ON DETERMINATION OF THE STRESS-STRAIN STATE OF A MULTI-WEDGE SYSTEM WITH THIN RADIAL DEFECTS UNDER ANTIPLANE DEFORMATION

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An elastic isotropic multi-wedge system with radially located thin defects under longitudinal shear is considered. The procedure of construction of asymptotics of the stress and displacement fields in the vicinity of the system apex using the apparatus of generalized functions and Mellin transform is presented. The notion of generalized stress intensity factor near the wedge system apex is introduced. The procedure proposed is applied to determine analytically the asymptotic distribution of stress and displacement fields in the three-wedge system peak. The generalized stress intensity factor near the three-wedge system apex is analyzed.

Keywords: wedge system, angular point, singularity order, stress assymptotics, generalized stress intensity factor.

1. INTRODUCTION

Modeling of mechanical systems based on the linear elasticity theory sometimes requires consideration the surfaces with angular points. It leads to the fact that some parameters of physico-mechanical fields are described by singular expressions. Generally speaking the presence of singularity conflicts with initial assumptions of the model of elastic continuum and results of contradictions that are included in the mathematical model of problems. Besides the solutions with singularities give the authentic qualitative picture of distribution and quantitative characteristics of the field outside of some very small vicinity of singularity point. In the cases when in the vicinity of irregular point the finite integral characteristics can be determined they are utilized successfully to analyze the physical properties of the field [6].

The analysis of physico-mechanical fields in the vicinity of angular points of material interface is carried out on the model regions, namely on the systems composed of a certain number of coupled wedges having one common point. In addition mainly the methods of complex Kolosov-Muskhelishvili potentials [3, 5], Airy stress functions [1, 8, 11] and the method of singular integral equations [12] are used. However it yields the cumbersome expressions which complicate obtaining the analytical writing of asymptotics of the stress-strain state even for the two-wedged composite. Therefore as a rule the attention is restricted to study only order of the stress field singularity. The Mellin transform use [14, 15, 16] somewhat simplifies the general form of expression and makes it possible to determine the stress intensity factors in the vicinity of the wedge system apex in special cases of antiplane problem of elasticity theory for two wedges [15]. But the problems with writing the expressions describing the physico-mechanical fields in the vicinity of the system peak, the tips of thin interphase defect and in the whole region occupied by a multi-wedge composite still remain. The same questions arise also during description of the fields of other physical nature (in particular, electromagnetic field) in a multi-component wedge system [9, 10].

In this paper the authors propose an efficient approach to solution of the problem on the stress-strain state of multi-wedge system with radially located thin inclusions under longitudinal shear. The approach bases on the method of generalized conjugate problem for the piecewise-homogeneous media, method of jump function, application of apparatus of generalized functions and it makes possible to write the stress and displacement fields in the composite composed of arbitrary number of wedges. Its utilization is illustrated on the example of elucidation of distribution of the stress field near the point of convergence of three wedges, loaded by concentrated force, with opening angle at the tip $\alpha_i = 2\pi/3$ $(i = 1,2,3)$ under conditions of the first and second boundary-value problems.

2. FORMULATION OF A CONJUGATE PROBLEM FOR PIECEWISE-HOMOGENEOUS WEDGE SYSTEM

Consider a composite composed of an arbitrary number of heterogeneous isotropic coupled wedges S_i $(i = 1,2,...,n)$ with opening angles at the tip $\alpha_i (\alpha_1 + \alpha_2 + ... + \alpha_n \le 2\pi)$ and wedge-shaped notch S_{n+1} (Fig. 1) which is under the longitudinal shear $u = 0$, $v = 0$, $w = w(r,\varphi)$. Loading of the notch edges is described by the corresponding boundary conditions. Thin linear defects occupying the region $r \in [a_i; b_i]$ are on the coupling lines of the wedges

$\varphi = \varphi_i = \alpha_1 + \alpha_2 + ... + \alpha_i$. Their presence is modeled by the jump functions [16, 17] $f_{\sigma i}(r)$, $f_{wi}(r)$ and the generalized function [7] as

$$\sigma_{\varphi z}\big|_{\varphi_i+0} - \sigma_{\varphi z}\big|_{\varphi_i-0} = f_{\sigma i}(r)\phi(a_i,b_i), \quad w\big|_{\varphi_i+0} - w\big|_{\varphi_i-0} = f_{wi}(r)\phi(a_i,b_i),$$

$$\phi(a_i,b_i) = \left[S_+(r-a_i) - S_+(r-b_i) \right], \quad S_+(\xi) = \{1\,(\xi > 0);\ 0\,(\xi \le 0)\}. \tag{1}$$
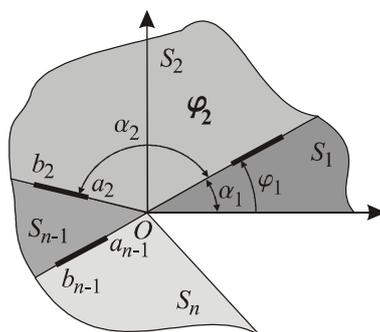

