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Nonlinear boundary condition in electromagnetic theory – analytical scheme of formulation

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Abstract: The paper presents a method of how the nonlinear boundary condition [1] may be applied in nonlinear problems of electromagnetic field theory. It is introduced for problems with nonlinear conductivity. An analytical procedure has been constructed, which seeks to reduce calculations related with the nonlinear region. In order to verify the proposed solutions, two problems have been formulated: one of linear and the other of cylindrical symmetry. These have been additionally solved by the authors' modification of the perturbation method that has been described in previous papers [7, 8, 10]. The electromagnetic field distribution obtained thereby has served as a referential result since it can obtain very accurate solutions [10]. Relative errors of electric and magnetic field strength are introduced to verify the results.

Key words: nonlinear boundary condition, nonlinear conductivity, analytical scheme

1. Introduction

The paper concerns an application of a certain proposed method in electromagnetic problems with nonlinear environment parameters. In order to reduce calculations related with the nonlinear region, a nonlinear boundary condition is applied. Preliminary considerations [1, 2] have led the authors to various implementation variants of this idea.

In a parallel study, a numerical-symbolic scheme was constructed [3] while this paper focuses on an analytical scheme applied to obtain the nonlinear boundary condition.

An application of the nonlinear boundary condition is depicted in Figure 1.
A linear region is presented, which is covered by one with nonlinear conductivity. A boundary condition is applied on the edge of this nonlinear region. If the interest is focused on the linear part then the nonlinear area can be skipped by imposing a nonlinear boundary condition. If so, it should approximate the effect of the nonlinear region.

Examples of linear and cylindrical symmetry are brought forth and analyzed to support the thesis of applying the scheme of the nonlinear boundary condition (Fig. 2).

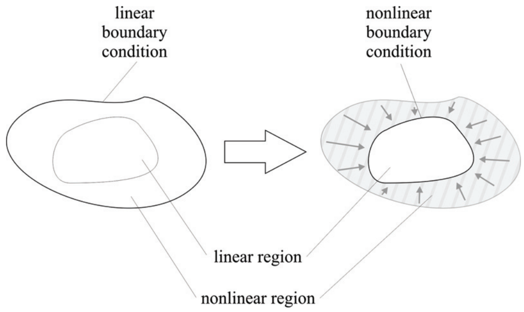

Fig. 1. Nonlinear boundary condition application scheme

In their work, the authors take only into account electromagnetic field components that are periodic in time with a low frequency. In addition, only geometrically simple problems are analyzed (as the ones presented on Figure 2) with field strengths described by only a single axial component each. The state variable for the electromagnetic field is assumed to be an axial component A of the magnetic vector potential $\vec{A}$, for which the equations are formulated in the following sections of this paper.

The presented considerations can be useful when analyzing structures with nonlinear conductive regions like for example high-temperature superconductors: this includes cable design with superconducting tapes covering linear conductors [4, 5].

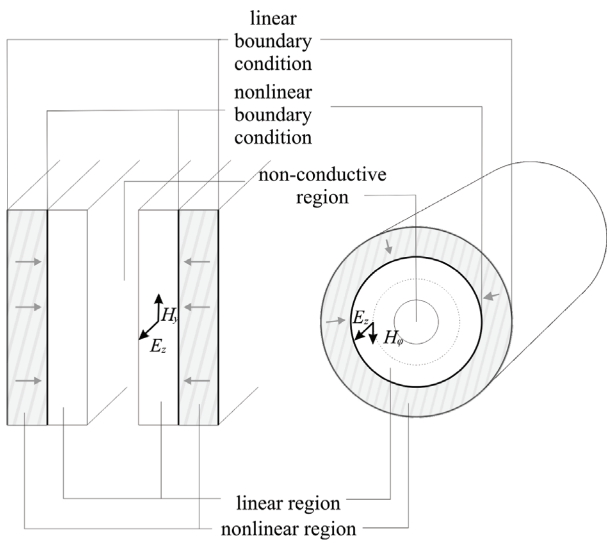

Fig. 2. Examples of nonlinear problems where the nonlinear boundary condition scheme can be used

To reduce the description of the problems, which are solved in a similar way (linear, cylindrical symmetry), Lamé coefficients L_u, L_v, L_w are applied and the orthogonal curvilinear coordinate systems are denoted with the coordinates u, v, w. Changes in the field are assumed to occur only along u and time. The magnetic vector potential is represented as follows:

$$\vec{A} = A\,\vec{1}_w. \tag{1}$$

The variable u refers to the axis orthogonal to the problem boundary (x in Cartesian coordinates, r in cylindrical coordinates).

In general, the nonlinear boundary condition is assumed in the form:

$$f\left(A(t,u_{\mathrm{b}}), \frac{\partial A(t,u_{\mathrm{b}})}{\partial u}\right) = \Xi_{\mathrm{n}}(t), \tag{2}$$

where f is a nonlinear function dependent on:

– material properties,
– problem geometry (physical structure dimensions),
– field constraints (frequency, boundary condition function, external fields).

