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ANALYSIS OF STRONGLY NON-LINEAR FREE VIBRATIONS OF BEAMS USING PERTURBATION METHOD

Roman LEWANDOWSKI
Poznan University of Technology, Institute of Structural Engineering
Piotrowo St. 5, 60-965 Poznań, Poland

The possibilities of application of the perturbation method to the analysis of strongly non-linear free vibrations of beams are discussed. The geometrical non-linearity is taken into account. The finite element method is used for the description of the dynamic behaviour of beams. The first order perturbation equation is solved and the obtained solution is compared with the solution found with the help of the harmonic balance method which is widely used and applicable to the analysis of strongly non-linear dynamic systems. It was proved that both solutions are almost identical and differences are negligibly small. The numerical procedure enabling determination of backbone curves is also briefly described. Theoretical results are supplemented by a description of the results of typical calculations.

Keywords: strongly non-linear vibrations, free vibrations, perturbation method, beam structures

1. INTRODUCTION

The perturbation method is one of the oldest methods used to analyse the dynamic behaviour of non-linear systems. Descriptions of the method can be found in many textbooks (see, for example, [1]). There are many versions of the perturbation method but many of them apply to weakly non-linear cases only. To overcome this limitation, many new techniques have been proposed recently. Cheung et al. [2], Lim et al. [3] and Hu [4] proposed some modifications which make possible the analysis of strongly non-linear systems with one degree of freedom only.

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In this paper, a possibility of using the perturbation method to analyse strongly non-linear free vibrations of beams is discussed. Beams are treated as geometrically non-linear systems. The von Karman theory is used to describe non-linear effects. The transformation into modal co-ordinates is used when the perturbation method is applied to the analysis of dynamic systems with many degrees of freedom. This transformation is not used in the paper. The finite element method is adopted to discetize the beam structures and the motion equation is written in a matrix form as it was derived in [5,6].

2. EQUATION OF MOTION

Consider the beam structures with immovable ends, of which one example is shown in Fig. 1. Large displacements and small rotations of the beams are assumed. The beam with immovable ends experiences in-plane stretching when deflected. The influence of this stretching on the response increased with the amplitude of vibrations. It could be described by the following non-linear axial strain – displacements equation

$$\varepsilon = u_{,x} + \tfrac{1}{2} w_{,x}^2 \ , \tag{2.1}$$

where symbols $\varepsilon,\ u,\ w$ denote the axial strain, the axial and the transverse displacements, respectively. Taking into account that ends of beam are immovable in the horizontal direction and ignoring the horizontal inertia forces we can write the beam axial force $N(t)$ in the form:

$$N(t) = \frac{EA}{2l} \int_0^l w_{,x}^2(x,t)dx \ . \tag{2.2}$$

where $E,\ A,\ l$ denote the Young's constant, the area of the beam cross-section and the beam length, respectively.

In the case of undamped free vibration, the motion equation of simple beam can be are written in the following:

$$m\ddot{w}(x,t) + EJw_{,xxxx}(x,t) - \frac{EA}{2l} \int_0^l w_{,x}^2(x,t)dx \ \ w_{,xx}(x,t) = 0 \ . \tag{2.3}$$

where symbols $m,\ J$ denote the mass of beam per unit length and the moment of inertia of the cross-section, respectively. Moreover, $(\circ)_{,x}$ denotes differentiation with respect to x and dots denote differentiation with respect to the time variable *t*.

Now, the beam structures with immovable ends are treated as the discrete systems obtained with a help of the finite element method. The motion equation can be written in the following matrix form (see [5,6]):

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}_0\mathbf{q}(t) + \frac{EA}{2l}\mathbf{B}\mathbf{q}(t)\mathbf{q}^T(t)\mathbf{B}\mathbf{q}(t) = \mathbf{0} \ , \tag{2.4}$$

where $\mathbf{M}$, $\mathbf{K}_0$ and $\mathbf{B}$ denote the global mass matrix, the global linear stiffness matrix, the global „geometric" stiffness matrix, respectively. Moreover, $\mathbf{q}(t)$ is the global vector of nodal parameters. The dimensions of the above introduced matrices and vectors are *(nxn)* and *(nx1)*, respectively.

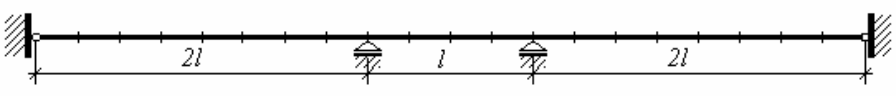

Fig. 1. Example of beam structure

On the elementary level, transverse displacements $w(x,t)$ are written in the form

$$w(x,t) = \mathbf{N}^T(x)\mathbf{q}_e(t) \ , \tag{2.5}$$

where $\mathbf{N}(x) = col(N_1(x), N_2(x), N_3(x), N_4(x))$ is the vector of shape functions and $\mathbf{q}_e(t) = col(w_1(t), \varphi_1(t), w_2(t), \varphi_2(t))$ is the vector of nodal parameters. The Hermite polynomials and two-node element are used in this paper.