Fig. 1. General scheme of the problem

For convenience all transformations are realized in the polar coordinate system r, φ with the system apex as center point O. Then in each of wedges S_i $\left(i = \overline{1,n}\right)$ that form the system the Cauchy relations, Hook's law and equilibrium equation that reads

$$\mu_i \Delta w_i = \mu_i \frac{\partial^2 w_i}{\partial r^2} + \frac{\mu_i}{r} \frac{\partial w_i}{\partial r} + \frac{\mu_i}{r^2} \frac{\partial^2 w_i}{\partial^2 \varphi} = 0 \ \left(i = \overline{1,n-1}\right) \tag{2}$$

are realized and on the line of wedge coupling $\varphi = \varphi_i$ the conjugation conditions

$$\left(\sigma_{\varphi z}^{i+1} - \sigma_{\varphi z}^i\right)\Big|_{\varphi=\varphi_i} = f_{\sigma i}(r)\phi(a_i,b_i), \quad \left(w_{i+1} - w_i\right)\Big|_{\varphi=\varphi_i} = f_{wi}(r)\phi(a_i,b_i) \tag{3}$$

are satisfied.

Here w_i, σ_{rz}^i, $\sigma_{\varphi z}^i$ are displacements and stresses in the wedge S_i; μ_i is the shear modulus of the wedge S_i material.

Depending on the load type the boundary conditions are given on the system surfaces.

Thus following the procedure given in [7] the wedge system is to be considered as an integral region $S = S_1 \bigcup S_2 \bigcup ... \bigcup S_n$ composed of an arbitrary

number n of regions S_i within the limits of which the Cauchy conditions, Hooke's law and the equilibrium conditions (2) are realized and on the boundary surfaces $\varphi = \varphi_i$ the conjugation conditions (3) are given.

Extend the displacement function $w_i(r, \varphi)$, differential operator $\dfrac{\partial^2 w_i}{\partial r^2}$, $\dfrac{\partial w_i}{\partial r}$ and shear moduli μ_i which are constant in the region S_i to the whole region S in the form

$$\left\{ \frac{\partial^2 w_i}{\partial r^2}, \frac{\partial w_i}{\partial r}, w_i, \mu_i \right\} \quad f(r, \varphi) = f_1 + \sum_{i=1}^{n-1} (f_{i+1} - f_i) S_+(\varphi - \varphi_i) \tag{4}$$

Using the connection between the generalized and classical derivatives [6] and the conjugation conditions (3) we obtain a partly degenerated differential equation

$$\Delta w(r, \varphi) = \frac{1}{r^2} \sum_{i=1}^{n-1} C_1^i(r) \delta_+'(\varphi - \varphi_i) + \frac{1}{r^2} \sum_{i=1}^{n-1} C_2^i(r) \delta_+(\varphi - \varphi_i), \tag{5}$$

where

$$C_1^i(r) = f_{wi}(r) \left[S_+(r - a) - S_+(r - b) \right]$$

$$C_2^i(r) = \frac{2r f_{wi}(r)}{\mu_{i+1}} \left[S_+(r - a) - S_+(r - b) \right] - \frac{\mu_{i+1} - \mu_i}{\mu_{i+1}} \frac{\partial w}{\partial \varphi} \bigg|_{\varphi = \varphi_i - 0}$$

with the following boundary conditions:
1) in the case of the first boundary-value problem –

$$\frac{\partial w}{\partial \varphi} \bigg|_{\varphi=0} = \frac{r}{\mu_1} \tau_0(r), \quad \frac{\partial w}{\partial \varphi} \bigg|_{\varphi=\varphi_n} = \frac{r}{\mu_n} \tau_{n+1}(r); \tag{6}$$

2) in the case of the second boundary-value problem –

$$w \big|_{\varphi=0} = w_0(r), \quad w \big|_{\varphi=\varphi_n} = w_{n+1}(r); \tag{7}$$

3) in the case of a mixed boundary-value problem two variants are possible -

$$\text{a)} \quad \frac{\partial w}{\partial \varphi} \bigg|_{\varphi=0} = \frac{r}{\mu_1} \tau_0(r), \quad w \big|_{\varphi=\varphi_n} = w_{n+1}(r), \tag{8}$$

$$b)\ w\big|_{\varphi=0}=w_0\left(r\right),\quad \frac{\partial w}{\partial \varphi}\bigg|_{\varphi=\varphi_n}=\frac{r}{\mu_n}\tau_{n+1}\left(r\right).\tag{9}$$

The partly-degenerated equation (5) together with boundary conditions (6) - (9) we shall call (similarly to [6]) a generalized conjugate problem as regards the wedge composite with thin radial defects at longitudinal shear.

