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Shift-M_{split} estimation

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Abstract: The method that is proposed in the present paper is a special case of squared M_{split} estimation. It concerns a direct estimation of the shift between the parameters of the functional models of geodetic observations. The shift in question may result from, for example, deformation of a geodetic network or other non-random disturbances that may influence coordinates of the network points. The paper also presents the example where such shift is identified with a phase displacement of a wave. The shift is estimated on the basis of wave observations and without any knowledge where such displacement took place. The estimates of the shift that are proposed in the paper are named Shift-M_{split} estimators.

Keywords: geodetic adjustment, M_{split} estimation, Shift-M_{split} estimation

1. Introduction

Wiśniewski (2009a, 2009b, 2009c) proposed a new method of estimation that is generally called M_{split} estimation. The theoretical basis of the method, later extended to q-dimensional case in (Wiśniewski, 2010), is the assumption that a measurement result can be a realization of either of two random variables Y_α or Y_β with the expected values $E\{Y_\alpha\}$ and $E\{Y_\beta\}$, respectively. Thus, the observation set $\Omega = \{y_i\colon i = 1, 2, \ldots, n\}$ is a disordered mixture of the elements assigned to the random variables Y_α or Y_β in an unknown way. For that reason, each observation y_i may have either of two competitive expected values $E_\alpha\{y_i\} = E\{Y_\alpha\}$ or $E_\beta\{y_i\} = E\{Y_\beta\}$. M_{split} estimation, at this stage of its development, assumes that the expected values in question are described by the following models:

$$E_\alpha\{y_i\} = \mathbf{a}_i\mathbf{X}_\alpha \tag{1}$$

$$E_\beta\{y_i\} = \mathbf{a}_i\mathbf{X}_\beta \tag{2}$$

where $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$ are two different parameter vectors, and $\mathbf{a}_i$ is the i^{th} row of a known design matrix $\mathbf{A} = [\mathbf{a}_1^{\mathrm{T}}, \cdots, \mathbf{a}_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathfrak{R}^{n,r}$. Thus it is assumed that models presented above are linear or at least result from linearization of other nonlinear models. In the estimation process, the parameters compete with each other for each observation that belongs to the set Ω. Such competition is based on the global split potential, which was proposed in (Wiśniewski, 2009a), and the objective function of the optimization problem that follows it.

Now, let the observation set Ω be a mixture of realizations of two random vectors $\mathbf{Y}_\alpha$ and $\mathbf{Y}_\beta$, with the respective expected values $E\{\mathbf{Y}_\alpha\}$ and $E\{\mathbf{Y}_\beta\}$. The vector of observations $\mathbf{y} \in \mathfrak{R}^n$, which contains disordered elements of the vectors in question, may be assigned to either of two competitive vectors of expected values

$$E_\alpha\{\mathbf{y}_i\} = \mathbf{A}\mathbf{X}_\alpha \tag{3}$$

$$E_\beta\{\mathbf{y}_i\} = \mathbf{A}\mathbf{X}_\beta \tag{4}$$

From the practical point of view, one is often interested in the difference $\Delta_\mathbf{X}$ between the parameters $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$. For example, if we analyze deformation of a network, which was measured twice, at two different epochs (1) and (2), we usually estimate the difference $\boldsymbol{\delta}_\mathbf{X} = \mathbf{X}_{(2)} - \mathbf{X}_{(1)}$, where $\mathbf{X}_{(1)}$ and $\mathbf{X}_{(2)}$ are vectors of coordinates of the network points at these epochs. Estimation of the vector $\delta_\mathbf{X}$, or common estimation of the parameters $\mathbf{X}_{(1)}$ and $\mathbf{X}_{(2)}$, by the least-squares method (*LS*-method) requires a common functional model of all observations (see, e.g. Chen, 1983; Caspary, 1988; Singh and Setan, 2001). Such traditional approach is usually good and sufficient, however, it may be troublesome if the observation set is affected by gross errors (this problem was pointed out in (Wiśniewski, 2009a, 2009c). Another problem may result from random disturbances that are not reflected in statistic and functional models of the observations. Such disturbances may adversely affect the process of identification and estimation itself. One can expect this type of random disturbances, which are difficult to be formally written and described, for example, in deformation analyses based on GPS measurements. Here, it mainly results from the fact that the random disturbances vary in time (e.g. Mertikas and Damianidis, 2007).

Estimation of the difference between parameters of the functional models (models of the expected values) is of greater significance for surveying theory and practice as well. Such approach is applied, for example, in estimation that is affected by outliers (e.g. Cross and Price, 1985; Schaffrin and Wang, 1994; Gui et al., 2007; Duchnowski, 2008) or in estimation that considers deterministic errors described by different potential models (e.g. Kubáčkowá and Kubáček, 1991; Wiśniewski, 1985, 2010). Similar problems may also occur when two observation sets combine or in the case of mixed pixels in laser scanning (e.g. Lichti et al., 2005; Gordon and Lichti, 2007). In GPS technique, such approach may be applied to estimate the phase shift of the measurement wave, which propagation is unexpectedly disturbed (see, e.g. Mubarak and Dempster, 2010; Pavelyev et al., 2010).

M_{split} estimators compete with each other for every single observation, thus the order of observation in the whole set does not matter. The objective function of M_{split} estimation, which is based on the global split potential, allows for the determination of such M_{split} estimates of the parameters $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$ that are not affected by outliers or other unexpected random disturbances. These two main properties of M_{split} estimates are also very important in estimation of the shift between parameters of functional models.

Creating M_{split} estimation of the shift, which can solve the problems mentioned above or other similar ones, one should note that if the competitive parameters $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$ differ from each other then this difference must result from the shift between sets of realizations of random variables $\mathbf{Y}_\alpha$ and $\mathbf{Y}_\beta$. To prove this, let us assume that the non-random quantity $\varDelta = Y_\alpha - Y_\beta$ is a shift between random variables Y_α and Y_β (e.g. Rousseeuw and Verboven, 2002; Duchnowski, 2008, 2009, 2010). Since in M_{split} estimation, the observation y_i may be regarded as a realization of either of two random variable Y_α or Y_β, thus the shift between these variables is equal to the following shift:

$$\forall i : \varDelta = (y_i \in Y_\alpha) - (y_i \in Y_\beta) \tag{5}$$

Since $E\{\varDelta\} = \varDelta$, $\varDelta = Y_\alpha - Y_\beta$ and considering Eq. (5) one can write that

$$E\{\varDelta\} = E\{Y_\alpha\} - E\{Y_\beta\} = E\{y_i \in Y_\alpha\} - E\{y_i \in Y_\beta\} = E_\alpha\{y_i\} - E_\beta\{y_i\} = \varDelta \tag{6}$$

where $E_\alpha\{y_i\}$ and $E_\beta\{y_i\}$ are the competitive expected values of the observation y_i. Similarly, one can write down the following relationship concerning vectors $\mathbf{Y}_\alpha$ and $\mathbf{Y}_\beta$

$$\boldsymbol{\Delta} = E\{\mathbf{Y}_\alpha\} - E\{\mathbf{Y}_\beta\} = E_\alpha\{\mathbf{y}\} - E_\beta\{\mathbf{y}\} \tag{7}$$

By putting the competitive models of the expected values (3) and (4) into the above equation, one obtains ($\boldsymbol{\Delta}_\mathbf{X} = \mathbf{X}_\beta - \mathbf{X}_\alpha$ is the shift between the parameter vectors)

$$\boldsymbol{\Delta} = \mathbf{A}\mathbf{X}_\alpha - \mathbf{A}\mathbf{X}_\beta = -\mathbf{A}\ \boldsymbol{\Delta}_\mathbf{X} \tag{8}$$

The main objective of the present paper is to present the method of direct estimation of the shift $\boldsymbol{\Delta}_\mathbf{X}$ by applying the squared M_{split} estimation. The method proposed will be called Shift-M_{split} estimation, and its theoretical foundations are presented in the second section of the paper. That section focuses on the dual optimization problem and its solution that refers to the traditional systems of the normal equations of the least squares method. The duality of the optimization problem is the basis for an iterative process that is effective and easy to be carried out. The section 3 presents numerical examples of how to compute Shift-M_{split} estimates. The first example (section 3.1) presents estimation of the phase shift based on wave observations. In the second example (section 3.2), the estimated shift is regarded as the change of the point heights in the levelling network that is measured at two different epochs. Thus, it is assumed that the deformation vector $\boldsymbol{\delta}_\mathbf{X}$ is equal to the shift $\boldsymbol{\Delta}_\mathbf{X}$ of the point heights.