On the finite element level, the definitions of the mass, the linear stiffness and the „geometric" stiffness matrices are

$$\mathbf{M}_e = \int_0^L m\mathbf{N}(x)\mathbf{N}^T(x)dx \ , \quad \mathbf{K}_e = \int_0^L EJ\mathbf{N}_{,xx}(x)\mathbf{N}_{,xx}^T(x)dx \ , \tag{2.6}$$

$$\mathbf{B}_e = \int_0^L \mathbf{N}_{,x}(x)\mathbf{N}_{,x}^T(x)dx \ , \tag{2.7}$$

where L is the length of the finite element. The above mentioned global matrices $\mathbf{M}, \mathbf{K}_0$ and $\mathbf{B}$ are determined in a usual way.

In a matrix form, the beam axial force $N(t)$ is given by (see [5, 6])

$$N(t) = \frac{EA}{2l}\mathbf{q}^T(t)\mathbf{B}\mathbf{q}(t) \ . \tag{2.8}$$

The initial conditions which must be fulfilled by the solution to the equation of motion are $\mathbf{q}(0) = \tilde{\mathbf{a}}$ and $\dot{\mathbf{q}}(0) = \mathbf{0}$.

3. SOLUTION TO THE EQUATION OF MOTION USING THE PERTURBATION METHOD

First of all, the motion equation (2.4) is rewritten in a slightly different form:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}_0\mathbf{q}(t) + \varepsilon\frac{EA}{2l}\mathbf{B}\mathbf{q}(t)\mathbf{q}^T(t)\mathbf{B}\mathbf{q}(t) = \mathbf{0}\ , \tag{3.1}$$

where ε is the artificially introduced small parameter $0 \le \varepsilon \le 1$. For $\varepsilon = 0$ Eq. (3.1) describes linear vibrations of beams while for $\varepsilon = 1$ the geometrically non-linear effects are fully taken into account.

The slightly refined version of the perturbation method will be used to obtain the solution to the motion equation. The solution to Eq. (3.1) is assumed in the form:

$$\mathbf{q}(t) = \mathbf{q}_0(t) + \varepsilon\,\mathbf{q}_1(t) + \varepsilon^2\mathbf{q}_2(t) + \ldots\ldots\ . \tag{3.2}$$

Moreover, the unknown matrix $\mathbf{K}_s$ is introduced. It is assumed that this matrix can be also expanded in a following power series with respect to ε

$$\mathbf{K}_s = \mathbf{K}_0 + \varepsilon\,\mathbf{K}_1 + \varepsilon^2\mathbf{K}_2 + \ldots\ldots \tag{3.3}$$

where $\mathbf{K}_1, \mathbf{K}_2$ and so on are also unknown matrices.

Usually, the perturbation method is applied to solve the single equation describing behaviour of a one-degree-of-freedom system or one modal coordinate of multi-degree-of-freedom systems. In this case, the square of non-linear frequency of vibration ω^2 is given in the following form:

$$\omega^2 = \omega_0^2 + \varepsilon\,\omega_1 + \varepsilon^2\omega_2 + \ldots\ldots \tag{3.4}$$

where, ω_0 is the linear frequency of vibration and ω_1 and ω_2 are some constants which must be found.

In this paper, transformation to modal coordinates is not introduced and Eq. (3.3) could be considered as a generalisation of Eq. (3.3) when the systems with many-degrees-of-freedom are considered.

Substituting Eqs. (3.2) and (3.3) into Eq. (3.1) gives

$$\begin{aligned}
&\mathbf{M}(\ddot{\mathbf{q}}_0 + \varepsilon\,\ddot{\mathbf{q}}_1 + \varepsilon^2\ddot{\mathbf{q}}_2 + ..) + \\
&(\mathbf{K}_s - \varepsilon\,\mathbf{K}_1 - \varepsilon^2\mathbf{K}_2 - ..)\,(\mathbf{q}_0 + \varepsilon\,\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + ..) + \\
&\varepsilon\,\frac{EA}{2l}\mathbf{B}(\mathbf{q}_0 + \varepsilon\,\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + ..)\,(\mathbf{q}_0 + \varepsilon\,\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + ..)^T \\
&\mathbf{B}(\mathbf{q}_0 + \varepsilon\,\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + ..) = \mathbf{0}
\end{aligned} \quad . \tag{3.5}$$