Thus elucidation of the stress-strain state in a wedge system under longitudinal shear is reduced to solution of equation (5) with corresponding boundary conditions (6) - (9)

3. CONSTRUCTION OF SOLUTION TO THE GENERALIZED CONJUGATE PROBLEM

Having applied the Mellin transform to equation (5) we proceed to solution of the problem

$$\frac{\partial^2 \tilde{w}}{\partial \varphi}+p^2\tilde{w}=\sum_{i=1}^{n-1}\tilde{C}_1^i\left(r\right)\delta_+'\left(\varphi-\varphi_i\right)+\sum_{i=1}^{n-1}\tilde{C}_2^i\left(r\right)\delta_+\left(\varphi-\varphi_i\right)\tag{10}$$

in the space images

where $\qquad \tilde{C}_1^i\left(p\right)=\tilde{f}_{wi}\left(p\right),\qquad \tilde{C}_2^i\left(p\right)=\dfrac{\tilde{f}_{\sigma i}\left(p+1\right)}{\mu_{i+1}}-\dfrac{\mu_{i+1}-\mu_i}{\mu_{i+1}}\dfrac{\partial \tilde{w}}{\partial \varphi}\bigg|_{\varphi=\varphi_i-0},$

$\tilde{f}_{ki}=\int\limits_0^\infty f_{ki}\left(r\right)\phi\left(a_i,b_i\right)r^{p-1}dr\ \ \left(k=w,\sigma\right),\ \tilde{\psi}=\int\limits_0^\infty \psi r^{p-1}dr\qquad$ is the Mellin transform of the corresponding functions.

The general solution of equation (10) is of the form

$$\tilde{w}\left(p,\varphi\right)=A_1\left(p\right)\left(\cos p\varphi-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_1^i\sin\left[p\left(\varphi-\varphi_i\right)\right]S_+\left(\varphi-\varphi_i\right)\right)+$$

$$+B_1\left(p\right)\left(\sin p\varphi-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_2^i\sin\left[p\left(\varphi-\varphi_i\right)\right]S_+\left(\varphi-\varphi_i\right)\right)+$$

$$\begin{aligned}&+\sum_{i=1}^{n-1}\left(\tilde{f}_{wi}\left(p\right)\cos\left[p\left(\varphi-\varphi_i\right)\right]+\frac{\tilde{f}_{\sigma i}\left(p+1\right)}{p\mu_{i+1}}\sin\left[p\left(\varphi-\varphi_i\right)\right]-\right.\\&\left.-\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_3^i\sin\left[p\left(\varphi-\varphi_i\right)\right]\right)S_+\left(\varphi-\varphi_i\right);\end{aligned}\tag{11}$$

$$L_1^i = -p\sin p\varphi_i - \sum_{k=1}^{i-1} \frac{\mu_{k+1} - \mu_k}{\mu_{k+1}} L_1^k \cos\big[p(\varphi_i - \varphi_k) \big],$$

$$L_2^i = p\cos p\varphi_i - \sum_{k=1}^{i-1} \frac{\mu_{k+1} - \mu_k}{\mu_{k+1}} L_2^k \cos\big[p(\varphi_i - \varphi_k) \big], \tag{12}$$

$$L_3^1 = 0, \; L_3^i = \sum_{k=1}^{i-1} \Bigg(\frac{\tilde{f}_{\sigma k}(p+1)}{\mu_{k+1}} \cos\big[p(\varphi_i - \varphi_k) \big] -$$

$$- p\tilde{f}_{wk}(p)\sin\big[p(\varphi - \varphi_i) \big] - \frac{\mu_{k+1} - \mu_k}{\mu_{k+1}} L_3^k \cos\big[p(\varphi_i - \varphi_k) \big] \Bigg).$$

Depending on the load nature on the notch surface, to determine the functions $A_1(p)$, $B_1(p)$ the following expressions are written:

1) if the boundary conditions are in the form $\dfrac{\partial \tilde{w}}{\partial \varphi}\Big|_{\varphi=0} = \dfrac{\tilde{\tau}_0(p+1)}{\mu_1}$,

$\dfrac{\partial \tilde{w}}{\partial \varphi}\Big|_{\varphi=\varphi_n} = \dfrac{\tilde{\tau}_{n+1}(p+1)}{\mu_n}$ then

$$A_1(p) = \frac{\tilde{\tau}_{n+1}(p+1)}{\mu_n \Delta_1} + \frac{1}{\Delta_1} \sum_{i=1}^{n-1} p\tilde{f}_{wi}(p)\sin\big[p(\varphi_n - \varphi_i) \big] -$$

$$- \frac{\tilde{\tau}_0(p+1)}{\Delta_1 p \mu_1} \Bigg(p\cos p\varphi_n - \sum_{i=1}^{n-1} \frac{\mu_{i+1} - \mu_i}{\mu_{i+1}} L_2^i \cos\big[p(\varphi_n - \varphi_i) \big] \Bigg) -$$