Ξ denotes the appropriate boundary value imposed.

For one time harmonic, (2) assumes the complex form:

$$f_{\mathrm{c}}\left(\underline{A}(u_{\mathrm{b}}), \frac{\mathrm{d}\underline{A}(u_{\mathrm{b}})}{\mathrm{d}u}\right) = \underline{\Xi}_{\mathrm{n}}, \tag{3}$$

where complex quantities are underlined. Two different derived forms of the nonlinear boundary condition are presented as Equations (27) and (28) in Section 3.

2. Test problems

Auxiliary contours have been added to the exemplary problems shown previously in Figure 2. Both problems can now be solved simultaneously subject to the unified notation (Fig. 3).

A nonlinear conductivity γ is introduced to the region $u \in [u_1, u_2]$:

$$\gamma(E) = \sum_{k=1,3,5\ldots}^{m} \gamma_k E^{k-1}, \tag{4}$$

which yields a nonlinear dependence of the current density J on electric field strength E:

$$J(E) = \sum_{k=1,\,3,\,5\ldots}^{m} \gamma_k E^{k}. \tag{5}$$

E and J are the electric field strength and current density along the w coordinate respectively:

$$\vec{E} = E\,\vec{1}_w, \tag{6}$$

and:

$$\vec{J} = J\,\vec{1}_w. \tag{7}$$

The most suited coordinate systems have been applied in both analyzed structures i.e. respectively Cartesian and cylindrical.

The upper part of Figure 3 represents a busbar configuration of length l, leading an AC current of the amplitude I. The electromagnetic field for $x < 0$ is an exact reflection of the distribution at $x > 0$ subject to the applied simplifications (which is why the contours for $x < 0$ have the same names as for $x > 0$ with an added apostrophe symbol). In addition, an assumption is made that $l \gg h \gg x_{C_2}$, which justifies the changes only along the x-axis.

The lower side of Figure 3 depicts a two-layer cylindrical conductor leading a total current of amplitude I. Displacement currents are omitted, hence an internal condition is assumed for the magnetic field strength H:

$$H_v(t, u_{C_0}) = 0. \tag{8}$$

The magnetic vector potential links to both the electric and magnetic field in the following ways:

$$\frac{\partial A(t,u)}{\partial t} = \frac{\partial A_w(t,u)}{\partial t} = -E_w(t,u) = -E(t,u), \tag{9}$$

$$\frac{\partial A(t,u)}{\partial u} = -\mu H_v(t,u) = -\mu H(t,u). \tag{10}$$

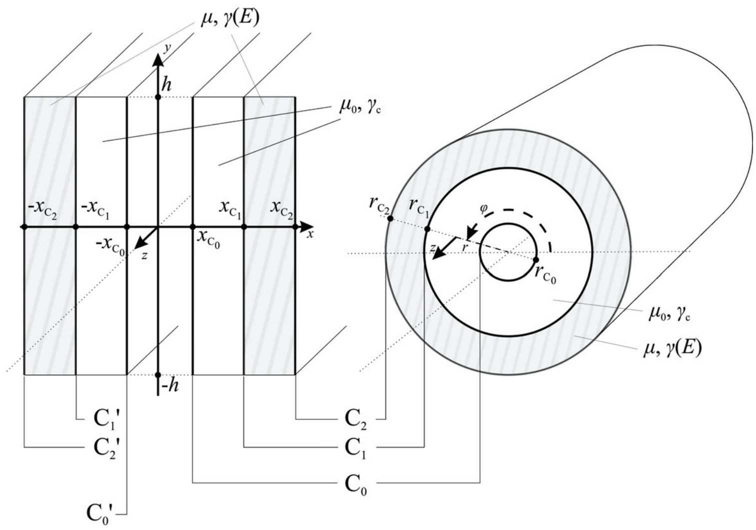

Fig. 3. Test problems of linear and cylindrical symmetry with auxiliary contours (denoted by C and consecutive numbers)

The differential equation for the nonlinear region is formulated:

$$\nabla^2 A = \mu\gamma(E)\frac{\partial A}{\partial t}, \tag{11}$$

which subject to the applied simplifications can be expanded into:

$$\frac{1}{L_u L_v}\frac{\partial}{\partial u}\left(\frac{L_v}{L_u L_w}\frac{\partial(L_w A)}{\partial u}\right)=\mu\gamma(E)\frac{\partial A}{\partial t}. \tag{12}$$

Equation 8 and the assumption of a known total current yield the boundary conditions:

$$\frac{\partial A(u_{C_0})}{\partial u}=0, \tag{13}$$

$$\frac{\partial A(u_{C_2})}{\partial u}=-\frac{\mu I}{o}, \tag{14}$$

where o varies for both coordinate systems:

$$o\big|_{\text{Cartesian}}=2h,\quad o\big|_{\text{cylindrical}}=2\pi r_{C_2}. \tag{15}$$

With the differential equation and boundary conditions kept in mind, one can proceed to obtaining the nonlinear boundary condition.