The above equation is satisfied by setting the coefficients of the powers of ε equal to zero. It results in the following set of matrix differential equations written below only for ε^0, ε and ε^2, respectively

$$\mathbf{M}\ddot{\mathbf{q}}_0(t) + \mathbf{K}_s\mathbf{q}_0(t) = \mathbf{0} \ , \tag{3.6}$$

$$\mathbf{M}\ddot{\mathbf{q}}_1(t) + \mathbf{K}_s\mathbf{q}_1(t) - \mathbf{K}_1\mathbf{q}_0(t) + \frac{EA}{2l}\mathbf{B}\mathbf{q}_0\mathbf{q}_0{}^T\mathbf{B}\mathbf{q}_0 = \mathbf{0} \ , \tag{3.7}$$

$$\begin{aligned}
&\mathbf{M}\ddot{\mathbf{q}}_2(t) + \mathbf{K}_s\mathbf{q}_2(t) - \mathbf{K}_1\mathbf{q}_1(t) + \frac{EA}{2l}\mathbf{B}\mathbf{q}_0(t)\mathbf{q}_0{}^T(t)\mathbf{B}\mathbf{q}_1(t) + \\
&\frac{EA}{2l}\mathbf{B}\mathbf{q}_0(t)\mathbf{q}_1{}^T(t)\mathbf{B}\mathbf{q}_0(t) + \frac{EA}{2l}\mathbf{B}\mathbf{q}_1(t)\mathbf{q}_0{}^T(t)\mathbf{B}\mathbf{q}_0(t) == \mathbf{0}
\end{aligned} \quad . \tag{3.8}$$

It is assumed that $\mathbf{q}_0(t)$ and $\mathbf{q}_i(t)$ $(i = 1,2,....)$ fulfil the following initial conditions $\mathbf{q}_0(0) = \tilde{\mathbf{a}}$, $\dot{\mathbf{q}}_0(0) = \mathbf{0}$ and $\mathbf{q}_i(0) = \dot{\mathbf{q}}_i(0) = \mathbf{0}$.

3.1. Solution to Equation (3.6)

Equation (3.6) is linear and its solution is given by

$$\mathbf{q}_0(t) = \mathbf{a}_0\cos\omega t \ , \tag{3.9}$$

where, at this point $\mathbf{a}_0$ and ω are the unknown vector and the non-linear frequency of vibration, respectively. The vector $\mathbf{a}_0$ and ω must be the solution of the following linear eigenvalue problem

$$\left(\mathbf{K}_s - \omega^2\mathbf{M}\right)\mathbf{a}_0 = \mathbf{0} \ . \tag{3.10}$$

Please note that, in fact, the eigenvalue problem cannot be solved because the matrix $\mathbf{K}_s$ is still unknown. The eigenvalue problem can be solved when the matrix $\mathbf{K}_s$ is approximated by $\mathbf{K}_0$. It makes a difference between the method presented here and the classic perturbation method where the relations

(3.3) or (3.4) are not introduced. The eigenvalue problem (3.10) has a set of solutions for ω and $\mathbf{a}_0$ denoted by ω_i and $\mathbf{a}_{0i}$, $(i = 1,2,..,n)$, respectively, and the general solution to Eq. (3.6) can be written as

$$\mathbf{q}_0(t) = \sum_{i=1}^{n} \mathbf{a}_{0i}(c_i \cos\omega_i t + d_i \sin\omega_i t) \ . \tag{3.11}$$

The aim of this paper is to find periodic solutions and to determine the dynamic characteristics (i.e. the non-linear frequencies and modes of vibration) of the considered beams. For this reason the $\tilde{\mathbf{a}}$ vector of initial conditions is chosen in such a way that $\tilde{\mathbf{a}} = \mathbf{a}_{0l}$ (i.e. is identical with the *l*-th mode of vibration). The constants c_i and d_i resulting from the initial conditions mentioned above are $d_i = 0$, $c_l = 1$ and $c_i = 0$ if $i \neq l$ and, finally, the solution to Eq. (3.6) can be written as:

$$\mathbf{q}_0(t) = \mathbf{a}\cos\omega t \ , \tag{3.12}$$

where now $\mathbf{a} = \mathbf{a}_{0l} = \tilde{\mathbf{a}}$ and $\omega = \omega_l$.

3.2. Solution to Equation (3.7)

Substituting Eq. (3.12) into Eq. (3.7) and taking into account that

$$4\cos^3\omega t = 3\cos\omega t + \cos 3\omega t \ , \tag{3.13}$$

the following equation is obtained

$$\mathbf{M}\ddot{\mathbf{q}}_1(t) + \mathbf{K}_s\mathbf{q}_1(t) = \mathbf{K}_1\mathbf{a}\cos\omega t - \frac{EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a}\left(3\cos\omega t + \cos 3\omega t\right) . \tag{3.14}$$

The homogenous part of the solution to Eq. (3.14), denoted by $\mathbf{q}_1^*(t)$, is the solution to the following homogenous equation

$$\mathbf{M}\ddot{\mathbf{q}}_1^*(t) + \mathbf{K}_s\mathbf{q}_1^*(t) = \mathbf{0} \ , \tag{3.15}$$

which is identical to Eq. (3.6) after replacing $\mathbf{q}_1^*(t)$ by $\mathbf{q}_0(t)$.