$$- \frac{1}{\Delta_1} \sum_{i=1}^{n-1} \Bigg[\Bigg(\frac{\tilde{f}_{\sigma i}}{\mu_{i+1}} - \frac{\mu_{i+1} - \mu_i}{\mu_{i+1}} L_3^i \Bigg) \cos\big[p(\varphi_n - \varphi_i) \big] \Bigg], \; B_1(p) = \frac{\tilde{\tau}_0(p+1)}{p\mu_1}, \tag{13}$$

$$\Delta_1 = \Delta_1(p) = -p\sin p\varphi_n - \sum_{i=1}^{n-1} \frac{\mu_{i+1} - \mu_i}{\mu_{i+1}} L_1^i \cos\big[p(\varphi_n - \varphi_i) \big];$$

2) if $\tilde{w}_1\big|_{\varphi=0} = \tilde{w}_0(p)$, $\tilde{w}_n\big|_{\varphi=\varphi_n} = \tilde{w}_{n+1}(p)$, then

$$A_1(p) = \tilde{w}_0(p),$$

$$B_1(p) = \frac{\tilde{w}_{n+1}(p)}{\Delta_2} - \frac{1}{\Delta_2} \sum_{i=1}^{n-1} \tilde{f}_{wi}(p)\cos\big[p(\varphi_n - \varphi_i) \big] -$$

$$-\frac{\tilde{w}_0(p)}{\Delta_2}\left(\cos p\varphi_n - \sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_1^i\sin\left[p\left(\varphi_n-\varphi_i\right)\right]\right)-$$

$$-\frac{1}{\Delta_2}\sum_{i=1}^{n-1}\left[\left(\frac{\tilde{f}_{\sigma i}(p+1)}{p\mu_{i+1}}-\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_3^i\right)\sin\left[p\left(\varphi_n-\varphi_i\right)\right]\right],\tag{14}$$

$$\Delta_2 = \Delta_2(p) = \sin p\varphi_n - \sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_2^i\sin\left[p\left(\varphi_n-\varphi_i\right)\right];$$

3a) if $\tilde{w}\big|_{\varphi=0}=\tilde{w}_0(p)$, $\dfrac{\partial\tilde{w}}{\partial\varphi}\bigg|_{\varphi=\varphi_n}=\dfrac{\tilde{\tau}_{n+1}(p+1)}{\mu_n}$, then

$$B_1(p) = \frac{\tilde{\tau}_{n+1}(p+1)}{\Delta_3\mu_n}+\frac{1}{\Delta_3}\sum_{i=1}^{n-1}p\tilde{f}_{wi}(p)\sin\left[p(\varphi_n-\varphi_i)\right]-$$

$$-\frac{1}{\Delta_3}\sum_{i=1}^{n-1}\frac{\left[\tilde{f}_{\sigma i}(p+1)-(\mu_{i-1}-\mu_i)L_3^i\right]\cos\left[p(\varphi_n-\varphi_i)\right]}{\mu_{i+1}}-\frac{\tilde{w}_0(p)}{\Delta_{31}}\times$$

$$\times\left(-p\sin p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i-1}-\mu_i}{\mu_{i+1}}L_1^i\cos\left[p(\varphi_n-\varphi_i)\right]\right),\ A_1(p)=\tilde{w}_0(p),\tag{15}$$

$$\Delta_3 = \Delta_3(p) = p\cos p\varphi_n - \sum_{i=1}^{n-1}\frac{\mu_{i-1}-\mu_i}{\mu_{i+1}}L_2^i\cos\left[p(\varphi_n-\varphi_i)\right],$$

3b) if $\dfrac{\partial\tilde{w}}{\partial\varphi}\bigg|_{\varphi=0}=\dfrac{\tilde{\tau}_1(p+1)}{\mu_1}$, $\tilde{w}\big|_{\varphi=\varphi_n}=\tilde{w}_{n+1}(p)$, then

$$A_1(p) = \frac{w_{n+1}}{\Delta_4}-\frac{1}{\Delta_4}\sum_{i=1}^{n-1}\tilde{f}_{wi}(p)\cos\left[p(\varphi_n-\varphi_i)\right]+$$

$$+\frac{1}{\Delta_4}\sum_{i=1}^{n-1}\frac{\left[\tilde{f}_{\sigma i}(p+1)-(\mu_{i+1}-\mu_i)L_3^i(p)\right]\sin\left[p(\varphi_n-\varphi_i)\right]}{p\mu_{i+1}}-\frac{\tilde{\tau}_0(p+1)}{p\Delta_{32}\mu_1}\times$$

$$\times\left(\sin p\varphi_n-\sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_2^i\sin\left[p(\varphi_n-\varphi_i)\right]\right),\ B_1(p)=\frac{\tilde{\tau}_0(p+1)}{p\mu_1},\tag{16}$$

$$\Delta_4 = \Delta_4(p) = \cos p\varphi_n - \sum_{i=1}^{n-1}\frac{\mu_{i+1}-\mu_i}{p\mu_{i+1}}L_1^i\sin\left[p(\varphi_n-\varphi_i)\right].$$