3. Nonlinear boundary condition formulation

The proposed method uses a formula of an approximate solution in order to assess the effect of the nonlinear region. It is obtained in a similar way as in the analytical method applied by the authors in their previous papers [7, 8 10]. It is based on a scheme of a perturbation method [9], but involves symbolic calculations and numerical support for complicated expressions.

In certain problems, when taking into account the relationships of field components on the nonlinear region boundary connecting it with the linear region, the amount of coefficients that determine the electromagnetic field can be reduced [7]. In the case of the first time harmonic, a single complex coefficient can determine the distribution of the magnetic vector potential. In some cases, a direct relationship can be obtained between the remaining coefficient and a linear combination of the boundary potential and its spatial derivative (in this case – along the variable u). For example in the case of a tubular conductor of radius r_b, leading a total AC current of amplitude I, the relationship between the linear combination and the coefficient $\underline{c}$ is:

$$\underline{c}=\underline{b}_0\underline{A}(r_b)+\underline{b}_1\frac{d\underline{A}(r_b)}{du}. \tag{16}$$

The explanation of how the coefficients $\underline{b}_0$ and $\underline{b}_1$ are obtained is given in Appendix 1. The parameters of (16) are therefore dependent on the modified Bessel functions of the first and second kind (I and K respectively):

$$\underline{b}_0=\frac{K_1(\underline{\Gamma}R_2)}{I_1(\underline{\Gamma}R_2)K_0(\underline{\Gamma}R_2)+I_0(\underline{\Gamma}R_2)K_1(\underline{\Gamma}R_2)}, \tag{17}$$

$$\underline{b}_1 = \frac{K_0(\underline{\Gamma} R_2)}{\underline{\Gamma}\left(I_1(\underline{\Gamma} R_2)K_1(\underline{\Gamma} R_2) + I_0(\underline{\Gamma} R_2)K_1(\underline{\Gamma} R_2)\right)}. \tag{18}$$

A similar derivation can be performed for the nonlinear problems in the previous section, as the internal regions are linear.

An approximate solution was found in [7] where the coefficient's relationship with the boundary value is associated in the following manner:

$$\mathrm{e}^{\mathrm{j}\theta}\sum_{k=1}^{n}\underline{a}_k c^k = \underline{\Xi}. \tag{19}$$

where $\underline{c} = c\ \mathrm{e}^{\mathrm{j}\theta}$. The $\underline{a}_k$ coefficients are presented above with a general relation kept in mind, that is – they can vary in SI units according to the imposed boundary condition. Because of only odd power terms appearing in the *J-E* relationship (5), Equation (19) becomes:

$$\mathrm{e}^{\mathrm{j}\theta}\sum_{k=1,\,3,\,5,\ldots}^{n}\underline{a}_k c^k = \underline{\Xi}. \tag{20}$$

The Equation (19) is split into a linear and nonlinear part (denoted by the function f_n):

$$\underline{\Xi} - \underline{a}_1\underline{c} = f_\mathrm{n}(\underline{\mathrm{c}}), \tag{21}$$

which subject to the notation given in (16) gives:

$$\underline{\Xi} - \underline{a}_1\underline{b}_0\underline{A}(u_\mathrm{b}) - \underline{a}_1\underline{b}_1\frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u} = f_\mathrm{n}(\underline{\mathrm{c}}). \tag{22}$$

The nonlinear part of (22) emerges as a consequence of the nonlinear region of the problem. With it gone, (22) turns into the Hankel (third kind) boundary condition.

The following relationships can be calculated between the coefficient $\underline{c}$ and the magnetic vector potential [7]:

$$\underline{A}(u_{\mathrm{C}_1}) = \underline{c}\underline{\Delta}_\mathrm{p}, \tag{23}$$

or the radial derivative:

$$\frac{\mathrm{d}\underline{A}(u_{\mathrm{C}_1})}{\mathrm{d}u} = \underline{c}\underline{\Delta}_\mathrm{d}, \tag{24}$$

on the boundary $u = u_{\mathrm{C}_1}$. $\underline{\Delta}_\mathrm{p}$ and $\underline{\Delta}_\mathrm{d}$ are coefficients that express the respective relationships. This gives two preliminary forms of the nonlinear boundary condition:

$$\underline{a}_1\underline{b}_0\underline{A}(u_\mathrm{b}) + \underline{a}_1\underline{b}_1\frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u} + f_{\mathrm{n\,p}}\left(\underline{A}(u_\mathrm{b})\right) = \underline{\Xi}_\mathrm{p}, \tag{25}$$

and:

$$\underline{a}_1 \underline{b}_0 \underline{A}(u_\mathrm{b}) + \underline{a}_1 \underline{b}_1 \frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u} + f_{\mathrm{n\,d}}\left(\frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u}\right) = \underline{\Xi}_\mathrm{d}, \tag{26}$$

