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ANALYSIS OF FREE VIBRATIONS OF A PLATE AND FLUID IN CONTAINER

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The paper concerns the free vibrations of a plate in contact with a fluid in a container. The plate is supported at container walls. The analysed problem is a coupled problem of the structure-liquid type. It is assumed that the fluid is inviscid and incompressible. The side walls of the container are treated as rigid. Action of the fluid on the vibrating plate was described by the boundary integral equation. To solve it the boundary element method was used. Elements of constant type were applied. The plate mass was modelled by the masses concentrated in nodes. The plate flexibility matrix was determined using the FEM program. Numerical examples of the free vibrations of rectangular plates are solved.

Keywords: Boundary Element Method; fluid-structure interaction

1. INTRODUCTION

Free vibrations of a plate and a fluid in a container are analysed in this paper. The plate supported at container walls is in contact with the fluid free surface. A vibrating plate induces vibrations of a surrounding fluid which in turn generates additional inertia forces due to the fluid mass. The analysed problem is a coupled problem of the fluid-structure type. Such problems arise for example in the dynamic analysis of dams and liquid tanks.

The aim of this paper is to present a method for the solution of the free vibration problem for a plate interacting with fluid. The method describing the hydrodynamic pressure is based on the boundary integral equation, which is solved by the boundary element method. The most important advantage of the

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boundary integral equation formulations is the reduction of the computational dimension of the problem by one; e.g. three-dimensional problems are solved on two-dimensional surfaces enclosing the domain.

The constant type boundary elements are applied and the first order approximation is used for the calculations of integrals in this paper. The BEM was also used to find the fluid mass matrix in [1, 2]. In [3] vibrations of a circular plate on a liquid surface in a container were analysed using the Rayleigh-Ritz method.

2. PROBLEM FORMULATION

Let us consider a container of any shape with a free surface S_1 and bottom surface S_2 (Fig.1). A plate is in contact with the free surface of a fluid and is supported at container walls. It is assumed that the fluid is incompressible and inviscid. The perturbation fluid velocity potential $\Phi(x, y, z, t)$ satisfies Laplace's equation:

$$\nabla^2 \Phi(x, y, z, t) = 0 \tag{1}$$

The solution of equation (1) may be expressed as a single-layer and double-layer potential for the three-dimensional problem [4]

$$C(P)\Phi(P,t) = \int_S \frac{\partial \Phi(Q,t)}{\partial n(Q)} \Phi^*(P,Q) dS(Q) - \int_S \Phi(Q,t) \frac{\partial \Phi^*(P,Q)}{\partial n(Q)} dS(Q) \tag{2}$$

where: $C(P)$ is a coefficient, which for a smooth surface is equal to 0.5, t is time, $\Phi(P,Q) = \frac{1}{4\pi} \frac{1}{r(P,Q)}$ is the fundamental solution; P, Q are two arbitrary points on the surface S ($S=S_1+S_2$).

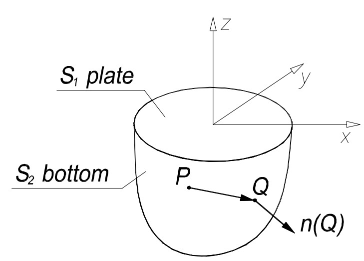

Fig.1. A container of any shape.

The bottom surface of the container is treated as rigid and the small vibrations of the plate and the fluid are considered. Boundary conditions on the surface S are of the Neumann type:

- bottom condition: $\dfrac{\partial \Phi}{\partial n} = 0$ on S_2 (3)
- plate-surface condition: $\dfrac{\partial \Phi}{\partial z} = \dfrac{\partial w_1}{\partial t}$ on S_1 (4)

where w_1 is the plate displacement, $w_1 = \tilde{w}_1 e^{i\omega t}$, ω is the circular frequency.
Boundary condition on the surface S_1 is a coupling condition between the structure and the fluid. The fluid in the container is in the gravity field, so on the wet surface of the plate a hydrostatic lift acts:

$$p_s = -\rho g w_1 \quad \text{on } S_1 \tag{5}$$

where: ρ is the fluid density, g is the gravity acceleration.
This relation is known as the free surface condition. The hydrodynamic pressure acting on the surface S is given by:

$$p_h = -\rho \frac{\partial \Phi}{\partial t} \tag{6}$$

where $p_h = \tilde{p}_h e^{i\omega t}$ is the hydrodynamic pressure.
Differentiating (2) with respect to time and using (4) and (6) we can rewrite (2) in the form:

$$C(P)\tilde{p}_h(P) = \omega^2 \rho \int_S \tilde{w}(Q)\Phi^*(P,Q)dS(Q) - \int_S \tilde{p}_h(Q)\frac{\partial \Phi^*(P,Q)}{\partial n(Q)}dS(Q) \tag{7}$$