2. Theoretical foundation

Consider M_{split} estimation that concerns the competitive expected values $E_\alpha\{\mathbf{y}\} = \mathbf{A}\mathbf{X}_\alpha$ and $E_\beta\{\mathbf{y}\} = \mathbf{A}\mathbf{X}_\beta$ of the observation vector $\mathbf{y} = [y_1, \cdots, y_n]^\mathrm{T}$. In this case, the objective function in respect of the parameters $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$ has the following form (Wiśniewski, 2009a, 2009b):

$$\Phi(\mathbf{y}; \mathbf{X}_\alpha, \mathbf{X}_\beta) = \sum_{i=1}^{n} \rho_\alpha(y_i; \mathbf{X}_\alpha)\rho_\beta(y_i; \mathbf{X}_\beta) \tag{9}$$

where ρ_α and ρ_β are arbitrary, at least twice differentiable convex functions. Note that in the classic M-estimation $\Phi_M(\mathbf{y};\mathbf{X}) = \sum_{i=1}^{n} \rho(y_i;\mathbf{X})$. The cited papers focus on the squared M_{split} estimation where $\rho_\alpha(y_i;\mathbf{X}_\alpha) = v_{i;\alpha}^2$ and $\rho_\beta(y_i;\mathbf{X}_\beta) = v_{i;\beta}^2$. Here, $v_{i;\alpha}$ and $v_{i;\beta}$ are competitive random errors of the observation y_i. These quantities can be described by the following functional models:

$$v_{i;\alpha} = y_i - E_\alpha\{y\} = y_i - \mathbf{a}_i\mathbf{X}_\alpha \quad \overset{i=1,\dots,n}{\rightarrow} \quad \mathbf{v}_\alpha = \mathbf{y} - \mathbf{A}\mathbf{X}_\alpha \tag{10}$$

$$v_{i;\beta} = y_i - E_\beta\{y\} = y_i - \mathbf{a}_i\mathbf{X}_\beta \quad \overset{i=1,\dots,n}{\rightarrow} \quad \mathbf{v}_\beta = \mathbf{y} - \mathbf{A}\mathbf{X}_\beta \tag{11}$$

Given these models, the objective function of Eq. (9) can be written as

$$\Phi(\mathbf{y};\mathbf{X}_\alpha,\mathbf{X}_\beta) = \sum_{i=1}^{n} (y_i - \mathbf{a}_i\mathbf{X}_\alpha)^2(y_i - \mathbf{a}_i\mathbf{X}_\beta)^2 = \sum_{i=1}^{n} v_{i;\alpha}^2 v_{i;\beta}^2 \tag{12}$$

On the one hand squared M_{split} estimation is a special case of M_{split} estimation, but on the other hand it is a kind of development of the classical LS-method with the objective function $\Phi_{LS}(\mathbf{y};\mathbf{X}) = \sum_{i=1}^{n} v_i^2$. It follows that squared M_{split} estimates have certain interesting properties and are rather easy to be calculated, which was described and illustrated with numerical tests presented in (Wiśniewski, 2009a, 2009c). The optimization problem

$$\min_{\mathbf{X}_\alpha,\mathbf{X}_\beta}\left[\Phi(\mathbf{y};\mathbf{X}_\alpha,\mathbf{X}_\beta) = \sum_{i=1}^{n} v_{i;\alpha}^2 v_{i;\beta}^2\right] = \sum_{i=1}^{n} (y_i - \mathbf{a}_i\hat{\mathbf{X}}_\alpha)^2(y_i - \mathbf{a}_i\hat{\mathbf{X}}_\beta)^2 = \sum_{i=1}^{n} \hat{v}_{i\alpha}^2 \hat{v}_{i\beta}^2 \tag{13}$$

that is solved by applying the models of Eqs. (10) and (11) leads to the competitive estimates $\hat{\mathbf{X}}_\alpha$ and $\hat{\mathbf{X}}_\beta$ of the parameters $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$, respectively. In this case, the parameter shift $\mathbf{\Delta}_\mathbf{X}$ can be estimated indirectly by $\mathbf{\Delta}\hat{\mathbf{X}} = \hat{\mathbf{X}}_\beta - \hat{\mathbf{X}}_\alpha$.

A direct estimation of the shift may be very important from the practical point of view. For example, it might be applied in displacement estimation in a free network or when reference points are supposed to be unstable. In such cases it would be sometimes better to estimate shift of the coordinates than the coordinates themselves. To determine the method of the estimation in question, one should note that a constant shift between elements of the set Ω, which is equal to the shift of the expected values $E_\alpha\{y\}$ and $E_\beta\{y\}$, can be associated with the difference Δ_v between the errors $v_{i;\alpha}$ and $v_{i;\beta}$, i.e.

$$\Delta = E_\alpha\{y\} - E_\beta\{y\} = v_{i;\beta} - v_{i;\alpha} = \Delta_v \tag{14}$$

This relationship can also be written for the error vectors $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$. Thus

$$\mathbf{\Delta} = E_\alpha\{\mathbf{y}\} - E_\beta\{\mathbf{y}\} = \mathbf{v}_\beta - \mathbf{v}_\alpha = \mathbf{\Delta}_v \tag{15}$$

Since $\mathbf{v}_\alpha = \mathbf{y} - \mathbf{A}\mathbf{X}_\alpha$ and $\mathbf{v}_\beta = \mathbf{y} - \mathbf{A}\mathbf{X}_\beta$, then

$$\mathbf{\Delta} = \mathbf{\Delta}_v = \mathbf{v}_\beta - \mathbf{v}_\alpha = -\mathbf{A}(\mathbf{X}_\beta - \mathbf{X}_\alpha) = -\mathbf{A}\mathbf{\Delta}_\mathbf{X} \tag{16}$$

It follows that the original optimization problem of M_{split} estimation in Eq. (13), which concerns competitive parameters $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$, can be replaced with the following equivalent problem

$$\min_{\boldsymbol{\Delta}_\mathbf{X}}\left[\Phi(\mathbf{y};\boldsymbol{\Delta}_\mathbf{X})=\sum_{i=1}^{n} v_{i;\alpha}^2 v_{i;\beta}^2\right] \tag{17}$$

Considering the expression of Eq. (16), one can write that

$$\mathbf{v}_\alpha=\mathbf{v}_\beta+\mathbf{A}\boldsymbol{\Delta}_\mathbf{X} \tag{18}$$

$$\mathbf{v}_\beta=\mathbf{v}_\alpha-\mathbf{A}\boldsymbol{\Delta}_\mathbf{X} \tag{19}$$

hence, for $i=1,2,\ldots,n$

$$v_{i\alpha}=v_{i\beta}+\mathbf{a}_i\boldsymbol{\Delta}_\mathbf{X} \tag{20}$$

$$v_{i\beta}=v_{i\alpha}-\mathbf{a}_i\boldsymbol{\Delta}_\mathbf{X} \tag{21}$$

Thus, the objective function from Eq. (17) can be changed into the following form:

$$\Phi(\mathbf{y};\boldsymbol{\Delta}_\mathbf{X})=\sum_{i=1}^{n} v_{i;\alpha}^2(\underbrace{v_{i;\alpha}-\mathbf{a}_i\boldsymbol{\Delta}_\mathbf{X}}_{v_{i;\beta}})^2=\sum_{i=1}^{n}(\underbrace{v_{i;\beta}+\mathbf{a}_i\boldsymbol{\Delta}_\mathbf{X}}_{v_{i;\alpha}})^2 v_{i;\beta}^2 \tag{22}$$

which is the basis for the following optimization problem:

$$\min_{\boldsymbol{\Delta}_\mathbf{X}}\Phi(\mathbf{y};\boldsymbol{\Delta}_\mathbf{X})=\Phi(\mathbf{y};\hat{\boldsymbol{\Delta}}_{\mathbf{X};\mathrm{split}}) \tag{23}$$

The estimator $\hat{\boldsymbol{\Delta}}_{\mathbf{X};\mathrm{split}}$ that solves this optimization problem is a direct M_{split} estimate of the shift $\boldsymbol{\Delta}_\mathbf{X}$. It is obtained by applying the conventional condition

$$\mathbf{g}(\boldsymbol{\Delta}_\mathbf{X})=\frac{\partial}{\partial\boldsymbol{\Delta}_\mathbf{X}}\Phi(\mathbf{y};\boldsymbol{\Delta}_\mathbf{X})\bigg|_{\boldsymbol{\Delta}_\mathbf{X}=\hat{\boldsymbol{\Delta}}_{\mathbf{X};\mathrm{split}}}=\mathbf{0} \tag{24}$$