The solution to Eq. (3.15) has the form

$$\mathbf{q}_1^*(t) = \mathbf{z}\cos\lambda t \ , \tag{3.16}$$

and the vector $\mathbf{z}$ and λ must be again determined from

$$\left(\mathbf{K}_s - \lambda^2 \mathbf{M}\right)\mathbf{z} = \mathbf{0} \ . \tag{3.17}$$

In the eigenvalue problems (3.17) and (3.10) identical matrices appear, which means that eigenvalues and eigenvectors of both problems are also identical (i.e. $\mathbf{z}_i = \mathbf{a}_{0i}$ and $\lambda_i = \omega_i$ for $i = 1,2,..,n$), and finally

$$\mathbf{q}_1^*(t) = \sum_{i=1}^{n} \mathbf{a}_{0i}(c_i \cos\omega_i t + d_i \sin\omega_i t) \ , \tag{3.18}$$

where c_i and d_i *(i=1,2,..,n)* are some unknown constants.

Now, the non-homogenous part of $\mathbf{q}_1(t)$ denoted by $\mathbf{q}_1^{"}(t)$ will be determined. Please note that the right hand side of Eq. (3.14) contains the secular term $\mathbf{K}_1\mathbf{a}\cos\omega t - 3EA/(8l)\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a}\cos\omega t$. This term must be eliminated. It happens when the unknown matrix $\mathbf{K}_1$ is given by

$$\mathbf{K}_1 = \frac{3EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B} = \frac{3EA}{8l}\mathbf{a}^T\mathbf{B}\mathbf{a}\mathbf{B} \ . \tag{3.19}$$

The second possible form of matrix $\mathbf{K}_1$ follows from the fact that the product of $\mathbf{a}^T\mathbf{B}\mathbf{a}$ is a scalar and in consequence the following relation holds

$$\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a} = \mathbf{a}^T\mathbf{B}\mathbf{a}\mathbf{B}\mathbf{a} \ . \tag{3.20}$$

In the end, the non-homogenous solution $\mathbf{q}_1^{"}(t)$ fulfils the following differential equation

$$\mathbf{M}\ddot{\mathbf{q}}_1^{"}(t) + \mathbf{K}_s\mathbf{q}_1^{"}(t) = -\frac{EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a}\cos 3\omega t \ . \tag{3.21}$$

The non-homogenous part of solution to Eq. (3.21) is assumed in the form

$$\mathbf{q}_1^{"}(t) = \mathbf{c}\cos 3\omega t \ , \tag{3.22}$$

and the unknown vector $\mathbf{c}$ can be determined from the following algebraic equation

$$\left(\mathbf{K}_s - 9\omega^2\mathbf{M}\right)\mathbf{c} = -\frac{EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a} \ . \tag{3.23}$$

Suppose that only two first terms of the power series (3.2) are enough to arrive at a solution to the problem with the required accuracy. In this case, the $\mathbf{K}_s$ matrix is given by

$$\mathbf{K}_s = \mathbf{K}_0 + \mathbf{K}_1 = \mathbf{K}_0 + \frac{3EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a} = \mathbf{K}_0 + \frac{3EA}{8l}\mathbf{a}^T\mathbf{B}\mathbf{a}\mathbf{B}\mathbf{a} \ , \tag{3.24}$$

and the unknown $\mathbf{a}$ and $\mathbf{c}$ vectors together with the non-linear frequency of vibration ω can be determined from Eqs (3.10) and (3.23). Equation (3.10) constitutes the non-linear eigenvalue problem which, after introducing (3.19) into it, can be rewritten in the form:

$$\left(\mathbf{K}_0 + \frac{3EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B} - \omega^2\mathbf{M}\right)\mathbf{a} = \mathbf{0} \ . \tag{3.25}$$

If the non-linear eigenvalue problem is solved the $\mathbf{a}$ vector and ω is known and the $\mathbf{c}$ vector can be easily determined from the linear algebraic equation (3.23).

One important exception is the case where the matrix $\mathbf{K}_0 + 3EA/(8l)\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B} - 9\omega^2\mathbf{M}$ is singular. It happens if, for a given $\mathbf{a}$ vector, one eigenvalue λ^2 of the linear eigenvalue problem (3.17) rewritten here for convenience

$$\left(\mathbf{K}_0 + \frac{3EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B} - \lambda^2\mathbf{M}\right)\mathbf{z} = \mathbf{0} \ , \tag{3.26}$$

is nine times as high as ω^2. This condition is very similar to the condition of existence of internal resonance 1:3 which is usually written in the form (see [7]) $3\omega \approx \omega_{lin}$, where ω_{lin} denote the frequency of vibration of the linearised system under consideration. The problem of existence of internal resonance will be illustrated later, after discussing the results of typical calculations. However, the elements of the $\mathbf{c}$ vector will be significant in comparison with the elements of $\mathbf{a}$ if the above mentioned matrix is nearly singular. It means that the $\mathbf{c}$ vector can be considered as an indicator of the existence of a region where the internal resonance can occur.