which in terms of the discussed problems gives respectively:

$$\lambda_\mathrm{d} \frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u} + \lambda_{\mathrm{p}\,1} \underline{A}(u_\mathrm{b}) + \mathrm{e}^{\mathrm{j}\vartheta} \lambda_{\mathrm{p}\,3} \left|\underline{A}(u_\mathrm{b})\right|^3 + \mathrm{e}^{\mathrm{j}\vartheta} \lambda_{\mathrm{p}\,5} \left|\underline{A}(u_\mathrm{b})\right|^5 + \ldots = \underline{\Xi}_\mathrm{p}, \tag{27}$$

and:

$$\lambda_\mathrm{p} \underline{A}(u_\mathrm{b}) + \lambda_{\mathrm{d}\,1} \frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u} + \mathrm{e}^{\mathrm{j}\phi} \lambda_{\mathrm{d}\,3} \left|\frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u}\right|^3 + \mathrm{e}^{\mathrm{j}\phi} \lambda_{\mathrm{d}\,5} \left|\frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u}\right|^5 + \ldots = \underline{\Xi}_\mathrm{d}, \tag{28}$$

where:

$$\vartheta = \arg\left(\underline{A}(u_\mathrm{b})\right), \tag{29}$$

and:

$$\phi = \arg\left(\frac{\mathrm{d}\underline{A}(u_\mathrm{b})}{\mathrm{d}u}\right). \tag{30}$$

An example of how to obtain the coefficients of Equation (27) for the exemplary problems of Section 2 is given in Appendix 2. An assumption that the linear term is dominant in both the nonlinear boundary conditions allows to formulate a linear boundary condition. This remaining Hankel boundary condition is used in the next section to check how a linear boundary condition will serve in comparison with a nonlinear boundary condition in the applied scheme.

4. Comparison with the referential solution

Both exemplary problems are analyzed and a chosen form of the nonlinear boundary condition is applied. For the Neumann boundary condition (14) electric and magnetic field strength values are obtained at $u = u_{\mathrm{C}_1}$ with the use of the nonlinear boundary condition (27). In order to verify the result a referential solution is used, which has been obtained by an analytical method with symbolic computation [7, 8, 10]. This method can solve similar nonlinear problems with a relatively high accuracy [10]. Figures 4 and 5 depict a comparison of obtained time functions by different methods for the linear symmetry problem. Additionally, a Hankel boundary condition is used in order to check if a linear boundary condition is sufficient to approximate the effect of the nonlinear region.

In the same manner, Figures 6 and 7 present the obtained time functions of electric and magnetic field strength for the cylindrical symmetry problem, where also a comparison is made of results obtained by various methods.

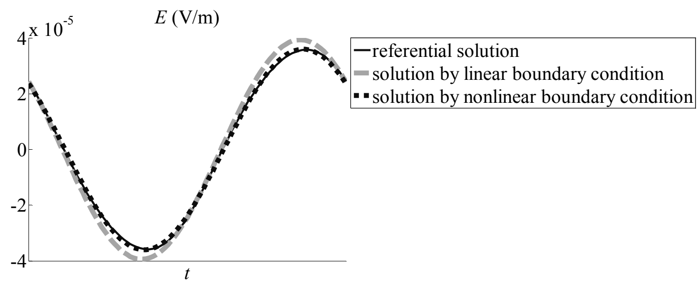

Fig. 4. Solution of busbars Cartesian coordinate problem at $x = x_{C1}$: comparison of electric field strength time function obtained by various methods

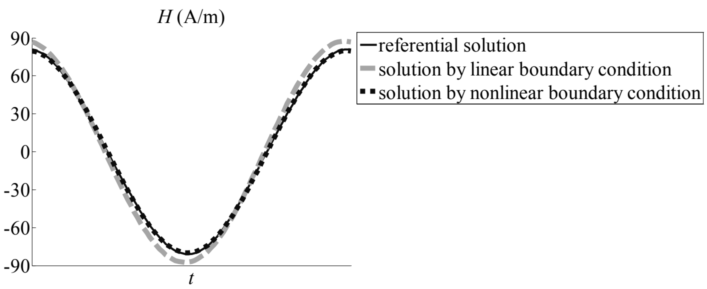

Fig. 5. Solution of busbars Cartesian coordinate problem at $x = x_{C1}$: comparison of magnetic field strength time function obtained by various methods

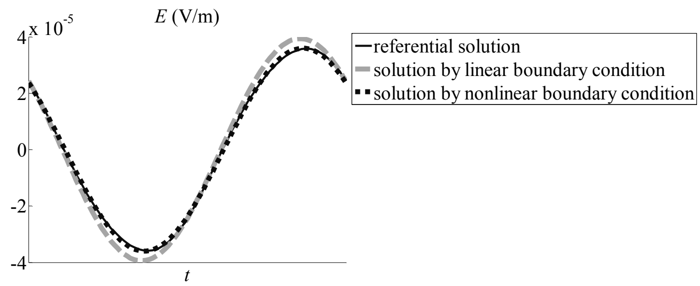

Fig. 6. Solution of cylindrical problem at $r = r_{C1}$: comparison of electric field strength time function obtained by various methods

