mixing properties. As the state of Eqn. (11) we will consider a function of an interval of length $h$ (delay) (see Fig. 8(a)). We will analyze its behaviour in subspaces of an infinite dimensional space of its values. A graphical example of such a subspace is presented in Fig. 8 (b). It is a six-dimensional space constructed by taking six arbitrary points of the functional state of Eqn. (11). Another solution is to equip the space $C_h$ with a proper norm; however, in this article apart, from one exception (see Fig. 5), we shall not consider this case. Results of computational analysis of Eqn. (11) in such spaces can be found in the work of Mitkowski (2011).

**3.3. Ergodicity of the flow.** Consider Fig. 9, showing a bunch of trajectories of Eqn. (11). First they evolve quite regularly but after some time the flow becomes very irregular, we could even say turbulent. Additionally, trajectories are bounded. Let us take two arbitrary subspaces form the infinite dimensional space we have introduced previously, e.g., the most natural space of values $N(t) \in \mathbb{R}$ and the space $N(t) \times N(t-h)$ (which is often used for delay differential equations). In Fig. 10 we can observe chosen moments of evolution of $10^4$ constant initial functions of Eqn. (11) distributed exponentially on some interval. Figure 10 (a),(c),(e),(g) shows the evolution on the space of $N(t) \in \mathbb{R}$ and Fig. 10(b), (d), (f), (h) on the space $N(t) \times N(t-h)$. After some time the normalized (to the probability density) histograms (counting the number of points of the ensemble in the subintervals of the space) tend to invariant histograms, i.e., some time after simulations they almost do not change their shape. This may indicate that we have reached some invariant density.

In order to check if this density tends to be smooth, we could calculate a significantly larger ensemble of tra-




(a) (b)

Fig. 8. Geometrical representation of state evolution given by Eqn. (11): an arbitrary state (a), an example of its representation in the six-dimensional space $N(t) \times N(t-h/2) \times N(t-h) \times N(t-h/5) \times N(t-2h/5) \times N(t-9h/10)$ (b).

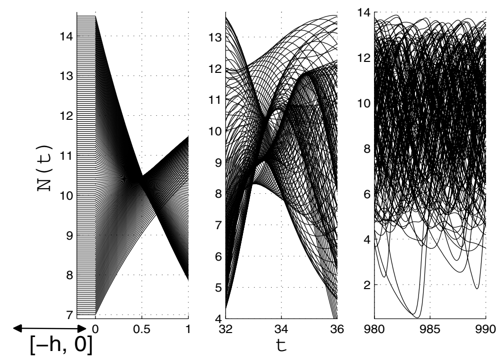



Fig. 9. Bunch of trajectories of Eqn. (11). First the flow is regular, then it becomes turbulent.

jectories, but then numerical calculations take a lot of time and become useless. However, we can examine if the flow exhibits the ergodic property, i.e., if histograms for single trajectories are similar to that of the ensemble. If that were true we could construct a histogram for a very long single trajectory and that would reflect also the average over the ensemble (see Theorem 1). Indeed, numerical simulations indicate that Eqn. (11) exhibits this typical property of ergodic flows; in Figs. 11(a), (b), we have more accurate histograms for single trajectories. We can see that the higher the accuracy the smoother the histograms. Each trajectory is also irregular (see Figs. 11(c), (d)), which is in accordance with the theory discussed in previous sections. The behaviour of the ensemble on the space $N(t) \times N(t-h)$ (see Figs. 10(b), (d), (f), (h)) as well as that of the single trajectory on this space (see Figs. 11(b),(d)) may suggest that there exists an attractor, which has a significant "volume", supporting the invariant ergodic measure.