3. SOLUTION OF THE PROBLEM

The hydrodynamic pressure acting on the surface S is described by the integral equation (7). The solution of this equation is obtained by means of the boundary element method. The surface of the plate and the bottom of the tank are discretized using triangular or quadrilateral elements of constant type. The collocation points are the centroids of the elements. The boundary element discretization of equation (7) results in the following matrix equation:

$$\mathbf{C}\tilde{\mathbf{p}}_h = \omega^2 \rho \mathbf{A}\tilde{\mathbf{w}} - \mathbf{B}\tilde{\mathbf{p}}_h \tag{8}$$

where: $\tilde{\mathbf{p}}_h = \begin{Bmatrix} \tilde{\mathbf{p}}_{1h} \\ \tilde{\mathbf{p}}_{2h} \end{Bmatrix}$, $\tilde{\mathbf{w}} = \begin{Bmatrix} \tilde{\mathbf{w}}_1 \\ \mathbf{0} \end{Bmatrix}$, $\tilde{\mathbf{p}}_h$ and $\tilde{\mathbf{w}}$ are the vectors of amplitudes of hydrodynamic pressure and displacements, respectively, **C** is the diagonal matrix of coefficients C, **A** and **B** are the (N×N)-quadratic matrices (N is the number of all boundary elements), vectors $\tilde{\mathbf{p}}_{1h}$ and $\tilde{\mathbf{w}}_1$ consist of M elements (M is the number of the plate surface boundary elements).

The elements A_{mn} and B_{mn} of the matrices **A** and **B** are given by:

$$A_{mn} = \int_{S_n} \Phi^*(m,Q)\, dS(Q), \tag{9}$$

$$B_{mn} = \int_{S_n} \frac{\partial \Phi^*(m,Q)}{\partial n(Q)} dS(Q) \tag{10}$$

In the paper, the first-order approximation is used to calculate the integrals (9) and (10) [5,6]. These integrals are evaluated as:

$$A_{mn} = \frac{S_n}{4\pi r_{mn}}, \qquad m \neq n \tag{11}$$

$$A_{nn} = \frac{1}{2}\sqrt{\frac{S_n}{\pi}} \tag{12}$$

$$B_{mn} = \frac{S_n}{4\pi r_{mn}^2} cos\, \alpha_{mn}, \qquad m \neq n \tag{13}$$

where: α_{mn} is defined as the angle between the normal vector **n** at the nodal point n and the vector **r** between nodal points m and n, S_n is the area of n-th element, r_{mn} is the distance between the points m and n.

From the equation (8) we can obtain:

$$\tilde{\mathbf{p}}_h = \omega^2 \rho\, \mathbf{H} \tilde{\mathbf{w}} \tag{14}$$

where $\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} = \mathbf{B}_1^+ \mathbf{A}$, $\mathbf{B}_1$=**C**+**B**, and $\mathbf{B}_1^+$ is the pseudo-inverse of $\mathbf{B}_1$.

Matrix $\mathbf{B}_1$ is singular, so we calculate the Moore-Penrose pseudo-inverse matrix using the SVD procedure [7]. It can be shown [4] that the diagonal elements B_{1ii} can be evaluated as:

$$B_{1ii} = -\sum_{j=1,\, j \neq i}^{N} B_{ij} \tag{15}$$

The amplitudes of the hydrodynamic forces acting at the plate collocation points can be determined as:

$$\tilde{\mathbf{P}} = \omega^2 \mathbf{M}_f \tilde{\mathbf{w}}_1 \tag{16}$$

where: $\mathbf{M}_f = \rho \mathbf{S}\mathbf{H}_{11}$ is the fluid mass matrix, $\mathbf{H}_{11}$ is the ($M \times M$)-submatrix of the matrix $\mathbf{H}$, $\mathbf{S} = \mathrm{diag}(S_1, \ldots, S_M)$, S_i is the area of the i-th element.

The action of the liquid on the vibrating plate consists of:

- a hydrostatic lift,
- an inertia force.

Hence, the equation of free vibrations of the plate takes the form:

$$D\nabla^4 w_1 + \rho g w_1 + \mu \ddot{w}_1 = p_{1h} \tag{17}$$

where: $D = \dfrac{Eh^3}{12\left(1-\nu^2\right)}$ - the plate stiffness, E – the Young's modulus, ν - the Poisson's ratio, h – the plate thickness, μ - the plate unit mass, p_{1h} – the hydrodynamic pressure.