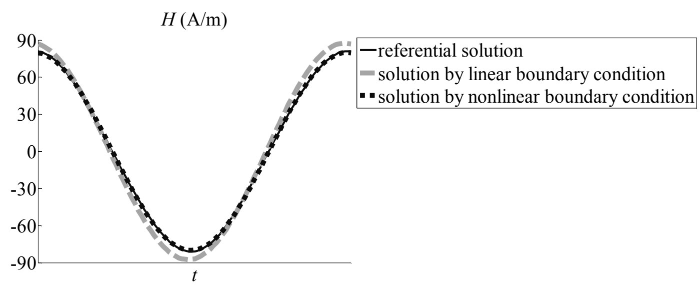

Fig. 7. Solution of cylindrical problem at $r = r_{C1}$: comparison of magnetic field strength time function obtained by various methods

The figures alone exhibit that the nonlinear boundary condition can approximate the effect of the nonlinear region. However, in order to perform a more detailed analysis of the differences, first time harmonic relative errors have been checked. Their results are presented in Table1. The amplitude error is calculated as follows:

$$e_{\text{amp}} = \left| 1 - \left| \underline{X} / \underline{X}_{\text{ref}} \right| \right| \cdot 100\%, \tag{31}$$

while the phase error is evaluated with the use of the complex numbers' argument difference:

$$e_{\text{ph}} = \left| (1/\pi) \arg(\underline{X} / \underline{X}_{\text{ref}}) \right| \cdot 100\%. \tag{32}$$

$\underline{X}$ is the chosen electromagnetic field component (E or H) calculated by the nonlinear boundary condition scheme, $\underline{X}_{\text{ref}}$ is the complex value of the respective field component obtained for the referential solution.

Table 1. Error calculation results for electric and magnetic field strength

Relative errors (%)	For electric field strength	For magnetic field strength
Linear symmetry		
e_{amp}	$9.92 \cdot 10^{-2}$	$1.04 \cdot 10^{-1}$
e_{ph}	$9.33 \cdot 10^{-4}$	$9.13 \cdot 10^{-4}$
Cylindrical symmetry		
e_{amp}	$1.84 \cdot 10^{-1}$	$1.54 \cdot 10^{-1}$
e_{ph}	$9.93 \cdot 10^{-4}$	$9.15 \cdot 10^{-4}$

The table results prove that the nonlinear boundary condition scheme is efficient for the selected problems. The figures clearly depict that a linear boundary condition could not obtain the correct solution.

5. Conclusions

A scheme of how to apply the nonlinear boundary condition in the electromagnetic field has been presented. It has been shown that it allows to approximate the effect of a nonlinear conductive region in the structures of certain problems i.e. busbars and power cable core.

The results were obtained for the nonlinear boundary condition obtained by an analytical scheme. They have been compared with a model solution obtained by an analytical method described in the authors' previous papers [7, 8, 10].

The applied method involves symbolic computation [11] in order to obtain the formula and parameters of the nonlinear boundary condition.

The nonlinear boundary condition allows to obtain the electromagnetic field components on the outer boundary of the linear region. This allows to omit or simplify estimation procedures when the electromagnetic field distribution is evaluated. Hence, the nonlinear boundary condition can be applied to aid numerical calculations and increase their efficiency.

The proposed scheme allows to obtain a very accurate reflection of the electromagnetic field components with respect to the referential solution.

An attempt was made in order to verify if a linear boundary condition could obtain the correct result. The result was inaccurate hence in order for the proper functions of electric and magnetic field strength to be obtained a nonlinear boundary must be applied.

Appendix 1

This appendix presents an example of how the relationship between the distribution coefficient $\underline{c}$ and a linear combination of the boundary potential and its spatial derivative is obtained. The relationship is presented as:

$$\underline{c} = \underline{b}_0 \underline{A}(r_\mathrm{b}) + \underline{b}_1 \frac{\mathrm{d}\underline{A}(r_\mathrm{b})}{\mathrm{d}r}, \tag{A1.1}$$

where further on $r_\mathrm{b} = R_2$. A tubular conductor, leading current $\underline{I}$ is used as an exemplary problem (Fig. 8).