At this stage of consideration the solution to Eq. (3.7) can be written as

$$\mathbf{q}_1(t) = \sum_{i=1}^{n} \mathbf{a}_{0i}(c_i \cos\omega_i t + d_i \sin\omega_i t) + \mathbf{c}\cos 3\omega t \ . \tag{3.27}$$

The c_i and d_i constants will be determined from initial conditions $\mathbf{q}_1(0) = \dot{\mathbf{q}}_1(0) = \mathbf{0}$. From condition $\dot{\mathbf{q}}_1(0) = \mathbf{0}$ it results that $d_i = 0$ for all *i*. The second condition leads us to the relation

$$\sum_{i=1}^{n} \mathbf{a}_{0i} c_i + \mathbf{c} = \mathbf{0} \ . \tag{3.28}$$

By pre-multiplying Eq. (3.28) by $\mathbf{a}_{0j}^T \mathbf{M}$ and taking into account the orthogonality properties of eigenvectors $\mathbf{a}_{0i}$ the c_i constants can be obtained from

$$c_i = -\frac{\tilde{c}_i}{\tilde{m}_i} \ , \tag{3.29}$$

where $\tilde{c}_i = \mathbf{a}_{0i}^T \mathbf{M} \mathbf{c}$ and $\tilde{m}_i = \mathbf{a}_{0i}^T \mathbf{M} \mathbf{a}_{0i}$.

Finally, the solution $\mathbf{q}_1(t)$ can be rewritten in the form

$$\mathbf{q}_1(t) = \sum_{i=1}^{n} \mathbf{a}_{0i} c_i \cos \omega_i t + \mathbf{c} \cos 3\omega t \ , \tag{3.30}$$

and if only two terms of the power series provide a solution to Eq. (3.1) with the required accuracy then

$$\mathbf{q}(t) = \mathbf{a} \cos \omega t + \varepsilon \sum_{i=1}^{n} \mathbf{a}_{0i} c_i \cos \omega_i t + \varepsilon \mathbf{c} \cos 3\omega t \ . \tag{3.31}$$

The solution to the considered problem of the free vibration of beams can be obtained by setting $\varepsilon = 1$ in (3.31) i.e.

$$\mathbf{q}(t) = \mathbf{a} \cos \omega t + \sum_{i=1}^{n} \mathbf{a}_{0i} c_i \cos \omega_i t + \mathbf{c} \cos 3\omega t \ . \tag{3.32}$$

Please note that the obtained solution is only approximately periodic because of the second term in (3.32). However, the influence of harmonics different from ω appearing in this term is small because the right hand side of Eq. (3.23) is approximately proportional to the chosen mode of vibration $\mathbf{a}$. Moreover, this means that the second term of (3.32) can be approximated by

$$\sum_{i=1}^{n} \mathbf{a}_{0i} c_i \cos \omega_i t \approx \mathbf{a}_{0l} c_l \cos \omega_l t = \mathbf{a} c_i \cos \omega t \ , \tag{3.33}$$

because $\mathbf{a}_{0l} = \mathbf{a}$ and $\omega_l = \omega$. Now, a sum of the second and the third terms of (3.32) can be written as

$$\mathbf{a} c_l \cos \omega t + \mathbf{c} \cos 3\omega t = -\tilde{c}_l / \tilde{m}_l \mathbf{a} \cos \omega t + \mathbf{c} \cos 3\omega t \ , \tag{3.34}$$

which means that these terms have been mutually cancelled in the approximation. This will be also illustrated in Section 6 where the results of typical calculations are discussed.

Therefore, the solution obtained with the help of the perturbation method can be approximated by

$$\mathbf{q}(t) \approx \mathbf{a}\cos\omega t \ . \tag{3.35}$$

4. STRONGLY NON-LINEAR VIBRATIONS OF BEAMS

The previously obtained solution is formally valid only for weakly non-linear beams because of the introduced small parameter ε. In this section, the applicability of solution (3.32) to the analysis of strongly non-linear vibrations will be discussed.

It is well known that there exist a few methods which can be used to analyse the dynamics of strongly non-linear systems. The harmonic balance method [6], the incremental harmonic balance method [8], the Galerkin method [6] and the Ritz method give accurate results in such cases. Moreover, it was found in [6, 9] that the non-linear eigenvalue problems resulting from all these methods are identical if the same harmonics are taken into account in the assumed solution of a motion equation.

In this paper, the harmonic balance method is used to solve the considered problem once again for comparison with the previously obtained solution. The one harmonic solution of the motion equation (2.4) is taken in the form:

$$\mathbf{q}(t) = \mathbf{a}\cos\omega t \ . \tag{4.1}$$

After substituting Eq. (4.1) into Eq. (2.4) and taking into account Eq. (3.13) the following equation is obtained

$$\left(\mathbf{K}_0 - \omega^2\mathbf{M}\right)\mathbf{a}\cos\omega t + \frac{EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a}\left(3\cos\omega t + \cos 3\omega t\right) = \mathbf{0} \ . \tag{4.2}$$