The hydrostatic lift is an analogue of a reaction force of an elastic foundation of the Winkler type. The flexibility matrix of the plate resting on the elastic foundation was calculated using the finite element method computer program PL-WIN. Triangular and quadrilateral finite elements were applied. The components of the plate flexibility matrix are the displacements of the collocation points evoked by the unit forces. The equation of motion of the plate in the matrix form reads:

$$\mathbf{I}\tilde{\mathbf{w}}_1 - \omega^2 \mathbf{D}\mathbf{M}\tilde{\mathbf{w}}_1 = \mathbf{D}\tilde{\mathbf{P}} \tag{18}$$

where $\mathbf{D}$ – the plate flexibility matrix, $\mathbf{M}$ – the plate mass matrix, $\tilde{\mathbf{P}}$ - the vector of the amplitudes of the hydrodynamic forces, $\mathbf{I}$ – the unit matrix.

The plate mass matrix was taken as diagonal with the masses corresponding to the collocation points. With (16) taken into account in (18) one gets:

$$(\mathbf{E} - \lambda \mathbf{I})\tilde{\mathbf{w}}_1 = \mathbf{0} \tag{19}$$

where: $\mathbf{E} = \mathbf{D}\left(\mathbf{M} + \mathbf{M}_f\right)$, $\lambda = \dfrac{1}{\omega^2}$.

The above equation represents the standard eigenproblem. The eigenvectors of the matrix $\mathbf{E}$ express the vibration modes of the plate and the eigenvalues λ allow to calculate the natural frequencies ω.

We shall now consider the calculation of pseudoinverse of matrix $\mathbf{B}_1$. A solution for the internal Neumann problem exists when:

$$\int_S \frac{\partial \Phi}{\partial n} dS = 0 \tag{20}$$

It is equivalent to the incompressibility condition of fluid. Matrix $\mathbf{B}_1$ is singular and the inverse matrix $\mathbf{B}_1^{-1}$ does not exist. Pseudoinverse matrix $\mathbf{B}_1^{+}$ is calculated from the singular value decomposition of the matrix $\mathbf{B}_1$ (SVD procedure [7])

$$\mathbf{B}_1 = \mathbf{U\Sigma}\ \mathbf{V}^T \tag{21}$$

where: $\mathbf{U}$, $\mathbf{V}$ - are orthogonal ($N\times N$)-matrices, $\mathbf{\Sigma} = diag(\sigma_1,...,\sigma_{N-1},0)$, σ_i is the i-th singular value. Hence:

$$\mathbf{B}_1^{+} = \mathbf{V\Sigma}^{+}\mathbf{U}^T \tag{22}$$

where $\mathbf{\Sigma}^{+} = diag(\sigma_1^{+},...,\sigma_{N-1}^{+},0)$, $\sigma_i^{+} = \dfrac{1}{\sigma_i}$, i=1, … , N-1.

In the singular decomposition (21) and (22) we substitute $\sigma_N = \varepsilon$, $\sigma_N^{+} = 1/\varepsilon$, instead of $\sigma_N = \sigma_N^{+} = 0$, where ε is a very small value when compared with other values σ_i. It is suggested to determine ε from the following formula:

$$\frac{1}{\varepsilon} = L\sum_{i=1}^{N-1}\frac{1}{\sigma_i}, \text{ where } L >> 1. \tag{23}$$

During numerical tests it was observed that when L increased, one of circular frequencies obtained from the equation (19) decreased to zero. This meant the lack of vibrations ($\omega_1\approx 0$) for the eigenvector corresponding to the condition (20). When L = 100 to 1000, the smallest circular frequency near zero was about two to three orders of magnitude smaller than next one and it was found to be satisfactory.

In the case of a tank with a flat bottom which is parallel to the free surface of liquid one can use the fundamental solution in the form:

$$\Phi^*(P,Q) = \frac{1}{4\pi r(P,Q)} + \frac{1}{4\pi r(P,Q^{'})} \tag{24}$$

where the point Q' is the mirror-reflection of the point Q with respect to the bottom plane. In this case the flat bottom does not undergo boundary element discretization and the number of unknowns in the problem is reduced.

4. NUMERICAL EXAMPLE

Calculations of the free vibrations of a plate resting on a liquid surface and supported at container walls were carried out. The container has the following dimensions: length 5.0m, width 4.0m, height 3.0m. The steel plate has following parameters: h =1.0cm, E =205GPa, ν=0.3, μ=78.5kg/m^2. The liquid density ρ=1000kg/m^3.

The two cases of plate boundary conditions are analysed: simply supported edges and clamped edges. Rectangular elements were used to discretize the plate and side-walls of the tank.

The obtained values of the natural frequencies of the simply supported plate are presented in the Table 1 and those of the clamped plate – in the Table 2. The last columns of both Tables show the results for the case without liquid.

Figures 2 and 3 present six first modes for both analysed plates.