The duality of the objective function of Eq. (22) enables to determine the gradient $\mathbf{g}(\boldsymbol{\Delta}_\mathbf{X})$ in two following equivalent ways:

$$\mathbf{g}(\boldsymbol{\Delta}_\mathbf{X})=\frac{\partial}{\partial\boldsymbol{\Delta}_\mathbf{X}}\Phi(\mathbf{y};\boldsymbol{\Delta}_\mathbf{X})=\frac{\partial\mathbf{v}_\beta}{\partial\boldsymbol{\Delta}_\mathbf{X}}\frac{\partial}{\partial\mathbf{v}_\beta}\left[\sum_{i=1}^{n} v_{i;\alpha}^2(\underbrace{v_{i;\alpha}-\mathbf{a}_i\boldsymbol{\Delta}_\mathbf{X}}_{v_{i,\beta}})^2\right] \tag{25}$$

$$\mathbf{g}(\boldsymbol{\Delta}_\mathbf{X})=\frac{\partial}{\partial\boldsymbol{\Delta}_\mathbf{X}}\Phi(\mathbf{y};\boldsymbol{\Delta}_\mathbf{X})=\frac{\partial\mathbf{v}_\alpha}{\partial\boldsymbol{\Delta}_\mathbf{X}}\frac{\partial}{\partial\mathbf{v}_\alpha}\left[\sum_{i=1}^{n}(\underbrace{v_{i;\beta}+\mathbf{a}_i\boldsymbol{\Delta}_\mathbf{X}}_{v_{i,\alpha}})^2 v_{i;\beta}^2\right] \tag{26}$$

These two variants of the gradient $\mathbf{g}(\boldsymbol{\Delta}_\mathbf{X})$ can also be presented in the following forms:

$$\mathbf{g}(\boldsymbol{\Delta}_\mathbf{X})=-2\mathbf{A}^\mathrm{T}\mathbf{w}(\mathbf{v}_\alpha)(\mathbf{v}_\alpha-\mathbf{A}\boldsymbol{\Delta}_\mathbf{X}) \tag{27}$$

$$\mathbf{g}(\boldsymbol{\Delta}_{\mathbf{X}}) = 2\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_{\beta})(\mathbf{v}_{\beta} + \mathbf{A}\boldsymbol{\Delta}_{\mathbf{X}}) \tag{28}$$

where $\mathbf{w}(\mathbf{v}_{\alpha}) = \mathrm{diag}(v_{1;\alpha}^2, \cdots, v_{n;\alpha}^2)$ and $\mathbf{w}(\mathbf{v}_{\beta}) = \mathrm{diag}(v_{1;\beta}^2, \cdots, v_{n;\beta}^2)$. Given these two forms of the gradient, the condition of Eq. (24) can be rewritten as

$$\mathbf{g}(\boldsymbol{\Delta}_{\mathbf{X}})|_{\hat{\boldsymbol{\Delta}}_{\mathbf{X};\mathrm{split}}} = \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} \mathbf{g}(\boldsymbol{\Delta}_{\mathbf{X}})|_{\hat{\boldsymbol{\Delta}}_{\mathbf{X};\alpha}} = \mathbf{0} \\ \mathbf{g}(\boldsymbol{\Delta}_{\mathbf{X}})|_{\hat{\boldsymbol{\Delta}}_{\mathbf{X};\beta}} = \mathbf{0} \end{cases} \tag{29}$$

Hence, two equivalent conditions are obtained for the optimality of the problem of Eq. (23)

$$\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_{\alpha})(\mathbf{v}_{\alpha} - \mathbf{A}\hat{\boldsymbol{\Delta}}_{\mathbf{X};\alpha}) = \mathbf{0} \tag{30}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_{\beta})(\mathbf{v}_{\beta} + \mathbf{A}\hat{\boldsymbol{\Delta}}_{\mathbf{X};\beta}) = \mathbf{0} \tag{31}$$

It is worth noting that such necessary conditions apply also to M_{split} estimation of the parameters $\mathbf{X}_{\alpha}$ and $\mathbf{X}_{\beta}$ from the competitive functional models of Eqs. (10) and (11) and objective function of Eq. (12). To prove it, let us compute the gradients of this objective function related to the competitive parameters $\mathbf{X}_{\alpha}$ and $\mathbf{X}_{\beta}$ (see also Wiśniewski, 2009a)

$$\mathbf{g}(\mathbf{X}_{\beta}) = \frac{\partial}{\partial \mathbf{X}_{\beta}}\Phi(\mathbf{y}; \mathbf{X}_{\alpha}, \mathbf{X}_{\beta}) = -2\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_{\alpha})\mathbf{v}_{\beta} = \mathbf{g}(\boldsymbol{\Delta}_{\mathbf{X}}) \tag{32}$$

$$\mathbf{g}(\mathbf{X}_{\alpha}) = \frac{\partial}{\partial \mathbf{X}_{\alpha}}\Phi(\mathbf{y}; \mathbf{X}_{\alpha}, \mathbf{X}_{\beta}) = -2\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_{\beta})\mathbf{v}_{\alpha} = -\mathbf{g}(\boldsymbol{\Delta}_{\mathbf{X}}) \tag{33}$$

Thus, the necessary conditions for solving the optimization problem of M_{split} estimation of the parameters $\mathbf{X}_{\alpha}$ and $\mathbf{X}_{\beta}$ have the following forms:

$$\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_{\alpha})\mathbf{v}_{\beta} = \mathbf{0} \tag{34}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_{\beta})\mathbf{v}_{\alpha} = \mathbf{0} \tag{35}$$

which is clearly consistent with the conditions of Eqs. (30) and (31). Note, that the necessary conditions from Eqs. (30) and (31) as well as the conditions of Eqs. (34) and (35) correspond to the necessary conditions of the *LS*-method with two following objective functions:

$$\Phi_{LS}(\mathbf{y}; \mathbf{X}_{\beta}) = \sum_{i=1}^{n} v_{i;\beta}^2 w(v_{i;\alpha}) \tag{36}$$

$$\Phi_{LS}(\mathbf{y}; \mathbf{X}_{\alpha}) = \sum_{i=1}^{n} v_{i;\alpha}^2 w(v_{i;\beta}) \tag{37}$$

Here, the quantities $w(v_{i;\alpha}) = v_{i;\alpha}^2$ and $w(v_{i;\beta}) = v_{i;\beta}^2$ are clearly regarded as the weights. The functions of Eqs. (36) and (37) illuminate the source of the properties of squared M_{split} estimates and furthermore they help to organize the computation process of squared M_{split} estimation.

The elementary transformations of Eqs. (30) and (31) result in two following systems of normal equations:

$$\mathbf{N}(\mathbf{v}_\alpha)\hat{\boldsymbol{\Delta}}_{\mathbf{X};\alpha} - \mathbf{b}(\mathbf{v}_\alpha) = \mathbf{0} \tag{38}$$

$$\mathbf{N}(\mathbf{v}_\beta)\hat{\boldsymbol{\Delta}}_{\mathbf{X};\beta} + \mathbf{b}(\mathbf{v}_\beta) = \mathbf{0} \tag{39}$$

where $\mathbf{N}(\mathbf{v}) = \mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v})\mathbf{A} \in \Re^{r,r}$, $\mathbf{b}(\mathbf{v}) = \mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v})\mathbf{v}$.

Let us assume that $\mathrm{rank}[\mathbf{N}(\mathbf{v})] = r$, for $\mathbf{v} = \mathbf{v}_\alpha$ and $\mathbf{v} = \mathbf{v}_\beta$. Then, the explicit solutions of the system of normal equations (38) and (39), which satisfy the conditions from Eqs. (29) and (30), are the following estimates:

$$\hat{\boldsymbol{\Delta}}_{\mathbf{X};\alpha} = [\mathbf{N}(\mathbf{v}_\alpha)]^{-1}\mathbf{b}(\mathbf{v}_\alpha) = [\mathbf{N}(\mathbf{v}_\alpha)]^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\alpha)\mathbf{v}_\alpha \tag{40}$$

$$\hat{\boldsymbol{\Delta}}_{\mathbf{X};\beta} = -[\mathbf{N}(\mathbf{v}_\beta)]^{-1}\mathbf{b}(\mathbf{v}_\beta) = -[\mathbf{N}(\mathbf{v}_\beta)]^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\beta)\mathbf{v}_\beta \tag{41}$$

Due to the duality of the objective function from Eq. (22), and hence the equivalence of the normal equations (38) and (39), the estimates that are proposed above must be equal to each other. The estimates $\hat{\boldsymbol{\Delta}}_{\mathbf{X};\mathrm{split}} = \hat{\boldsymbol{\Delta}}_{\mathbf{X};\alpha} = \hat{\boldsymbol{\Delta}}_{\mathbf{X};\beta}$ will be called Shift-M_{split} estimators.