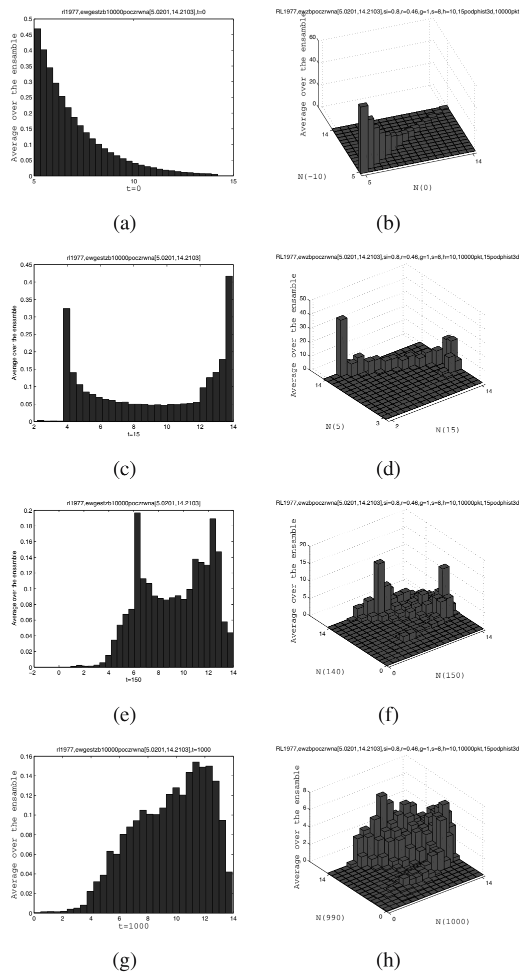



Fig. 10. Evolution of the initial exponential distribution of $10^4$ initial constant functions on $[-h, 0]$ in the space of $N(t)$ at time $t = 0$ (a), time $t = 15$ (c), time $t = 150$ (e), time $t = 1000$ and in the space $N(t) \times N(t-h)$ (g), time $t = 0$ (b), time $t = 15$ (d), time $t = 150$ (f), time $t = 1000$ (h).

**3.4. Mixing properties of the flow.** The flow generated by Eqn. (11) exhibits also properties typical for mixing systems. Numerical simulations indicate that each trajectory is unstable. In Fig. 5(a) we can see that the absolute value of the distance between the values $N(t)$ of two trajectories starting from very close initial functions is fluctuating irregularly. We have marked before that we will not consider any specific norm in the space, but here we will make an exception, because the unstability for Eqn. (11) is much better visible when we equip the space with the supremum or $L^1$ norm (see Fig. 5(b),(c)). Additionally, the correlation for the ensemble and for the single trajectory and its time shifts decreases rapidly (see Fig. 6), which is characteristic for mixing systems (see Section 2.2). The lack of correlation suggests that the attractor does not have a simple structure. It may also indicate that each trajectory is turbulent in the sense of Bass (Bass, 1974; Rudnicki, 2004; 1988). Computational results concerning the problem of turbulence for Eqn. (11) can be found in the work of Mitkowski (2011).

## 4. Concluding remarks

We have presented numerical computations suggesting that the delay differential equation (11) posseses an attractor of a nonsimple structure, supporting an invariant mixing measure. This verifies the conjecture of Lasota which, using the language of ergodic theory, poses the problem of the chaotic behaviour of Eqn. (11).

More computational analysis concerning ergodic properties of Eqn. (11) as well as new contributions to its

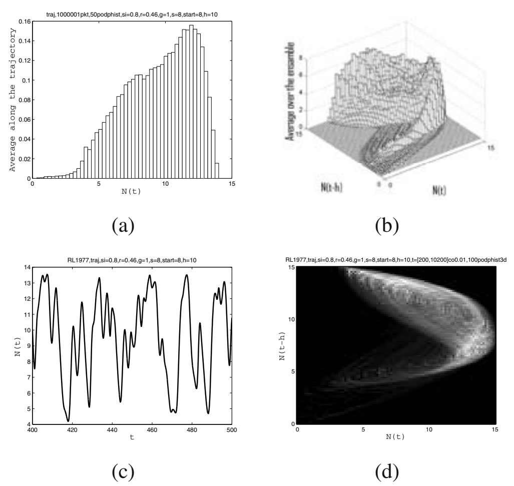



Fig. 11. Average along a single trajectory in the space of $N(t)$ (a), in the space $N(t) \times N(t-h)$ (b). Time evolution of a single trajectory (c), the projection onto the space $N(t) \times N(t-h)$ (d).

biological meaning can be found in the work of Mitkowski (2011).

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