Table 1. Natural frequencies of the simply supported plate

Number of mode shape	Natural frequency [rad/s]			
	with liquid			without liquid
	Number of degrees of freedom			
	25	49	81	81
1	8.672	8.817	8.859	15.664
2	12.370	12.527	12.564	34.041
3	19.367	19.845	19.954	44.403
4	19.566	20.328	20.498	62.842
5	30.715	31.878	32.083	64.776
6	31.293	33.135	33.530	92.417
7	34.723	38.685	39.424	93.547
8	38.990	41.477	41.948	107.903

Table 2. Natural frequencies of the clamped plate

Number of mode shape	Natural frequency [rad/s]			
	with liquid			without liquid
	Number of degrees of freedom			
	25	49	81	81
1	14.669	14.498	14.390	29.030
2	20.771	20.530	20.367	51.163
3	28.660	28.730	28.544	66.752
4	30.638	30.442	30.193	87.307
5	44.496	45.104	44.841	87.316
6	46.199	47.112	46.902	121.793
7	50.675	52.137	52.144	124.614
8	57.425	58.097	57.882	136.738

The results show a good convergence when the number of degrees of freedom increases. It is clear that the presence of liquid reduces significantly the natural frequencies.

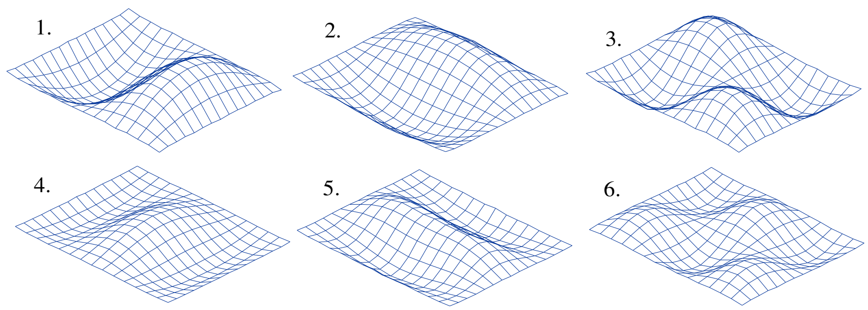

Fig. 2. The first six vibration modes of the simply supported plate

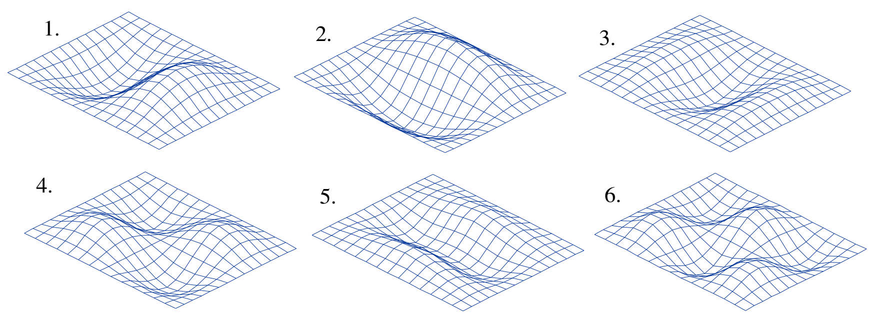

Fig. 3. The first six vibration modes of the clamped plate

5. CONCLUSIONS

A method of calculation of free vibrations of a plate in contact with a fluid in a container was presented in the paper. The hydrodynamic pressure of the liquid was described by the boundary integral equation and the boundary element method was applied to solve it. The constant type elements and the first-order approximation were used. The FEM computer program was used to determine the plate flexibility matrix. The obtained results show a good convergence. The influence of liquid on the plate free vibrations is significant.

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ANALIZA DRGAŃ WŁASNYCH PŁYTY I CIECZY W ZBIORNIKU

Streszczenie

W pracy analizuje się drgania płyty w kontakcie z cieczą znajdującą się w zbiorniku. Płyta zamocowana jest w ścianach bocznych zbiornika. Rozważany problem zalicza się do zagadnień drgań sprzężonych typu konstrukcja-ciecz. Przyjęto, że ciecz jest nieściśliwa, a ściany boczne zbiornika są sztywne. Oddziaływanie cieczy na drgającą płytę opisano brzegowym równaniem całkowym. Do jego rozwiązania wykorzystano metodę elementów brzegowych. Zastosowano elementy brzegowe typu „constans". Przyjęto model płyty z masami skupionymi w węzłach. Do wyznaczenia macierzy podatności płyty zastosowano program MES. Zamieszczono przykłady liczbowe drgań własnych prostokątnej płyty w kontakcie z cieczą w zbiorniku. Wpływ cieczy na drgania własne płyty jest duży.