It is easy to prove that the estimates (40) and (41) actually solve the systems of normal equations (38) and (39), respectively. On the other hand, they solve the alternative systems of normal equations too. To prove this, let us first recall that $\mathbf{v}_\alpha = \mathbf{v}_\beta + \mathbf{A}\boldsymbol{\Delta}_X$ and $\mathbf{v}_\beta = \mathbf{v}_\alpha - \mathbf{A}\boldsymbol{\Delta}_X$. Thus, when applying the estimate form of Eq. (40) to Eq. (39), then

$$\begin{aligned}\mathbf{N}(\mathbf{v}_\beta)\hat{\boldsymbol{\Delta}}_{\mathbf{X};\alpha} + \mathbf{b}(\mathbf{v}_\beta) &= \mathbf{N}(\mathbf{v}_\beta)[\mathbf{N}(\mathbf{v}_\alpha)]^{-1}\mathbf{b}(\mathbf{v}_\alpha) + \mathbf{b}(\mathbf{v}_\beta) = \\ &= \mathbf{N}(\mathbf{v}_\beta)[\mathbf{N}(\mathbf{v}_\alpha)]^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\alpha)(\mathbf{v}_\beta + \mathbf{A}\boldsymbol{\Delta}_{\mathbf{X}}) + \mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\beta)(\mathbf{v}_\alpha - \mathbf{A}\boldsymbol{\Delta}_{\mathbf{X}})\end{aligned} \tag{42}$$

In a similar manner and by applying the estimate of Eq. (41) to Eq. (38), one can also write that

$$\begin{aligned}\mathbf{N}(\mathbf{v}_\alpha)\hat{\boldsymbol{\Delta}}_{\mathbf{X},\beta} - \mathbf{b}(\mathbf{v}_\alpha) &= -\mathbf{N}(\mathbf{v}_\alpha)[\mathbf{N}(\mathbf{v}_\beta)]^{-1}\mathbf{b}(\mathbf{v}_\beta) - \mathbf{b}(\mathbf{v}_\alpha) = \\ &= -\mathbf{N}(\mathbf{v}_\alpha)[\mathbf{N}(\mathbf{v}_\beta)]^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\beta)(\mathbf{v}_\alpha - \mathbf{A}\boldsymbol{\Delta}_{\mathbf{X}}) - \mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\alpha)(\mathbf{v}_\beta + \mathbf{A}\boldsymbol{\Delta}_{\mathbf{X}})\end{aligned} \tag{43}$$

Now by using the conditions of Eqs. (34) and (35) one can finally write down

$$\begin{aligned}\mathbf{N}(\mathbf{v}_\beta)[\mathbf{N}(\mathbf{v}_\alpha)]^{-1}\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\alpha)\mathbf{v}_\beta}_{\mathbf{0}} + \mathbf{N}(\mathbf{v}_\beta)\boldsymbol{\Delta}_{\mathbf{X}} + \underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\beta)\mathbf{v}_\alpha}_{\mathbf{0}} - \mathbf{N}(\mathbf{v}_\beta)\boldsymbol{\Delta}_{\mathbf{X}} = \\ = \mathbf{N}(\mathbf{v}_\beta)\boldsymbol{\Delta}_{\mathbf{X}} - \mathbf{N}(\mathbf{v}_\beta)\boldsymbol{\Delta}_{\mathbf{X}} = \mathbf{0}\end{aligned} \tag{44}$$

and similarly

$$-\mathbf{N}(\mathbf{v}_\alpha)[\mathbf{N}(\mathbf{v}_\beta)]^{-1}\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\beta)\mathbf{v}_\alpha}_{\mathbf{0}}+\mathbf{N}(\mathbf{v}_\alpha)\mathbf{\Delta}_{\mathbf{X}}-\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{w}(\mathbf{v}_\alpha)\mathbf{v}_\beta}_{\mathbf{0}}-\mathbf{N}(\mathbf{v}_\alpha)\mathbf{\Delta}_{\mathbf{X}}=$$

$$=\mathbf{N}(\mathbf{v}_\alpha)\mathbf{\Delta}_{\mathbf{X}}-\mathbf{N}(\mathbf{v}_\alpha)\mathbf{\Delta}_{\mathbf{X}}=\mathbf{0} \tag{45}$$

Let us now consider the way of how to compute Shift-M_{split} estimates of Eqs. (40) and (41). It is obvious that the values of the estimates $\hat{\mathbf{\Delta}}_{\mathbf{X};\alpha}$ and $\hat{\mathbf{\Delta}}_{\mathbf{X};\beta}$ depend on the values of the vectors $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$, respectively. Thus, Shift-M_{split} estimation that leads to the estimate $\hat{\mathbf{\Delta}}_{\mathbf{X};\mathrm{split}}=\hat{\mathbf{\Delta}}_{\mathbf{X};\alpha}=\hat{\mathbf{\Delta}}_{\mathbf{X};\beta}$ seems to be an iterative process, however, other ways to solve the optimization problem are also not excluded (e.g. global optimization). Let us now assume that the value of $\mathbf{v}_\alpha^{j-1}$ is known in the next j^{th} iterative step. According to Eqs. (40) and (19), one can then compute ($j=1,2,\ldots,m$)

$$\mathbf{\Delta}^j_{\mathbf{X};\alpha}=[\mathbf{N}(\mathbf{v}_\alpha^{j-1})]^{-1}\mathbf{b}(\mathbf{v}_\alpha^{j-1}) \tag{46}$$

$$\mathbf{v}_\beta^j=\mathbf{v}_\alpha^{j-1}-\mathbf{A}\mathbf{\Delta}^j_{\mathbf{X};\alpha} \tag{47}$$

and afterwards (according to Eqs. (41) and (18))

$$\mathbf{\Delta}^j_{\mathbf{X};\beta}=-[\mathbf{N}(\mathbf{v}_\beta^{j})]^{-1}\mathbf{b}(\mathbf{v}_\beta^{j}) \tag{48}$$

$$\mathbf{v}_\alpha^j=\mathbf{v}_\beta^{j}+\mathbf{A}\mathbf{\Delta}^j_{\mathbf{X};\beta} \tag{49}$$

The iterative process that is described by Eqs. (36)-(39) is convergent, which results from earlier presented numerical tests and which is consistent with the theory of squared M_{split} estimation (see, Wiśniewski, 2009a, 2010). Thus, one can write that

$$\left.\begin{aligned}\lim_{j\to\infty}\mathbf{\Delta}^j_{\mathbf{X};\alpha}&=\hat{\mathbf{\Delta}}_{\mathbf{X};\alpha}\\ \lim_{j\to\infty}\mathbf{\Delta}^j_{\mathbf{X};\beta}&=\hat{\mathbf{\Delta}}_{\mathbf{X};\beta}\end{aligned}\right\}=\hat{\mathbf{\Delta}}_{\mathbf{X};\mathrm{split}} \tag{50}$$

and the estimate $\hat{\mathbf{\Delta}}_{\mathbf{X};\mathrm{split}}=\hat{\mathbf{\Delta}}_{\mathbf{X};\alpha}=\hat{\mathbf{\Delta}}_{\mathbf{X};\beta}$ is such quantity for which $\left|\mathbf{\Delta}^m_{\mathbf{X};\beta}-\mathbf{\Delta}^m_{\mathbf{X};\alpha}\right|<\varepsilon$, where ε is the assumed precision of the computations.

The theoretical basis is common for M_{split} estimation and Shift-M_{split} estimation, and it is their common link to *LS*-method. This has not only a formal significance (for example, in analyzing properties of Shift-M_{split} estimation). These relationships can also be applied, for example, at the beginning of the iterative process. Wisniewski (2009a, 2009c) proposed to start the iterative process of M_{split} estimation from the *LS*-estimates, namely $\hat{\mathbf{X}}_{LS}=(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{y}$ and $\hat{\mathbf{v}}_{LS}=\mathbf{y}-\mathbf{A}\hat{\mathbf{X}}_{LS}$. In the case of Shift-$M_{split}$ estimation, the initial iterative step (step "0") should of course be supplemented with the values of $\mathbf{\Delta}^0_{\mathbf{X};\beta}=-[\mathbf{N}(\hat{\mathbf{v}}_{LS})]^{-1}\mathbf{b}(\hat{\mathbf{v}}_{LS})$ and $\mathbf{v}^0_\alpha=\hat{\mathbf{v}}_{LS}+\mathbf{A}\mathbf{\Delta}^0_{\mathbf{X};\beta}$, which are computed on the basis of Eqs. (48) and (49).

3. Examples

Shift-M_{split} estimation that is presented in the previous sections can be applied in some geodetic problems where values of the parameters of the functional model vary, for example, in time. The present paper focuses especially on the theory of this method. Thus, the examples presented here show possible applications of Shift-M_{split} estimation, and are not related to details of the measurement theory or real technological conditions of the problems to which they refer.