So applying the relations (13) – (16), the presentation (11) and Hooke's law by means of the Mellin transform theorem the stress and displacement field components are determined as

$$w(r,\varphi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{w}(p,\varphi) r^{-p} dp, \ \ \sigma_{\varphi z} = \frac{\mu}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\partial \tilde{w}(p,\varphi)}{\partial \varphi} r^{-p-1} dp,$$

$$\sigma_{rz} = -\frac{\mu}{2\pi i} \int_{c-i\infty}^{c+i\infty} p\tilde{w}(p,\varphi) r^{-p-1} dp. \tag{17}$$

To calculate the obtained integrals it is reasonable to utilize the residue theorem, and finally we will obtain the stresses and displacements in the form of series in the poles of the integrand function

$$w(r,\varphi) = \sum_{p_i} \frac{Ta^{p_i}}{\Delta_j'(p_0)} g_3(p_i) g_1(p_i,\varphi) \left(\frac{r}{a}\right)^{-p_i},$$

$$\sigma_{\varphi z}(r,\varphi) = \sum_{p_i} \frac{T}{\Delta_j'(p_i)a} g_3(p_i) g_2(p_i,\varphi) \left(\frac{r}{a}\right)^{-(p_i+1)},$$

$$\sigma_{rz}(r,\varphi) = -\sum_{p_i} \frac{p_i T}{a\Delta_j'(p_i)} g_3(p_i) g_1(p_i,\varphi) \left(\frac{r}{a}\right)^{-(p_i+1)},$$

where $g_1(p_i,\varphi)$, $g_2(p_i,\varphi)$, $g_3(p_i)$ are the functions dependent on the particular statement of the problem, which are constructed on the basis of expressions (11) - (17), their general form is not given here for lack of space; p_i are the roots of the transcendental equation: $\Delta_1 = 0$ - for the first problem of elasticity theory; $\Delta_2 = 0$ - for the second problem of elasticity theory; $\Delta_3 = 0$ or $\Delta_4 = 0$ (depending on the boundary conditions) – for the mixed one.

According to the conclusions of [1] the stress field for $r \to 0$ will have the singularity of order $r^{-(1+\operatorname{Re} p)}$ if the denominator of the corresponding integrand has zeroes on the strip $-1 < \operatorname{Re}(p) < 0$. Hence, to determine the singularity order $\lambda = 1 + \operatorname{Re} p$ it is necessary to solve the corresponding transcendental equation $\Delta_j(p) = 0 \ \left(j = \overline{1,4}\right)$.

To determine the unknown jump functions one should use the conditions of interaction between a composite and inclusion what will yield the system of singular integral equations from which the unknown jump functions are to be defined.

It should be noticed that the equation $\Delta_j(p)=0$ $\left(j=\overline{1,4}\right)$ in its structure is identical to that obtained by G. Sulym, M. Makhorkin [2] and the results of calculations of maximal singularity order are the same in all cases. Also it should be noted that in special cases of a two-wedge system the equations coincide with that obtained by I. Butvinnik [3]. In the case of a crack located on the wedge bisectrix or between two wedges with identical opening angles the results obtained on the basis of (17) coincide with the ones obtained by M. Savruk [12] and A. Shahani [15].

If inside of one of the wedges S_j of the system at the point with polar coordinates $\varphi=\varphi_i$, $r=a$ $\left(\varphi_{j-1}<\varphi_i<\varphi_j,\quad j-1<i<j\right)$ the concentrated force T is applied and on the notch edges the homogeneous boundary conditions are given then this case is modeled so that on the line $\varphi=\varphi_i$ the stress jump $f_{\sigma i}(r)=T\delta(r-a)$ is to be considered when the displacement jump $f_{wi}(r)\equiv 0$ is absent (Fig. 2).
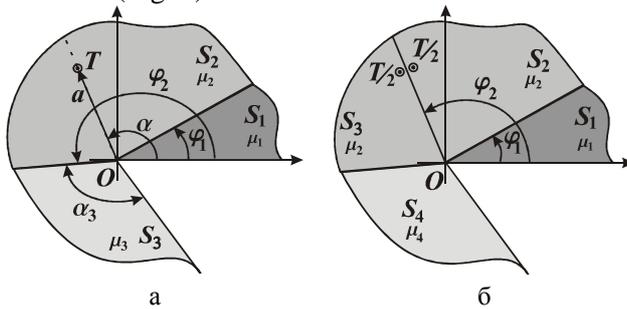