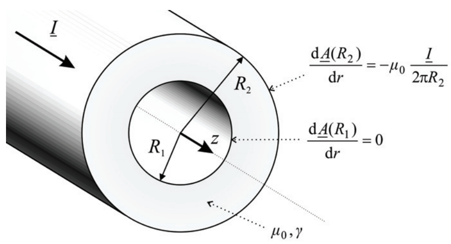

Fig. 8. Exemplary tubular conductor problem with imposed boundary conditions of the second kind

The z axis component of the magnetic vector potential is considered:

$$\vec{A} = A\,\vec{1}_z. \tag{A1.2}$$

The Neumann boundary condition determining the total current in the tubular conductor is:

$$\frac{\mathrm{d}\underline{A}(R_2)}{\mathrm{d}r} = -\mu_0 \frac{\underline{I}}{2\pi R_2} = \underline{\Xi}_\mathrm{N}. \tag{A1.3}$$

The magnetic vector potential distribution is described by the function:

$$\underline{A}(r) = \underline{c}_\mathrm{I} I_0(\underline{\Gamma} r) + \underline{c}_\mathrm{II} K_0(\underline{\Gamma} r), \tag{A1.4}$$

while its radial derivative is:

$$\frac{\mathrm{d}\underline{A}(r)}{\mathrm{d}r} = \underline{c}_\mathrm{I} \underline{\Gamma} I_1(\underline{\Gamma} r) - \underline{c}_\mathrm{II} \underline{\Gamma} K_1(\underline{\Gamma} r), \tag{A1.5}$$

where $\underline{\Gamma}$ is the complex propagation constant:

$$\underline{\Gamma} = \sqrt{\mathrm{j}\omega_1 \mu_0 \gamma}. \tag{A1.6}$$

The relation is sought out on the boundary $r = R_2$ hence the coefficients are reduced by the relation:

$$\underline{c}_{\mathrm{II}} = \frac{\dfrac{\mathrm{d}\underline{A}(R_2)}{\mathrm{d}r} - \underline{c}_{\mathrm{I}} \underline{\Gamma} I_1(\underline{\Gamma} R_2)}{-\underline{\Gamma} K_1(\underline{\Gamma} R_2)}, \tag{A1.7}$$

which when put into (A1.4) leads to the relationship at the boundary:

$$\begin{aligned} \underline{c}_{\mathrm{I}} = & \frac{K_1(\underline{\Gamma} R_2)}{I_1(\underline{\Gamma} R_2) K_0(\underline{\Gamma} R_2) + I_0(\underline{\Gamma} R_2) K_1(\underline{\Gamma} R_2)} \underline{A}(R_2) + \\ & + \frac{K_0(\underline{\Gamma} R_2)}{\underline{\Gamma}\left(I_1(\underline{\Gamma} R_2) K_1(\underline{\Gamma} R_2) + I_0(\underline{\Gamma} R_2) K_1(\underline{\Gamma} R_2)\right)} \frac{\mathrm{d}\underline{A}(R_2)}{\mathrm{d}r}. \end{aligned} \tag{A1.8}$$

which can be presented as (A1.1). In this case the corresponding parameters are:

$$\underline{b}_0 = \frac{K_1(\underline{\Gamma} R_2)}{I_1(\underline{\Gamma} R_2) K_0(\underline{\Gamma} R_2) + I_0(\underline{\Gamma} R_2) K_1(\underline{\Gamma} R_2)}, \tag{A1.9}$$

and:

$$\underline{b}_1 = \frac{K_0(\underline{\Gamma} R_2)}{\underline{\Gamma}\left(I_1(\underline{\Gamma} R_2) K_1(\underline{\Gamma} R_2) + I_0(\underline{\Gamma} R_2) K_1(\underline{\Gamma} R_2)\right)}. \tag{A1.10}$$

Appendix 2

This section lays out an example of the coefficients of Equation (27) in a simplified case when the first time harmonic is dominant and $m = 3$. The Equation (5) becomes:

$$J(E) = \gamma_1 E + \gamma_3 E^3. \tag{A2.1}$$

Taking into account Equation (9), this constitutes a simplified case of Equation (11):

$$\nabla^2 A - \mu\gamma_1 \frac{\partial A}{\partial t} = \mu\gamma_3 \left(\frac{\partial A}{\partial t}\right)^3. \tag{A2.2}$$

The magnetic vector potential is expanded into the series:

$$A = A_1 + \kappa A_2 + \kappa^2 A_3 + \kappa^3 A_4 + \dots . \tag{A2.3}$$

where κ is often referred to as the small parameter [9]. In this section, the lower index numbers denote the "correction term" number, all components are assumed to consist of the first time harmonic alone. The nonlinear term's multiplier is replaced by an expression linearly dependent on this parameter:

$$\gamma_3 = \gamma_3' \kappa. \tag{A2.4}$$

This allows to rewrite (A2.2) as:

$$\sum_{i=1}^{\infty} \kappa^{i-1} \left(\nabla^2 A_i - \mu \gamma_1 \frac{\partial A_i}{\partial t} \right) = \kappa \mu \gamma_3' \left(\sum_{i=1}^{\infty} \kappa^{i-1} \frac{\partial A_i}{\partial t} \right)^3. \tag{A2.5}$$