3.1. Example 1

Shift-M_{split} estimation can be, for example, applied to determine a phase shift $\Delta_\varphi = \varphi_\beta - \varphi_\alpha$ between two waves of the following equations:

$$\tilde{y}(t;\varphi_\alpha) = y_0 + A\sin(\omega t + \varphi_\alpha) \tag{51}$$

$$\tilde{y}(t;\varphi_\beta) = y_0 + A\sin(\omega t + \varphi_\beta) \tag{52}$$

where y_0 – unknown constant, A – amplitude of the wave, ω – angular frequency, t – time, φ – phase offset. Let us assume that the wave is measured at the time t_i, $i = 1, 2, \ldots, n$, and the result may be related to either of two waves: $\tilde{y}(t;\varphi_\alpha)$ or $\tilde{y}(t;\varphi_\beta)$. Thus, the only information about the observation y_i is that it belongs to either of two sets: $U_\alpha = \{\tilde{y}(t_i;\varphi_\alpha) + v_{i;\alpha}\}$ or $U_\beta = \{\tilde{y}(t_i;\varphi_\beta) + v_{i;\beta}\}$, where v_i is a random error of the measurement. Hence, there are two following competitive functional models:

$$y_i = \tilde{y}_\alpha(t_i) + v_{i;\alpha} = y_{0;\alpha} + A\sin(\omega t_i + \varphi_\alpha) + v_{i;\alpha} \tag{53}$$

$$y_i = \tilde{y}_\beta(t_i) + v_{i;\beta} = y_{0;\beta} + A\sin(\omega t_i + \varphi_\beta) + v_{i;\beta} \tag{54}$$

assigned to each observation y_i, $i = 1, 2, \ldots, n$, belonging to the set $\Omega \equiv U_\alpha \cup U_\beta$. The value y_0 is assumed as constant, however, according to the M_{split} estimation principles, there are two competitive values $y_{0;\alpha}$ and $y_{0;\beta}$ in the models of Eqs. (53) and (54). In order to simplify the further estimation process (and only for this reason), the functions from Eqs. (51) and (52) are brought to the following linear forms:

$$\tilde{y}(t,\varphi_\alpha) = y_0 + A\sin(\omega t) + \frac{d\tilde{y}(t,\varphi_\alpha)}{d\varphi_\alpha}\varphi_\alpha \cong y_0 + A\sin(\omega t) + \cos(\omega t)\varphi_\alpha \tag{55}$$

$$\tilde{y}(t,\varphi_\beta) = y_0 + A\sin(\omega t) + \frac{d\tilde{y}_\beta(t,\varphi_\beta)}{d\varphi_\beta}\varphi_\beta \cong y_0 + A\sin(\omega t) + \cos(\omega t)\varphi_\beta \tag{56}$$

Thus, the functional models (53) and (54) can then be written as follows:

$$v_{i;\alpha} = y_i - [y_{0;\alpha} + A\sin(\omega t_i) + \cos(\omega t_i)\varphi_\alpha] \quad \overset{i=1,\ldots,n}{\rightarrow} \quad \mathbf{v}_\alpha = \mathbf{l} - \mathbf{A}\mathbf{X}_\alpha \tag{57}$$

$$v_{i;\beta} = y_i - [y_{0;\beta} + A\sin(\omega t_i) + \cos(\omega t_i)\varphi_\beta] \quad \overset{i=1,\ldots,n}{\rightarrow} \quad \mathbf{v}_\beta = \mathbf{l} - \mathbf{A}\mathbf{X}_\beta \tag{58}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cos(\omega t_1) & \cos(\omega t_2) & \cdots & \cos(\omega t_n) \end{bmatrix}^T, \quad \mathbf{X}_\alpha = \begin{bmatrix} y_{0;\alpha} \\ \varphi_\alpha \end{bmatrix}, \quad \mathbf{X}_\beta = \begin{bmatrix} y_{0;\beta} \\ \varphi_\beta \end{bmatrix},$$

$$\mathbf{l} = [y_1 - A\sin(\omega t_1), \quad y_2 - A\sin(\omega t_2), \quad \cdots, \quad y_n - A\sin(\omega t_n)]^T$$

The linear functions, models and also the matrices presented above are valid for small values of the phase shifts φ_α and φ_β. However, they were used in the following computations for different values of φ_α and φ_β just to show the iterative process of Shift-M_{split} estimation.

In the case at hand, the shift of the parameter vector resulting from the difference $\mathbf{v}_\beta - \mathbf{v}_\alpha = -\mathbf{A}(\mathbf{X}_\beta - \mathbf{X}_\alpha) = -\mathbf{A}\boldsymbol{\Delta}_\mathbf{X}$ has the following form:

$$\boldsymbol{\Delta}_\mathbf{X} = \mathbf{X}_\beta - \mathbf{X}_\alpha = [\mathit{\Delta}_{y_0}, \ \mathit{\Delta}_\varphi]^T = [y_{0;\beta} - y_{0;\alpha}, \ \varphi_\beta - \varphi_\alpha]^T \tag{59}$$

where $\Delta_{y_0} = y_{0;\beta} - y_{0;\alpha}$ is the shift of the constant y_0, and $\mathit{\Delta}_\varphi = \varphi_\beta - \varphi_\alpha$ is the phase shift.

The wave observations are simulated by applying the following formula:

$$y_i = y_0 + A\sin(\omega t_i + \varphi) + v_i \tag{60}$$

and by assuming either of two variants of the phase offset: $\varphi = \varphi_\alpha$ or $\varphi = \varphi_\beta$. Furthermore, observations are simulated for $A = 1$ and $\omega t_i = i\pi/8$, $i = 1, 2, \ldots, 16$ (y_0 remains the same). The observation errors v_i are generated under the assumption that they are Gaussian errors with the mean of 0 and the standard deviation of 0.002. In order to avoid any influence of the set sizes, the sets U_α and U_β are simulated as the sets of the same number of elements. The Shift-M_{split} estimates, namely $\hat{\mathit{\Delta}}_{y_0;\mathrm{split}}$ and $\hat{\mathit{\Delta}}_{\varphi;\mathrm{split}}$ are compared with the following quantities:

$$\Delta\hat{y}_0 = \hat{y}_{0;\beta} - \hat{y}_{0;\alpha} \tag{61}$$

$$\Delta\hat{\varphi} = \hat{\varphi}_\beta - \hat{\varphi}_\alpha \tag{62}$$

where $\hat{y}_{0;\alpha}$, $\hat{y}_{0;\beta}$, $\hat{\varphi}_\alpha$, $\hat{\varphi}_\beta$ are M_{split} estimates of y_0 and φ, respectively.

Let the following vector be a vector of the wave observations:

$$\mathbf{y} = [2.706 \quad 2.834 \quad 2.995 \quad 2.983 \quad 2.704 \quad 2.555 \quad 2.005 \quad 1.803 \quad 1.289 \quad 1.174$$
$$0.998 \quad 1.023 \quad 1.290 \quad 1.446 \quad 1.997 \quad 2.200]^T$$

which is simulated under the assumptions that $y_0 = 2.000$, $\varphi_\alpha = \pi/16$, $\varphi_\beta = 2\pi/16$. Hence, the theoretical phase shift is equal to $\mathit{\Delta}_\varphi = \pi/16$, and the theoretical shift of the constant y_0 is equal to $\mathit{\Delta}_{y_0} = 0.000$. The iterative processes of M_{split} estimation and Shift-M_{split} estimation for the assumed vector $\mathbf{y}$ are presented in Table 1.