According to the harmonic balance procedure, the term with the higher harmonic (i.e. with $\cos 3\omega t$) is neglected and from relation (4.2) the following is obtained

$$\left(\mathbf{K}_0 - \omega^2\mathbf{M}\right)\mathbf{a} + \frac{3EA}{8l}\mathbf{B}\mathbf{a}\mathbf{a}^T\mathbf{B}\mathbf{a} = \mathbf{0} \ . \tag{4.3}$$

It is very important that in the course of derivation of Eq. (4.3) the assumption concerning the degree of non-linearity has not been used so the mentioned equation is valid also in the case of strong non-linearity. This fact is well

documented in the literature (see, for example [10,11]). On the other hand, Eq. (4.3) is identical with Eq. (3.25) resulting from the presented version of the perturbation method. It means that the accuracy of the perturbation solution is of the same order as the accuracy of the harmonic balance method. A remarkable difference exists when the influence of a third harmonic is significant. As demonstrated above, this is connected with the case of internal resonance. However, the assumed one harmonic solution postulated in the harmonic balance method is not accurate and two harmonics solution must be postulated. If the internal resonance does not exist, which is the most typical situation, the presented version of the perturbation method and the harmonic balance method provides solutions with almost equal accuracies.

5. DETERMINATION OF RESPONSE CURVE

Non-linear frequency of vibration depends on vibration amplitudes and for this reason the response curve (i.e. non-linear frequency versus amplitude) is determined, mostly in a numerical way. In the numerical procedure, starting from the given linear frequency and mode of vibration a set of solutions to Eqs. (3.25) and (3.23) is obtained for gradually increased vibration amplitudes. Often the vector of the mode of vibration is normalised in such a way that its maximal entry is equal to the prescribed value. This quantity is called the amplitude of vibration.

The iteration process is necessary to solve Eqn (3.25) because of its non-linearity with respect to $\mathbf{a}$. In a typical iteration some approximation of the $\mathbf{a}$ vector and ω, denoted by $\mathbf{a}_{(r)}$ and $\omega_{(r)}$ where the subscript (r) is a number of iteration, is needed. Starting the iteration process for the amplitude α_j, the $\mathbf{a}$ vector determined from the previous value of amplitude (say α_{j-1}) is normalised in such a way that its maximal value is now equal to α_j. The obtained vector is a starting approximation of $\mathbf{a}$ for the amplitude α_j. For a given $\mathbf{a}_{(r)}$ the r-th approximation of the matrix $\mathbf{K}_s$ denoted by $\mathbf{K}_{s(r)}$ is obtained from

$$\mathbf{K}_{s(r)} = \mathbf{K}_0 + \frac{3EA}{8l}\mathbf{B}\mathbf{a}_{(r)}\mathbf{a}_{(r)}^T\mathbf{B}\mathbf{a}_{(r)} \ . \tag{5.1}$$

Now, Eq. (3.25) can be rewritten in the form

$$\left(\mathbf{K}_0 + \frac{3EA}{8l}\mathbf{B}\mathbf{a}_{(r)}\mathbf{a}_{(r)}^T\mathbf{B} - \omega^2\mathbf{M}\right)\mathbf{a} = \mathbf{0} \ , \tag{5.2}$$

and treated as the linear eigenvalue problem solved with respect to $\mathbf{a}$ and ω. The eigenvector $\mathbf{a}_k$, appropriately normalised, and the corresponding eigenvalue that is the closest to $\mathbf{a}_{(r)}$ and $\omega_{(r)}$ are chosen as a next approximation of $\mathbf{a}$ and ω. Iterations are repeated until the following convergence criteria are fulfilled

$$\left\|\mathbf{a}_{(r+1)} - \mathbf{a}_{(r)}\right\| \le \mu_1 \left\|\mathbf{a}_{(r+1)}\right\| , \quad \left|\omega_{(\mathrm{r}+1)} - \omega_{(\mathrm{r})}\right| \le \mu_2 \omega_{(\mathrm{r}+1)} , \tag{5.3}$$

where μ_1 and μ_2 are the assumed accuracies of calculations. After determination of $\mathbf{a}$ and ω the $\mathbf{c}$ vector and c_i constants are obtained from Eqs. (3.23) and (3.29), respectively.

6. RESULTS OF TYPICAL CALCULATIONS

Example 1 – the three span beam

As a first example the three span beam shown in Fig.1 is considered. The beam is divided into twenty finite elements. The fundamental backbone curve is calculated. Results are shown in Figs. 2 and 3. In Fig.2 the non-dimensional, nonlinear fundamental frequency of vibration $\omega / \omega_{1,lin}$ versus the non-dimensional amplitude of vibration a/i is shown as the solid line. Here, $\omega_{1,lin}$ is the funda-