The above equation is split into linear equations by comparing multipliers of exponentiations of κ on both its sides. The first two equations are:

$$\nabla^2 A_1 - \mu \gamma_1 \frac{\partial A_1}{\partial t} = 0, \tag{A2.6}$$

$$\nabla^2 A_2 - \mu \gamma_1 \frac{\partial A_2}{\partial t} = \mu \gamma_3' \left(\frac{\partial A_1}{\partial t} \right)^3. \tag{A2.7}$$

Further derivations are made with the assumption of a complex form, taking into account one time harmonic. Assuming that:

$$\left| \underline{A}_1 + \kappa \underline{A}_2 \right| >> \left| \kappa^2 \underline{A}_3 + \kappa^3 \underline{A}_4 + \ldots \right|, \tag{A2.8}$$

one can assume that the approximate solution is accurate enough by solving (A2.6) and (A2.7). For a complex form, Equation (A2.6) has the solution:

$$\underline{A}_1(u) = \underline{c}_{\mathrm{I}} \underline{f}_1(u) + \underline{c}_{\mathrm{II}} \underline{f}_2(u), \tag{A2.9}$$

$\underline{f}_1$ and $\underline{f}_2$ depend on the problem geometry. For the problem of linear symmetry, these are exponential functions:

$$\underline{f}_1(x) = \exp(\underline{\Gamma} x), \tag{A2.10}$$

$$\underline{f}_2(x) = \exp(-\underline{\Gamma} x), \tag{A2.11}$$

while for the cylindrical symmetry problem these are modified Bessel functions of the first and second kind:

$$\underline{f}_1(r) = I_0(\underline{\Gamma} r), \tag{A2.12}$$

$$\underline{f}_2(r) = K_0(\underline{\Gamma} r). \tag{A2.13}$$

Equation (A2.7) (in a complex form for the first time harmonic) has the solution [1, 7]:

$$\underline{A}_2(u) = \underline{f}_1(u)\int_{u_{C1}}^{u}\underline{f}_2(u')\underline{W}(u')\underline{\varsigma}(u')\mathrm{d}u' - \underline{f}_2(u)\int_{u_{C1}}^{u}\underline{f}_1(u')\underline{W}(u')\underline{\varsigma}(u')\mathrm{d}u', \quad \text{(A2.14)}$$

where $\underline{\varsigma}$ also depends on the coordinate system. The $\underline{W}$ function represents the first time harmonic of the right-hand side of Equation (A2.7) (in complex form). For Cartesian coordinates:

$$\underline{\varsigma}(x) = \frac{1}{2\underline{\Gamma}}, \quad \text{(A2.15)}$$

while for a cylindrical coordinate system one obtains:

$$\underline{\varsigma}(r) = r. \quad \text{(A2.16)}$$

It can be noticed that subject to the definite integral in (A2.14):

$$\underline{A}(u_{C_1}) = \underline{A}_1(u_{C_1}). \quad \text{(A2.17)}$$

The same can be said for the spatial derivative along u:

$$\frac{\mathrm{d}\underline{A}(u_{C_1})}{\mathrm{d}u} = \frac{\mathrm{d}\underline{A}_1(u_{C_1})}{\mathrm{d}u}. \quad \text{(A2.18)}$$

Hence, the following relationship can be obtained:

$$\underline{c}_1 = \frac{\dfrac{\mathrm{d}\underline{f}_2(u_{C_1})}{\mathrm{d}u}}{\dfrac{\mathrm{d}\underline{f}_2(u_{C_1})}{\mathrm{d}u}\underline{f}_1(u_{C_1}) - \dfrac{\mathrm{d}\underline{f}_1(u_{C_1})}{\mathrm{d}u}\underline{f}_2(u_{C_1})}\underline{A}(u_{C_1}) + \frac{\underline{f}_2(u_{C_1})}{\dfrac{\mathrm{d}\underline{f}_1(u_{C_1})}{\mathrm{d}u}\underline{f}_2(u_{C_1}) - \dfrac{\mathrm{d}\underline{f}_2(u_{C_1})}{\mathrm{d}u}\underline{f}_1(u_{C_1})}\frac{\mathrm{d}\underline{A}(u_{C_1})}{\mathrm{d}u}. \quad \text{(A2.19)}$$