Table 1. The course of the iterative processes of M_{split} and Shift-M_{split}estimations of the vector $\mathbf{\Delta_X} = [\Delta_{y_0},\ \Delta_\varphi]^{\mathrm{T}}$ for assumed values: $y_0 = 2.000$, $\Delta_{y_0} = 0$, $\varphi_\alpha = \pi/16$, $\varphi_\beta = 2\pi/16$, $\Delta_\varphi = \pi/16$

	M_{split}				Shift-M_{split}	
Step	$\mathbf{g}_\alpha(\mathbf{X}_\alpha^{j-1}, \mathbf{X}_\beta^{j-1})$ $\mathbf{g}_\beta(\mathbf{X}_\alpha^{j}, \mathbf{X}_\beta^{j-1})$	$d\mathbf{X}_\alpha^j$ $d\mathbf{X}_\beta^j$	$\mathbf{X}_\alpha^j$ $\mathbf{X}_\beta^j$	$\Delta\mathbf{X}^j$ $= \mathbf{X}_\beta^j - \mathbf{X}_\alpha^j$	$\mathbf{b}(\mathbf{v}_\alpha^{j-1})$ $\mathbf{b}(\mathbf{v}_\beta^{j-1})$	$\mathbf{\Delta}_{\mathbf{X};\alpha}^j$ $\mathbf{\Delta}_{\mathbf{X};\beta}^j$
LS	32.0020		2.0001			
	2.3130		0.2891			
0_β	0.0000	0.0006	2.0007	0.0006	0.0000	0.0006
	-0.0010	0.0246	0.3137	0.0246	-0.0014	0.0246
1_α	0.0000	-0.0010	1.9991		0.0000	0.0010
	0.0029	-0.0477	0.2414		0.0029	0.0477
1_β	0.0000	0.0004	2.0011	0.0020	0.0000	0.0017
	-0.0050	0.0718	0.3855	0.1441	-0.0050	0.0900
2_α	-0.0004	0.0019	2.0010		0.0000	0.0013
	0.0041	-0.0370	0.2044		0.0117	0.1410
2_β	0.0000	-0.0021	1.9990	-0.0020	0.0000	0.0007
	-0.0010	0.0135	0.3990	0.1946	-0.0090	0.1384
3_α	-0.0001	0.0007	2.0017		-0.0004	-0.0007
	0.0001	-0.0009	0.2036		0.0194	0.1781
3_β	0.0000	-0.0002	1.9989	-0.0028	0.0010	-0.0022
	-0.0000	0.0000	0.3990	0.1954	-0.0190	0.1942
4_α	0.0000	0.0000	2.0017		-0.0014	-0.0027
	0.0000	0.0000	0.2036		0.0251	0.1954
4_β	0.0000	0.0000	1.9988	-0.0029	0.0010	-0.0028
	0.0000	0.0000	0.3990	0.1954	-0.0200	0.1955
5_α	0.0000	0.0000	2.0017		-0.0014	-0.0029
	0.0000	0.0000	0.2036		0.0251	0.1954
5_β	0.0000	0.0000	1.9988	-0.0029	0.0010	-0.0029
	0.0000	0.0000	0.3990	0.1954	-0.0200	0.1954

Theoretical value	Results	
	M_{split} estimators	Shift-M_{split} estimators
$y_0 = 2.000$	$\hat{y}_{0;\alpha} = 2.0017$ $\hat{y}_{0;\beta} = 1.9988$	
$\Delta_{y_0} = 0.000$	$\Delta\hat{y}_0 = \hat{y}_{0;\beta} - \hat{y}_{0;\alpha} = -0.0029$	$\hat{\Delta}_{y_0;\alpha} = \hat{\Delta}_{y_0;\beta} = \hat{\Delta}_{y_0;\mathrm{split}} = -0.0029$
$\varphi_\alpha = \pi/16 = 0.196$ $\varphi_\beta = 2\pi/16 = 0.393$	$\hat{\varphi}_\alpha = 0.2036$ $\hat{\varphi}_\beta = 0.3990$	
$\Delta_\varphi = \pi/16 = 0.196$	$\Delta\hat{\varphi} = \hat{\varphi}_\beta - \hat{\varphi}_\alpha = 0.1954$	$\hat{\Delta}_{\varphi;\alpha} = \hat{\Delta}_{\varphi;\beta} = \hat{\Delta}_{\varphi;\mathrm{split}} = 0.1954$

The graphical interpretation of the theoretical assumptions and the estimation results are also shown in Figure 1.

Now, let us consider other values of φ_α and φ_β, hence other values of the phase shift (the other assumptions remain the same). The results of M_{split} and Shift-M_{split} estimations are presented in Table 2. Additionally, Table 2 presents the differences e between theoretical values and their estimates, namely *for* M_{split} *estimates*:

$$e_{\hat{\varphi}_\alpha} = \varphi_\alpha - \hat{\varphi}_\alpha, \quad e_{\hat{\varphi}_\beta} = \varphi_\beta - \hat{\varphi}_\beta, \quad e_{\Delta\hat{\varphi}} = \Delta_\varphi - \Delta\hat{\varphi}$$

$$e_{\hat{y}_{0;\alpha}} = y_0 - \hat{y}_{0;\alpha}, \quad e_{\hat{y}_0;\beta} = y_0 - \hat{y}_{0;\beta}, \quad e_{\Delta\hat{y}_0} = \Delta_{y_0} - \Delta\hat{y}_0$$

for Shift-M_{split} *estimates*:

$$e_{\hat{\Delta}_\varphi} = \Delta_\varphi - \hat{\Delta}_{\varphi;\mathrm{split}}$$

$$e_{\hat{\Delta}_{y_0}} = \Delta_{y_0} - \hat{\Delta}_{y_0;\mathrm{split}}$$

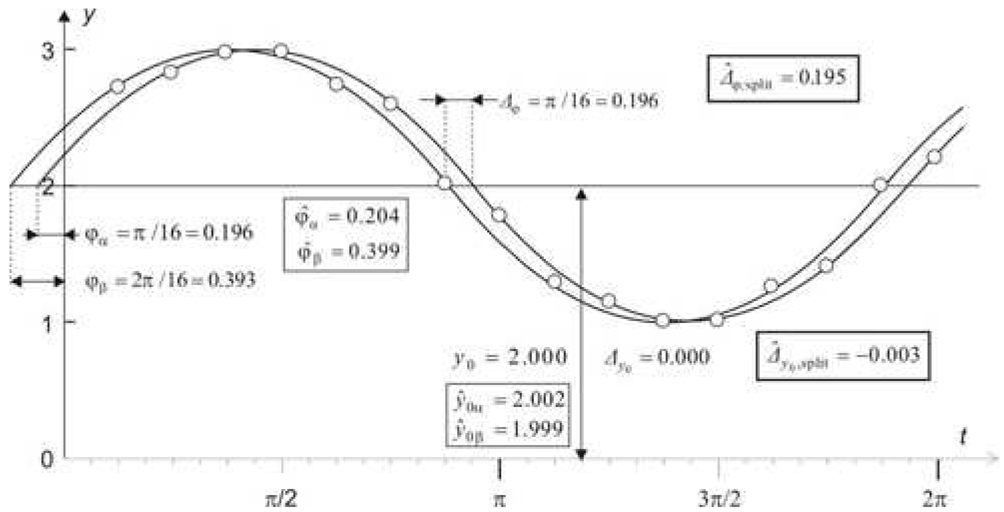

Fig. 1. Simulated observations of the waves (o) and the results of M_{split} and Shift-M_{split} estimations

The assumed linear models of Eqs. (57) and (58) are not suitable for bigger values of φ_α and φ_β, however, the results obtained are close to the theoretical values. Thus this assumption did not influence the estimation process in a significant way, however, one can note that usually the larger values of the phase shift are, the bigger the differences e are obtained.

3.2. Example 2

The shift of the parameters is often the basis in deformation analysis and sometimes it can be regarded as displacements of network points. As it was already mentioned, there are several methods used for such purposes (e.g. Caspary, 1988; Chen, 1983; Singh and Setan, 2001). Some examples were also presented in the papers that concern M_{split} estimation (Wiśniewski, 2009a, 2009c, 2010). The example presented here refers to Example 4, section 4 in (Wiśniewski, 2009a) and Example 4, section 3 in (Wiśniewski, 2009c).

Table 2. The final estimates and their errors for the different values of the phase offsets φ_α and φ_β and for the constant y_0 = 2.000 (Δ_{y_0} = 0.000)