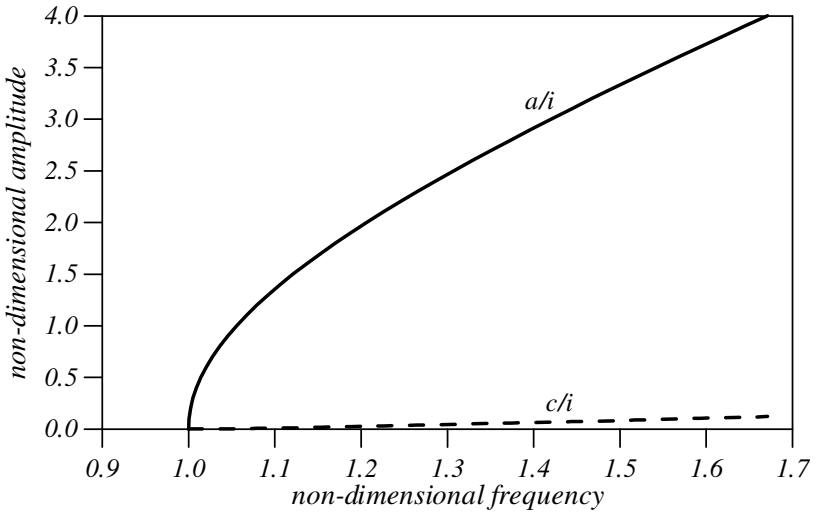

Fig. 2. Three span beam – the fundamental backbone curve

mental, linear frequency of beam and i is the radius of inertia of a beam cross-section. The quantity a denotes the amplitude of vibration in the middle of the first span of the beam. It is obvious that the beam can be considered as the strongly non-linear system when it vibrates with non-dimensional amplitudes of the order 4. The element of $\mathbf{c}$ vector versus non-dimensional frequency is shown as the dashed line. In fact, the entry of the $\mathbf{c}$ vector corresponding to the middle of the first span of the beam, divided by the radius of inertia of the beam cross-section is drawn versus the non-dimensional frequency. On the basis of the presented results, it can be concluded that contribution of the third term of the solution (3.32) is rather small in comparison with the first one. Moreover, this influence is significantly decreased by the second term of the solution (3.32). It is presented in Fig.3 where the non-dimensional transverse displacement in the middle of the first span ($a/i = 4$) is shown in a time domain as the solid line. The dashed line represents the results of calculation when only the first term of the solution (3.32) is taken into account. Differences between both curves are almost imperceptible. It indicates a very good accuracy of results obtained by means of the perturbation method because the dashed line represents also the solution obtained with the help of the harmonic balance method.

Example 2 – the simply-supported-fixed beam

The simply supported-fixed beam is also analysed because of the existence of internal resonance 1:3 (see [6])**.** The beam is divided into eight finite elements. The fundamental backbone curve is determined. The results of calculation are

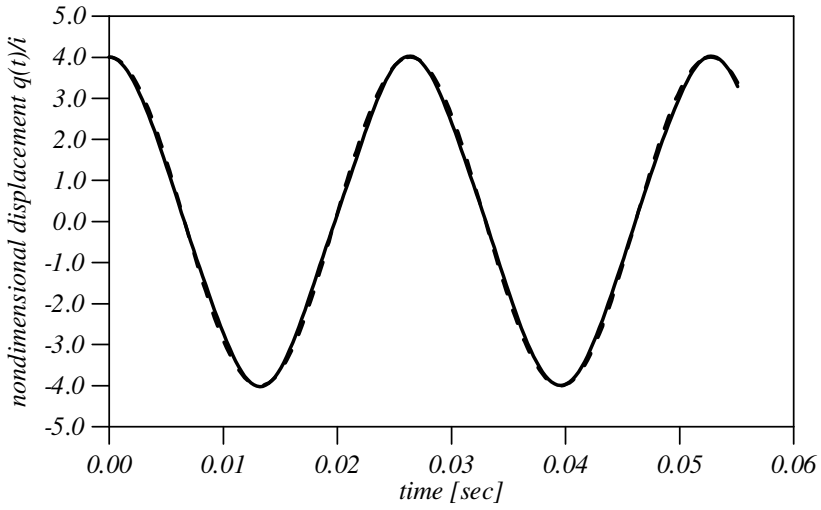

Fig. 3 Three span beam – in time variation of displacement in the middle of first span