The outer boundary value of the spatial derivative for the first term of (A2.3) is:

$$\frac{\mathrm{d}\underline{A}_1(u_{C_2})}{\mathrm{d}u} = \underline{\lambda}_{p1}\underline{A}(u_{C_1}) + \underline{\lambda}_{d}\frac{\mathrm{d}\underline{A}(u_{C_1})}{\mathrm{d}u}, \quad \text{(A2.20)}$$

where the boundary condition coefficients can be obtained as follows:

$$\underline{\lambda}_{p1} = \frac{\dfrac{\mathrm{d}\underline{f}_2(u_{C_1})}{\mathrm{d}u}\dfrac{\mathrm{d}\underline{f}_1(u_{C_2})}{\mathrm{d}u} - \dfrac{\mathrm{d}\underline{f}_2(u_{C_2})}{\mathrm{d}u}\dfrac{\mathrm{d}\underline{f}_1(u_{C_1})}{\mathrm{d}u}}{\dfrac{\mathrm{d}\underline{f}_2(u_{C_1})}{\mathrm{d}u}\underline{f}_1(u_{C_1}) - \dfrac{\mathrm{d}\underline{f}_1(u_{C_1})}{\mathrm{d}u}\underline{f}_2(u_{C_1})}, \quad \text{(A2.21)}$$

and:

$$\underline{\lambda}_{\mathrm{d}} = \frac{\underline{f}_2(u_{\mathrm{C}_1})\dfrac{\mathrm{d}\underline{f}_1(u_{\mathrm{C}_2})}{\mathrm{d}u} - \dfrac{\mathrm{d}\underline{f}_2(u_{\mathrm{C}_2})}{\mathrm{d}u}\underline{f}_1(u_{\mathrm{C}_1})}{\dfrac{\mathrm{d}\underline{f}_1(u_{\mathrm{C}_1})}{\mathrm{d}u}\underline{f}_2(u_{\mathrm{C}_1}) - \dfrac{\mathrm{d}\underline{f}_2(u_{\mathrm{C}_1})}{\mathrm{d}u}\underline{f}_1(u_{\mathrm{C}_1})}., \tag{A2.22}$$

The above already allows to formulate the Hankel boundary condition. A more difficult task is to obtain the coefficients of the nonlinear part.

A relation of the form (23) needs to be derived. E.g., such a dependence has been obtained in [7]. The $\underline{W}$ function in (A2.14) takes the form:

$$\underline{W}(u) = \frac{3}{4}\mathrm{j}\omega_1{}^3\mu\,\gamma_3^{'}\left|\underline{e}(u)\right|^3\left|\frac{1}{\underline{\Delta}_{\mathrm{p}}}\right|^3\left|\underline{A}(u_{\mathrm{C}_1})\right|^3\mathrm{e}^{\mathrm{j}\left(\arg(\underline{e}(u)\frac{1}{\underline{\Delta}_{\mathrm{p}}})+\vartheta\right)}. \tag{A2.23}$$

where:

$$\underline{e}(u) = \underline{\chi}\underline{f}_1(u) + \underline{f}_2(u). \tag{A2.24}$$

where χ can been obtained through potential and derivative relations on the boundary between the linear region and the nonlinear region [7]. Additionally, the coefficient describing the dependence between the magnetic vector potential and the $\underline{c}$ coefficient is:

$$\underline{\Delta}_{\mathrm{p}} = \underline{e}(u_{\mathrm{C}_1}). \tag{A2.25}$$

The outer boundary value of the potential's second term is:

$$\underline{A}_2(u_{\mathrm{C}_2}) = \left|\underline{A}(u_{\mathrm{C}_1})\right|^3\left|\underline{e}(u_{\mathrm{C}_2})\right|^3\left|\frac{1}{\underline{\Delta}_{\mathrm{p}}}\right|^3\mathrm{e}^{\mathrm{j}\left(\arg(\underline{e}(u_{\mathrm{C}_2})\frac{1}{\underline{\Delta}_{\mathrm{p}}})+\vartheta\right)}\frac{3}{4}\mathrm{j}\omega_1{}^3\mu\gamma_3'\ \underline{\xi}(u_{\mathrm{C}_2}), \tag{A2.26}$$

where the auxiliary function $\underline{\xi}$ is:

$$\underline{\xi}(u) = \underline{f}_1(u)\int_{u_{\mathrm{C}1}}^{u}\underline{f}_2(u')\left|\underline{e}(u')\right|^2\underline{e}(u')\underline{\varsigma}(u')\,\mathrm{d}u' - \\ - \underline{f}_2(u)\int_{u_{\mathrm{C}1}}^{u}\underline{f}_1(u')\left|\underline{e}(u')\right|^2\underline{e}(u')\underline{\varsigma}(u')\,\mathrm{d}u'. \tag{A2.27}$$

Through a comparison of the corresponding terms, one can finally obtain the coefficient of the nonlinear term of the boundary condition (27):

$$\underline{\lambda}_{\mathrm{p}3} = \frac{\frac{3}{4}\mathrm{j}\omega_1{}^3\mu\,\gamma_3\underline{\xi}(u_{\mathrm{C}_2})}{\left|\underline{\Delta}_{\mathrm{p}}\right|^2\underline{\Delta}_{\mathrm{p}}}. \tag{A2.28}$$

When dealing with a more difficult problem, that is when *m* is greater or there is a more significant effect of nonlinearity (more terms of (A2.3) need to be taken into account), the formulae are much more complicated. For the computations made in this paper, their obtainment was made possible with the support of specially written C++ programs featuring specialized symbolic computation [11].

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