Theoretical values	M_{split}				Shift$-M_{split}$			
φ_α φ_β Δ_φ	$\hat{\varphi}_\alpha$ $\hat{\varphi}_\beta$ $\Delta\hat{\varphi}$	$e_{\hat{\varphi}_\alpha}$ $e_{\hat{\varphi}_\beta}$ $e_{\Delta\hat{\varphi}}$	$\hat{y}_{0;\alpha}$ $\hat{y}_{0;\beta}$ $\Delta\hat{y}_0$	$e_{\hat{y}_0;\alpha}$ $e_{\hat{y}_0;\beta}$ $e_{\Delta\hat{y}_0}$	$\hat{\Delta}_{\varphi;\text{split}}$	$e_{\hat{\Delta}_\varphi}$	$\hat{\Delta}_{y_0;\text{split}}$	$e_{\hat{\Delta}_{y_0}}$
1/32 π = 0.0982 2/32 π = 0.1963 1/32 π = 0.0982	0.1010 0.1955 0.0945	0.0028 -0.0008 -0.0037	2.0021 1.9990 -0.0031	0.0021 -0.0010 -0.0031	0.0945	**-0.0037**	-0.0030	**-0.0030**
1/32 π = 0.0982 3/32 π = 0.2945 2/32 π = 0.1963	0.1029 0.2945 0.1916	0.0047 -0.0001 -0.0048	2.0019 1.9991 -0.0028	0.0019 -0.0009 -0.0028	0.1916	**-0.0048**	-0.0028	**-0.0028**
1/32 π = 0.0982 4/32 π = 0.3927 3/32 π = 0.2945	0.1060 0.2937 0.2878	0.0078 0.0010 -0.0068	2.0018 1.9990 -0.0028	0.0018 -0.0010 -0.0028	0.2878	**-0.0068**	-0.0028	**-0.0028**
1/16 π = 0.1963 2/16 π = 0.3927 1/16 π = 0.1963	0.2036 0.3990 0.1954	0.0072 0.0063 -0.0009	2.0017 1.9988 -0.0029	0.0017 -0.0012 -0.0029	0.1954	**-0.0009**	-0.0029	**-0.0029**
1/16 π = 0.1963 3/16 π = 0.5890 2/16 π = 0.3927	0.2150 0.6024 0.3874	0.0187 0.0134 -0.0053	2.0014 1.9986 -0.0028	0.0014 -0.0014 -0.0028	0.3874	**-0.0053**	-0.0028	**-0.0028**
1/16 π = 0.1963 4/16 π = 0.7854 3/16 π = 0.5890	0.2306 0.8074 0.5768	0.0343 0.0220 -0.0123	2.0011 1.9982 -0.0028	0.0011 -0.0018 -0.0029	0.5768	**-0.0123**	-0.0028	**-0.0028**
1/8 π = 0.3927 2/8 π = 0.7854 1/8 π = 0.3927	0.4061 0.8376 0.4315	0.0134 0.0522 0.0388	2.0009 1.9971 -0.0039	0.0009 -0.0029 -0.0039	0.4315	**0.0388**	-0.0039	**-0.0039**
1/8 π = 0.3927 3/8 π = 1.1781 2/8 π = 0.7854	0.4250 1.2485 0.8238	0.0323 0.0704 0.0381	2.0006 1.9958 -0.0048	0.0006 -0.0042 -0.0048	0.8235	**0.0381**	-0.0048	**-0.0048**
1/8 π = 0.3927 4/8 π = 1.5708 3/8 π = 1.1781	0.3948 1.5694 1.1746	0.0021 -0.0014 -0.0035	2.0007 1.9941 -0.0066	0.0007 -0.0059 -0.0069	1.1746	**-0.0035**	-0.0068	**-0.0068**

The levelling network that we are interested in is presented in Figure 2. Two network points named as P_1 and P_2 are the fixed ones with heights $H_{P_1} = 0.00$ and $H_{P_2} = 0.00$. There are also three unknown points named as A, B and C.

Let the height differences h_i, $i = 1, 2, \ldots, 8$, be measured two times at two different epochs (1) and (2). Two following functional models correspond to the measurement epochs

$$\mathbf{v}_1 = \mathbf{y}_1 - \bar{\mathbf{A}}\mathbf{H}_1 \tag{63}$$

$$\mathbf{v}_2 = \mathbf{y}_2 - \bar{\mathbf{A}}\mathbf{H}_2 \tag{64}$$

where

$$\bar{\mathbf{A}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

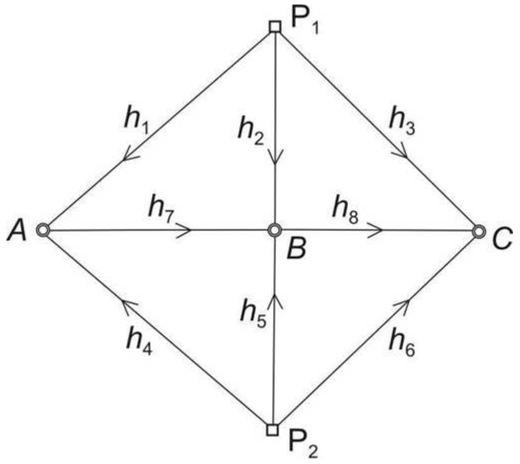

Fig. 2. Tested levelling network

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are errors of the following observation vectors (h_{ij} is the i^{th} height difference measured at the epoch $j = 1, 2$):

$$\mathbf{y}_1 = [h_{11}, h_{21}, h_{31}, h_{41}, h_{51}, h_{61}, h_{71}, h_{81}]$$

$$\mathbf{y}_2 = [h_{12}, h_{22}, h_{32}, h_{42}, h_{52}, h_{62}, h_{72}, h_{82}]$$

The parameter vectors of the models from Eqs. (63) and (64), namely $\mathbf{H}_1 = [H_{A1}, H_{B1}, H_{C1}]^{\mathrm{T}}$ and $\mathbf{H}_2 = [H_{A2}, H_{B2}, H_{C2}]^{\mathrm{T}}$ are vectors of the point heights in two measurement epochs. The functional models from Eqs. (63) and (64) are the basis for the traditional model of network deformation

$$\mathbf{\Delta v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{y}_2 - \mathbf{y}_1 - \bar{\mathbf{A}}(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{y}_2 - \mathbf{y}_1 - \bar{\mathbf{A}}\mathbf{\Delta}_{\mathbf{H}} \tag{65}$$

where $\mathbf{\Delta}_{\mathbf{H}} = \mathbf{H}_2 - \mathbf{H}_1 = [\varDelta_{H_A}, \varDelta_{H_B}, \varDelta_{H_C}]^{\mathrm{T}}$ is the shift of the point heights. The *LS* estimator of this shift will be denoted later as $\hat{\mathbf{\Delta}}_{\mathbf{H};LS}$.

Now, let the shift vector $\mathbf{\Delta}_H$ be estimated by applying Shift-M_{split} estimation. First, let the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ combine to form one observation vector

$$\mathbf{y} = [h_{11}, h_{12}, h_{21}, h_{22}, h_{31}, h_{32}h_{41}, h_{42}, h_{51}, h_{52}, h_{61}, h_{62}, h_{71}, h_{72}, h_{81}, h_{82}]^{\mathrm{T}}$$

Such combined vector corresponds to two competitive functional models

$$\mathbf{v}_\alpha = \mathbf{y} - \mathbf{A}\mathbf{H}_\alpha \tag{66}$$

$$\mathbf{v}_\beta = \mathbf{y} - \mathbf{A}\mathbf{H}_\beta \tag{67}$$

where $\mathbf{A} = [\bar{\mathbf{a}}_1^{\mathrm{T}}, \bar{\mathbf{a}}_1^{\mathrm{T}} \vdots\, \bar{\mathbf{a}}_2^{\mathrm{T}}, \bar{\mathbf{a}}_2^{\mathrm{T}} \vdots \ldots \vdots\, \bar{\mathbf{a}}_6^{\mathrm{T}}, \bar{\mathbf{a}}_6^{\mathrm{T}}]^{\mathrm{T}}$ is formed by repeating certain rows of the matrix $\bar{\mathbf{A}}$. Since the observation vector $\mathbf{y}$ is the same in both models of Eqs. (66) and (67) then the shift model (65) can be written as

$$\Delta \mathrm{v} = \mathbf{v}_\beta - \mathbf{v}_\alpha = \mathbf{y} - \mathbf{y} - \mathbf{A}(\mathbf{H}_\beta - \mathbf{H}_\alpha) = -\mathbf{A}\mathbf{\Delta}_{\mathbf{H}} \tag{68}$$

The observation vector **y** is simulated under assumptions that measurement errors are Gaussian errors with the mean of 0 and standard deviation of 0.01, and the theoretical heights of the points A, B and C at the epoch (1) are as follows $H_{A1} = 1.0$, $H_{B1} = 1.0$, $H_{C1} = 1.0$, i.e. $\mathbf{H}_1 = [1.0,\ 1.0,\ 1.0]^{\mathrm{T}}$. It is also assumed that some of the network points moved downwards at the epoch (2) and let us consider two following variants of such displacements (see Wiśniewski, 2009c)

Variant 1: $\boldsymbol{\Delta}^t_{\mathbf{H}} = \mathbf{H}_2 - \mathbf{H}_1 = [0.0,\ 0.0,\ -\underline{1.0}]^{\mathrm{T}}$

Variant 2: $\boldsymbol{\Delta}^t_{\mathbf{H}} = \mathbf{H}_2 - \mathbf{H}_1 = [0.0,\ -\underline{1.0},\ -\underline{1.0}]^{\mathrm{T}}$

where $\boldsymbol{\Delta}^t_{\mathbf{H}}$ is a vector of the theoretical displacements (underlined – the values for the displaced points). The results of Shift-M_{split} estimation for both proposed variants are presented in Table 3.