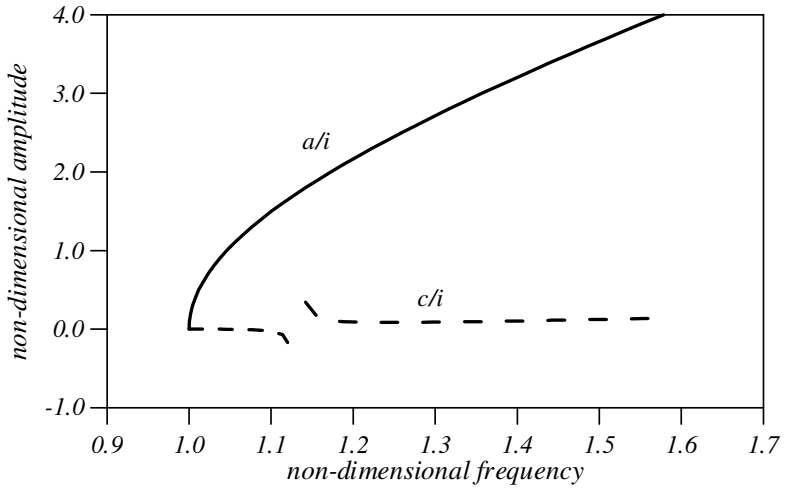

Fig. 4. Simply supported – fixed beam – the fundamental backbone curve

presented in Figs. 4 and 5. In Fig. 4 the variation of non-dimensional frequency ($\omega/\omega_{1,lin}$) versus non-dimensional amplitude (a/i) in the middle of the beam is shown as the solid line. Moreover, the dashed line shows the variation of the non-dimensional element of $\mathbf{c}$ vector, denoted by (c/i) in Fig.4, and corresponding to the middle of the beam. The discontinuity of the above mentioned curve indicates the region of existence of internal resonance. As previously, now eventually excluding the region of internal resonance, a contribution of the $\mathbf{c}$ vector to the global solution is rather insignificant and additionally cancelled by the influences of the second term of the solution (3.32).

This is illustrated in Fig. 5 where the in time variation of the non-dimensional displacement in the point at $5/8l$ from the simple support of the beam is shown. The solution described by (3.32) is shown as the solid curve while the solution given by $\mathbf{q}(t) = \mathbf{a}\cos\omega t$ is shown as the dashed curve. The solution in the region of internal resonance is presented in Fig.5. In comparison with similar results presented previously in Fig.3, differences between the both above mentioned curves are more visible but still small.

7. CONCLUDING REMARKS

Up to now, the applicability of the perturbation method has been restricted to the analysis of weakly non-linear systems while the harmonic balance method is considered as one able to solve the motion equation of strongly non-linear sys-

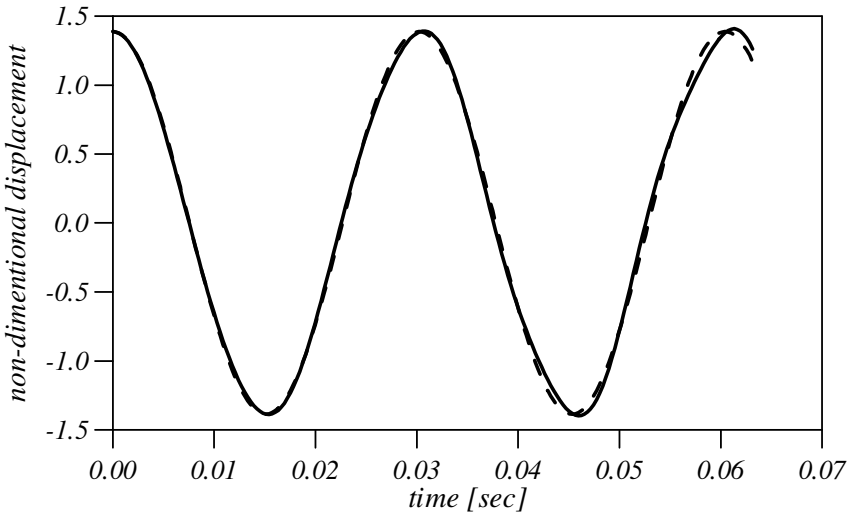

Fig. 5. Simply supported – fixed beam – in time variation of solution in the region of internal resonance

tems with a very good accuracy. In the paper, the possibility of application of the perturbation method to the dynamic analysis of strongly non-linear free vibrations of beams is discussed. The first order perturbation solution of the motion equation is derived and compared with the solution obtained using the harmonic balance method. On the basis of similarities discovered in the both solutions, it was concluded that, for $\varepsilon = 1$ the solution obtained by means of the perturbation method is almost identical to the one given by the harmonic balance method. The presented results of typical calculations confirm these observations. Finally, it is concluded that the perturbation solution has also enough accuracy when the strongly non-linear systems are considered. It is believed that the reason of success of the presented perturbation method comes from a feedback which must be taken into account when the $\mathbf{K}_s$ matrix is introduced. However, at this stage, the presented results are not general and they are valid only in the case of strongly non-linear free vibrations of beams.

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ANALIZA SILNIE NIELINIOWYCH DRGAŃ WŁASNYCH BELEK ZA POMOCĄ METODY PERTURBACJI

S t r e s z c z e n i e

W pracy omawia się możliwości zastosowania metody perturbacji do analizy silnie nieliniowych drgań własnych belek. Uwzględnia się nieliniowości geometryczne. Użyto metody elementów skończonych do opisu dynamicznego zachowania belek. Podano rozwiązanie równania pierwszego przybliżenia metody perturbacji. Wykazano, że jest ono bardzo bliskie rozwiązaniu uzyskiwanemu za pomocą metody bilansu harmonicznych; metody powszechnie stosowanej do analizy układów silnie nieliniowych. Opisano procedurę numeryczną umożliwiającą wyznaczanie krzywych szkieletowych. Rozważania teoretyczne uzupełniono omówieniem wyników przykładowych obliczeń.