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Fig. 2. The scheme of a system loaded by internal concentrated force
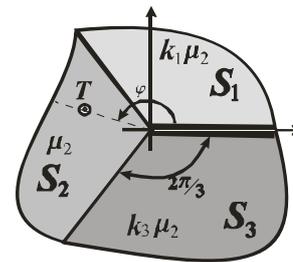
Fig. 3. A three-wedge system

Taking into consideration our profound interest in the behavior of the stress and displacement fields in the vicinity of the stress concentrators where the stress field is singular we study the stress asymptotics in the vicinity of the system peak. In the neighborhood of irregular point of material interface the stress field nature is determined by the component of asymptotic series containing the maximum peculiarity. Thus, in order to determine the stress state it is sufficient to calculate the residual in that pole the value of which provides the greatest peculiarity of stresses in the vicinity of the system apex. As a result it is elucidated that the stresses and displacements in a small vicinity of the system apex can be described asymptotically by such expressions

$$w(r,\varphi) = \frac{Ta^{p_0}}{\Delta_j'(p_0)} g_3(p_0) g_1(p_0,\varphi) \left(\frac{r}{a}\right)^{-p_0} = \frac{r^{1-\lambda^*}\tilde{K}_3}{\sqrt{2\pi}} f_1(\varphi),$$

$$\sigma_{\varphi z}(r,\varphi) = \frac{T}{\Delta_j'(p_0)} g_3(p_0) g_2(p_0,\varphi) \left(\frac{r}{a}\right)^{-(p_0+1)} = \frac{\tilde{K}_3 f_2(\varphi)}{\sqrt{2\pi} r^{\lambda^*}}, \qquad (18)$$

$$\sigma_{rz}(r,\varphi) = -\frac{pT}{\Delta_j'(p)} g_3(p) g_1(p,\varphi) \left(\frac{r}{a}\right)^{-(p+1)} = \frac{\tilde{K}_3 f_3(\varphi)}{\sqrt{2\pi} r^{\lambda^*}},$$

where $p_0 = \max \operatorname{Re} p \in (-1;0)$ is the solution of equation $\Delta_j(p) = 0$ $\left(j = \overline{1,4}\right)$, which ensures the maximal value of the stress peculiarity in the vicinity of the system peak; $\Delta_j(p) = 0$ is the equation which is constructed according to the boundary conditions given on the edges of the wedge-shaped notch on the basis of equations (13) - (16); $g_3(p_0)$, $g_i(p_0,\varphi)$ $(i = 1,2)$ are the functions constructed on the basis of equations (11) - (12) (their general form is not presented because of inconvenience); $\tilde{K}_3$ is a constant coefficient which characterizes the type and way of loading; $f_i(\varphi)$ is an angular function near the maximal value of the singularity order $(i = 1,2,3)$, which characterizes the angular variation of the displacement and stress distribution and it does not depend on the way of the system load; λ^* is the maximal order of singularity.

Granting that at passage to the limit from a wedge system with a wedge-shaped notch to a crack or a rigid inclusion in homogeneous medium ($\alpha_1 = ... = \alpha_{j-1} = \alpha_{j+1} = ... = \alpha_{n+1} = 0$, or $\mu_1 = \mu_2 = .. = \mu_n$, $\alpha_{n+1} = 0$) the expressions (18) are same as the known expressions for the stress and displacement asymptotics in the vicinity of the crack tip or rigid inclusion in homogeneous material [17] and the value $\tilde{K}_3$ is the same as that of a classical stress intensity factor (SIF) K_3. We can conclude that the coefficient $\tilde{K}_3$ is the analogy of the SIF for inclusion in homogeneous material. Therefore according to the definition in [17] it will be right to call $\tilde{K}_3$ a generalized stress intensity factor for a wedge system (GSIFWS).

4. THREE-WEDGE SYSTEM LOADED BY CONCENTRATED SHEAR FORCE

As noted above in a general cases the equations $\Delta_j(p) = 0$ are transcendental and to find their roots one needs the numerical methods. However in some cases

of particular configuration of the system the transcendental equation converts into a trigonometric one the roots of which can be found analytically. For example in the case when the system consists of three wedges with opening angle $\alpha_i = 2\pi/3$ $(i = 1,2,3)$ (Fig. 3) the characteristic equation will be of the form

$$f_3(p)\sin\alpha p\left[f_1(\mu_1,\mu_2,...,\mu_n)+f_2(\mu_1,\mu_2,...,\mu_n)\cos\beta p\right]=0, \qquad (19)$$

where the functions $f_1(\mu_1,\mu_2,\mu_3)$, $f_2(\mu_1,\mu_2,\mu_3)$, $f_3(p)$, depending on the boundary conditions, are of the form:
in the case of the first boundary-value problem –

$$f_1(\mu_1,\mu_2,\mu_3)=1+k_1+k_3(1-k_1), \quad f_2(\mu_1,\mu_2,\mu_3)=(1+k_1)(1+k_3),$$
$$f_3(q)=q;$$

in the case of the second one –

$$f_1(\mu_1,\mu_2,\mu_3)=k_1-1+k_3(1+k_1), \quad f_2(\mu_1,\mu_2,\mu_3)=(1+k_1)(k_3+1),$$
$$f_3(p)=1; \quad k_1=\mu_1/\mu_2, \quad k_3=\mu_3/\mu_2, \quad \alpha=2\pi/3, \quad \beta=4\pi/3.$$