Table 3. The course of the iterative process of Shift-M_{split} estimation

Variant 1: $\mathbf{y} = [1.01,\ 0.98 \vdots 1.00,\ 1.02 \vdots 0.98,\ 0.01 \vdots 0.97,\ 0.99 \vdots 1.00,\ 1.01 \vdots 0.99,\ -0.01 \vdots 0.02,\ -0.01 \vdots -0.01, -1.01]^{\mathrm{T}}$; $\boldsymbol{\Delta}^t_{\mathbf{H}} = [0.0,\ 0.0,\ -\underline{1.0}]^{\mathrm{T}}$											
	step j										
	0	1	2	3	4	5	6	7	8	9	10
$\boldsymbol{\Delta}^j_{\mathbf{H}_\alpha}$	0.000	0.006	0.013	-0.046	-0.181	-0.298	-0.275	-0.177	-0.005	0.032	**0.033**
	0.000	-0.006	-0.026	-0.101	-0.332	-0.533	-0.544	-0.412	-0.061	0.015	**0.015**
	0.000	-0.002	-0.010	-0.039	-0.130	-0.255	-0.431	-0.771	-0.977	-0.985	**-0.985**
$\boldsymbol{\Delta}^j_{\mathbf{H}_\beta}$	0.003	0.010	-0.003	-0.078	-0.174	-0.270	-0.275	-0.178	0.009	0.031	**0.033**
	-0.003	-0.013	-0.051	-0.194	-0.467	-0.549	-0.509	-0.230	0.008	0.015	**0.015**
	-0.001	-0.005	-0.020	-0.074	-0.196	-0.325	-0.586	-0.916	-0.985	-0.985	**-0.985**
Variant 2: $\mathbf{y} = [1.01,\ 0.98 \vdots 1.00,\ 0.02 \vdots 0.98,\ 0.01 \vdots 0.97,\ 0.99 \vdots 1.00,\ 0.01 \vdots 0.99,\ -0.01 \vdots 0.02,\ -1.01 \vdots -0.01,\ 0.01]^{\mathrm{T}}$; $\boldsymbol{\Delta}^t_{\mathbf{X}} = [0.0,\ -\underline{1.0},\ -\underline{1.0}]^{\mathrm{T}}$											
$\boldsymbol{\Delta}^j_{\mathbf{H}_\alpha}$	0.000	0.071	0.255	0.500	0.458	0.336	0.264	0.244	0.074	0.045	**0.045**
	0.000	0.035	0.130	0.290	0.326	0.225	-0.006	-0.487	-0.946	-0.985	**-0.985**
	0.000	-0.018	-0.071	-0.224	-0.438	-0.598	-0.748	-0.916	-0.984	-0.985	**-0.985**
$\boldsymbol{\Delta}^j_{\mathbf{H}_\beta}$	0.036	0.140	0.408	0.505	0.394	0.289	0.259	0.160	0.046	0.045	**0.045**
	0.018	0.070	0.217	0.329	0.289	0.129	-0.209	-0.779	-0.984	-0.985	**-0.985**
	-0.009	-0.037	-0.134	-0.335	-0.525	-0.669	-0.832	-0.967	-0.985	-0.985	**-0.985**

Similar results can also be obtained if the shift is estimated by the traditional *LS*-method, and by applying the model of Eq. (65). Let us now consider the case where the observation vector is affected by gross errors. To investigate this problem let us consider the first variant presented above and let the height difference h_{32} be disturbed with a gross error g. The values of the gross error are chosen in such a way that shows how the method proposed here responds to such disturbances. The results of both *LS*-estimation ($\hat{\boldsymbol{\Delta}}_{\mathbf{H};LS}$) and Shift-$M_{split}$ estimation ($\hat{\boldsymbol{\Delta}}_{\mathbf{H};split}$) are listed in Table 4.

The results show that both methods are not robust against gross errors, however, they respond to such disturbances in different ways. The gross error affects the *LS* estimates in a well known way, the bigger it is, the larger the influence becomes. For the Shift-M_{split} estimates, the influence grows bigger first but then it decreases. This reflects of how the observation h_{32} "moved" from the second group to the first one, namely if $g = 0.00$ then this observation is assigned to the second epoch but in the last variant where $g = 1.00$ the observation in question just "ignores" the displacement of the point C and it is regarded as the observation from the first epoch. Thus, in this case, the gross error does not affect the estimation process.

Table 4. *LS* and Shift-M_{split} estimates of the shift where observations are affected by gross errors

	Without a gross error	With the gross error g					
g	0.00	0.05	0.25	0.50	0.75	0.95	1.00
$\hat{\mathbf{\Delta}}_{\mathbf{H};LS}$	0.008 0.005 -0.988	0.010 0.010 -0.970	0.017 0.030 0.897	0.025 0.055 -0.805	0.033 0.080 -0.731	0.040 0.100 -0.640	0.042 0.105 -0.622
$\hat{\mathbf{\Delta}}_{\mathbf{H};\mathrm{split}}$	0.033 0.015 -0.985	-0.006 0.038 -0.961	0.049 0.115 -0.884	0.077 0.144 -0.853	0.039 0.086 -0.913	0.033 0.014 -0.986	0.030 0.003 -1.003

4. Conclusions

The method that is proposed in the present paper is based on squared M_{split} estimation that was presented in (Wiśniewski, 2009a, 2009c). Hence, the theoretical as well as practical properties of Shift-M_{split} estimates correspond to the general properties of squared M_{split} estimator.

The optimization problem of M_{split} estimation in relation to the shift of parameters is solved by the use of two equivalent systems of normal equations (see, Eq. (38) and Eq. (39)), which follow sufficient conditions of optimality. These systems are directly related to the system of normal equations in *LS*-method. Thus, the optimization problem of Shift-M_{split} estimation is solved without significant numerical problems. The present paper shows that M_{split} estimation, in the form that is proposed here, is not only a theoretical development of *LS*-method. It may also be solved in a similar practical way. The alternate iterative process, namely solving the equivalent systems of normal equations alternately, is in fact equivalent to *LS*-method that is carried out in two parallel processes with two different objective functions from Eqs. (36) and (37). These functions directly point at mutual cross-weighting of the competitive functional models. This property of M_{split} estimation was discussed, mainly from a theoretical point of view, in the papers (Wiśniewski, 2009a, 2009b).

The numerical examples that are presented in this paper are related to the real surveying problems. However, since the present paper focuses on theory rather than practice, they are not complete solutions of these problems. The estimation of the phase shift, for example in GPS techniques, requires more developed functional models. On the other hand, the use of such models would obscure the main properties of Shift-M_{split} estimation, and the diagnosis of such properties is, at this stage of research, the primary objective. It is also worth noting that the estimate values are very close to the assumed theoretical shifts. However, the estimation results are less similar to the theoretical values when the phase shift grows bigger (see Table 2). This results from the fact that the nonlinear wave model is expanded in a Taylor series confined to the first terms only. This example points also at small sensitiveness of Shift-M_{split} estimates to "fuzzy" differences between elements of the split observation set (see Fig. 1).

The results obtained in the second example are also very interesting from a practical point of view. This example illustrates possible application of Shift-M_{split} estimates in deformation analyses. It shows the iterative process and the organization of computations rather than gives a complete method of how to apply Shift-M_{split} estimation in such surveying problems. In practice, such application needs additional analyses that should concern, for example, comparison with conventional methods. One should pay special attention to two main properties of Shift-M_{split} estimates, namely the independence from the way of ordering of the observation set and in special cases robustness against outliers. In the case of geodetic networks, which are measured twice at two different epochs, there is no problem to arrange observations properly. If additionally no observation is affected by a gross error, then results of Shift-M_{split} estimation will be similar to traditional *LS*-estimates. This equivalence is true if only both presented conditions are met. The second property, namely robustness, needs some comments. In general, it is a natural property of all M_{split} estimates. This is due to the split of the observation set (the observations are also divided into "matching" and "outlying"). However, the presented examples show that Shift-M_{split} estimates are usually affected by gross errors and the method is robust only in special cases. Thus this special case of M_{split} estimates is generally not robust against gross errors. The possible robustness of Shift-M_{split} estimates requires other theoretical assumptions, hence also new solutions, which goes beyond the scope of this paper. It should also be studied in relation to inner reliability that is defined and described in the papers (Prószyński, 1997, 2010).

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Estymacja Shift-M_{split}

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Streszczenie

Przedstawiona w pracy metoda jest szczególnym przypadkiem kwadratowej M_{split} estymacji. Dotyczy ona bezpośredniej estymacji przesunięcia między wartości parametrów występujących w funkcjonalnych modelach obserwacji. Takie przesunięcie (shift) może na przykład wynikać z deformacji sieci geodezyjnej lub innych nielosowych zakłóceń obciążających jej współrzędne. W pracy przedstawiono także przykład, w którym shift jest utożsamiany z przesunięciem fazowym fali. Przesunięcie to jest estymowane na podstawie pomiarów wartości fali, realizowanych bez informacji o punktach, w których takie przesunięcie występuje. Zaproponowane estymatory nazwano ogólnie Shift-M_{split} estymatorami.