Thus the solutions of equation (19) are of the form

$$p_{1n}=\pm\frac{1}{\beta}\arccos\left(\frac{f_1(\mu_1,\mu_2,\mu_3)}{f_2(\mu_1,\mu_2,\mu_3)}\right)+\frac{2\pi n}{\beta},$$

$$p_{2n}=\frac{2\pi n}{\beta}+i\,\text{arch}\left(\frac{f_1(\mu_1,\mu_2,\mu_3)}{f_2(\mu_1,\mu_2,\mu_3)}\right), \quad q_{3n}=\frac{\pi n}{\alpha} \quad (n\in Z).$$

5. NUMERICAL STUDIES

Using the relations (18), (11) - (16) and solutions of equation (19) the asymptotics of stresses and displacements in a three-wedge system loaded by a concentrated shear force T at point $\varphi=\alpha$, $r=a$ (Fig. 3) were written. For different values of relation of shear moduli $k_1=\mu_1/\mu_2$, $k_3=\mu_3/\mu_2$ the dependence of distribution of tangential stresses $\sigma_{\varphi z}(r,\varphi)$ (the function of stress distribution according to (18) $\tau=f_2(\varphi)$) and values of GSIFWS $\tilde{K}_3$ ($K=\tilde{K}_3/T\mu_2 a^{p_0}$) on the location of point of application of concentrated force were studied. Some results of numerical studies are presented in Figs. 4 – 7.

As can be seen from the graphs of distribution of tangential stresses in the case of the first boundary value problem the maximal stresses will be approximately on the crack continuation (Fig. 4). In the case of absolutely rigid inclu-

sion and partly delaminated absolutely rigid inclusion (the second and mixed boundary-value problems) the stress maximum will be reached on the clamped edge in the wedge with larger shear modulus (Fig. 5). In addition under conditions of the second boundary-value problem the stress for some value of the angle is equal to zero and then changes its sign. If we consider a solid plate composed of wedges then the stress which is maximal in modulus will be reached on the coupling line of wedges with larger rigidity.
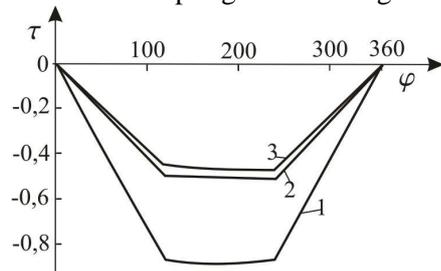

Fig. 4. The function of distribution of the tangential stresses τ under conditions of the first boundary-value problem

1 - $k_1 = 10$, $k_2 = 10$;

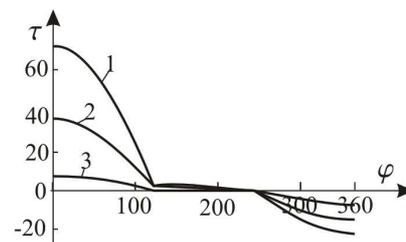2 - $k_1 = 10$, $k_2 = 100$;

3 - $k_1 = 10$, $k_2 = 1000$.

Fig. 5. The function of distribution of the tangential stresses τ under conditions of the second boundary-value problem

1 - $k_1 = 100$; $k_2 = 10$;

2 - $k_1 = 50$; $k_2 = 10$;

3 - $k_1 = 10$; $k_2 = 10$.

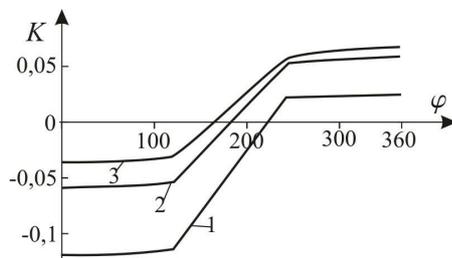
Fig. 6. The values of GSIFWS under conditions of the first boundary-value problem

1 - $k_1 = 10$, $k_3 = 50$;

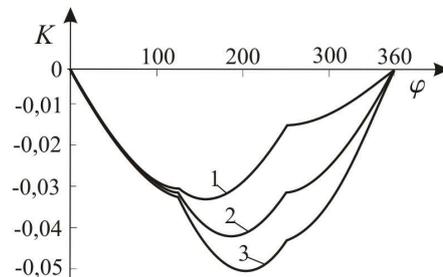2 - $k_1 = 10$, $k_3 = 10$;

3 - $k_1 = 10$, $k_3 = 5$.

Fig. 6. The values of GSIFWS under conditions of the second boundary-value problem

1 - $k_1 = 10$, $k_3 = 50$;

2 - $k_1 = 10$, $k_3 = 10$;

3 - $k_1 = 10$, $k_3 = 5$.

The results presented in Figs. 6-7 show that change of relation of shear moduli and point of application of concentrated force influences essentially the quantitative value of GSIFWS. For the case of a crack the maximal value of GSIFWS will be in the case when the force is applied on the notch edge (Fig.6).

As it is seen from other studies and for the case of partially-delaminated absolutely rigid inclusion we will have the maximal value of GSIFW if the force is applied on the free edge of a wedge system. For absolutely rigid inclusion we will have the GSIFWS maximal value if the force is applied on the line $\varphi \approx \pi$ (Fig. 7). For certain values of angle φ that determines the line of application of the concentrated force the GSIFWS value will be equal to zero (Fig. 6). This testifies that in these cases to evaluate the stress-strain state in the vicinity of the system apex one cannot utilize a component of asymptotic series which contains the maximal value of singularity (it is equal to zero). One must take the component of the asymptotic series containing the singularity which is subsequent in the modulus value.

6. CONCLUSIONS

Basing on the constitutive relations of elasticity theory for homogeneous body, legitimacy of representing the physico-mechanical characteristics of a piecewise-homogeneous system in the form of piecewise-homogeneous functions of polar angle and on the apparatus of the theory of generalized functions the analytical-numerical procedure is proposed to define the stress and strain field in the wedge composite with thin radial defects. Its application reduces the problem of study of the stress-strain state in the vicinity of irregular point of material interface to finding the solution of one partly-degenerated differential equation of the form (5).

A general solution of this equation is constructed. Utilizing this equation the analytical expressions of Mellin transformant of stresses and displacements in the system composed of arbitrary number of wedges with thin radially located inclusions are written. Besides the analytical presentation of asymptotics of stresses and displacements in the vicinity of multi-wedge system apex and equations to calculate the stress singularity order are written. In a special case of three-wedge system composed of wedges with opening angles $\alpha_i = 2\pi/3 \; (i = 1, 2, 3)$ the possibility to find such roots analytically is demonstrated.

Basing on the general form of asymptotics of stresses and displacements in the vicinity of the system apex and results of [2, 16, 17] we have introduced the concept of generalized stress intensity factor for a multi-wedge system and justified the legitimacy of this concept. Using the obtained results the numerical studies of variation of the GSIFWS value in a three-wedge system vs. the point of application of concentrated force were carried out.

It is shown that for some values of the angle which determines the line of application of the force the stress-strain state in the vicinity of the system apex is described by a component of asymptotic series which contains not the maximal singularity but the subsequent one in the modulus value. Thus we can select

the point the concentrated force at which will not cause singular stresses in the vicinity of the system apex.

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OKREŚLENIE STANU NAPRĘŻEŃ I PRZEMIESZCZEŃ W UKŁADZIE WIELOKLINOWYM Z PROMIENIOWYMI DEFEKTAMI PRZY ŚCINANIU WZDŁUŻNYM

S t r e s z c z e n i e

Przedstawione badania dotyczą nowego podejścia do rozwiązywania zagadnienia antypłaskiego dla układu wieloklinowego z ułożonymi promieniowo cienkimi niejednorodnościami. Metoda ta, wykorzystuje podejście uogólnionego zagadnienia sprzężenia materiałów do modelowania istnienia cienkich defektów za pomocą funkcji skoków. To daje możliwość otrzymania w postaci analitycznej transformat Mellina naprężeń i przemieszczeń w pakietach z dowolną ilością klinów. Wskutek stosowania metody wyznaczenie stanu naprężeniowo-odkształceniowego w układzie klinowym podczas ścinania wzdłużnego sprowadza się do rozwiązywania jednego częściowo zdegenerowanego równania różniczkowego z odpowiednimi warunkami brzegowymi. Do jego rozwiązania stosuje się transformacja całkowa Mellina, której transformaty dla dowolnej ilości komponentów układu klinowego można znaleźć w postaci analitycznej. W wyniku przeprowadzonych badań zostały otrzymane asymptotyki naprężeń i przemieszczeń w otoczeniu wierzchołka układu klinowego oraz transformaty Mellina przemieszczeń. Wprowadzono pojęcie uogólnionego współczynnika intensywności naprężeń w wierzchołku klina oraz dla szeregu układów wieloklinowych otrzymano postać analityczną dla naprężeń i przemieszczeń w dowolnym punkcie kompozytu. Również został dokładnie zbadany przypadek obciążenia skupioną siłą układu z trzech klinów.