Averaged models of pulse-modulated DC-DC power converters.
Part II. Models based on the separation of variables

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Abstract: The separation of variables approach to formulate the averaged models of DC-DC switch-mode power converters is presented in the paper. The proposed method is applied to basic converters such as BUCK, BOOST and BUCK-BOOST. The ideal converters or converters with parasitic resistances, working in CCM and in DCM mode are considered. The models are presented in the form of equation systems for large signal, steady-state and small-signal case. It is shown, that the models obtained by separation of variables approach differ in some situations from standard models based on switch averaging method.

Key words: power converters; pulse-width modulation; BUCK; BOOST; BUCK-BOOST; averaged models

1. Introduction

In paper [1] being the first part of this paper series, the basic methods of creating the averaged models of switch-mode DC-DC power converters are discussed. The averaged model of the power stage of switching converter is typically used in designing the control circuit of converter.

The state-space averaging method of obtaining the averaged model of converter is elegant and accurate but rather inconvenient in practical use, therefore the switch-averaging approach, leading to relatively simple equivalent circuits of converters is usually preferred. According to analysis presented in [1], the acceptable accuracy of switch averaging approach is obtained only for ideal converters working in continuous conduction mode (CCM). For real converters with parasitic resistances or operating in DCM mode, some inaccuracies of switch-averaging approach are observed.

The main purpose of the present paper is to propose another procedure of obtaining the averaged models of switch-mode converters. The starting point for the averaged model for-

mulation is to divide the variables (i.e. currents and voltages) in power stage of converter into two groups, according to the rule presented in Section 2. The principal formulas describing the power stage are obtained as dependencies of variables belonging to one group (denoted as "B") on the variables belonging to other group ("A"). The rule of the separation of variables is simple and the mathematical derivations leading to averaged models are straightforward.

The following sections of the paper are devoted to presentation of the proposed procedure of obtaining the averaged models. In Section 3 the averaged models of ideal converters in *CCM* mode are briefly discussed. The averaged models obtained by separation of variables are, in this case, consistent with models obtained by switch averaging and presented in the literature. By the applying the proposed method to converters working in *DCM* and converters with parasitic resistances, the series of new formulas for averaged large signal, steady-state and small-signal models of converters are obtained. Such formulas for ideal *BUCK, BOOST* and *BUCK-BOOST* converters working in *DCM* are presented in Section 4. Section 5 is devoted to converters with parasitic resistances, namely: *BUCK* and *BOOST* in *CCM* and *BUCK* in *DCM*. The figures representing the schemes of the converters under considerations are presented in the first part of this series [1] and are not repeated here.

2. Two groups of variables in the description of DC-DC converters

In textbooks, papers and application notes concerning the basic DC-DC, PWM- controlled converters, the simplified waveforms of currents and voltages are often presented. The example of idealized waveforms for simple *BUCK* converter working in *CCM* mode is shown in Figure 1. The constant values of v_L and i_C in given subinterval, for example *ON*, and resulting linear dependencies of i_L and v_C on time, are usually accepted approximation (see [1], Section 1).

It is seen, that the waveforms of inductor current i_L and output voltage v_O (or capacitor voltage $v_C = v_O$) differ fundamentally from waveforms of inductor voltage v_L and capacitor current i_C. The local average value $i_L(ON)$ of inductor current for the *ON* subinterval is the same as $i_L(OFF)$ for *OFF* subinterval and, consequently, is the same as the average value i_{LS} of inductor current over the full switching period T_S. The same feature is observed for capacitor voltage and for input voltage v_G. The changes of input voltage are very slow (as compared to fast switching of frequency f_S) and there are no reasons for the difference between $v_G(ON)$ and $v_G(OFF)$. The features of inductor voltage and capacitor current (as well as input current for *BUCK* and semiconductor switches currents and voltages) are different. Therefore it may be written, for example:

$$i_L(ON) = i_L(OFF) = i_{LS}, \tag{1}$$

$$v_O(ON) = v_O(OFF) = v_{OS} \tag{2}$$

and:

$$v_L(ON) \neq v_L(OFF) \neq v_{LS}, \tag{3}$$

$$i_C(ON) \neq i_C(OFF) \neq i_{CS} . \tag{4}$$

The variables in equations describing a converter are therefore divided into two groups. The group *A* contains the variables having the same average values in *ON* and *OFF* subintervals (equal to averages for the whole switching period T_S), for example input and output voltages and inductor current for *BUCK* in *CCM*. The group *B* contains variables with different values in subintervals *ON* and *OFF*: inductor voltage v_L, capacitor current i_C, input current i_G, currents and voltages of semiconductor switches.
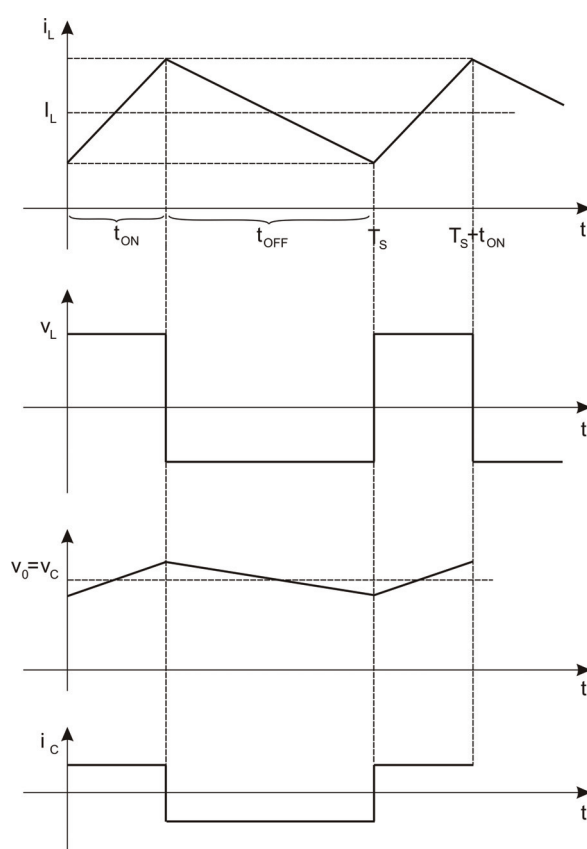

Fig. 1. Idealized waveforms of inductor current i_L, voltage v_L, and capacitor voltage v_C and current i_C for *BUCK* converter in *CCM*

Such separation of variables in the description of switch-mode converters is useful in derivation of averaged models of converters. Some equations describing a given converter scheme are independent of the converter state (*ON* or *OFF*) and are obtained directly from Kirchhoff laws. Other equations depend on the state of switches. These equations should be written as dependencies of the variables belonging to the group *B* on the variables of the group *A*. Applying the general formula for the average value w_S of a variable $w(t)$ in *CCM*:

$$w_S = d_A \cdot w(ON) + (1 - d_A) \cdot w(OFF),$$ (5)

(or properly modified formula for *DCM*), one obtains the equations for the average values of circuit variables for period T_S.

The inductor current for basic converters working in *CCM* belongs to the group *A*. It is not true for *DCM* mode, because the waveform of i_L corresponds to Figure 2. The inductor current in *DCM* belongs to the group *B*.
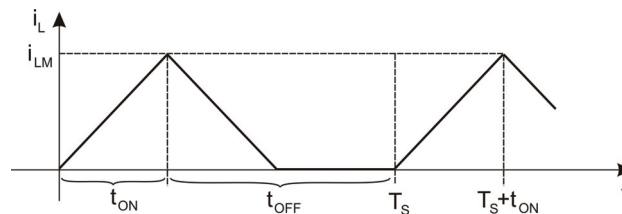

Fig. 2. The typical waveform of inductor current (for *BUCK* or *BOOST)* in *DCM*

3. Ideal converters in *CCM*

In this section, the derivation of averaged models of ideal *BUCK* and *BOOST* converters based on the separation of variables, is presented. The large-signal nonlinear averaged models are considered in Section 3.1. Steady-state and small-signal description is given in Section 3.2. The schemes of the converters under considerations (as well as their equivalent circuits for particular subintervals) have been presented in [1].

3.1. Large signal averaged description

In the equations given below, the notation used for variables of group *B* distinguishes the values for *ON* or *OFF* state and the average values for period T_S, for example $v_L(ON)$, $v_L(OFF)$, v_{LS}. Symbols of variables belonging to the group *A* do not contain the indicator of state (*ON, OFF*), nor the subscript "*S*" for average values, because the values of these variables for state *ON* and *OFF* are equal to average value.

3.1.1. BUCK converter

The equations independent of the switches state for ideal *BUCK*, true also for averaged values (see Fig. 3a in [1]) are:

$$v_L = L \cdot \frac{di_L}{dt},$$ (6)

$$i_L = G \cdot v_O + C \frac{dv_O}{dt}.$$ (7)

The dependencies of the variables of group *B* on variables of group *A*:

- *ON* (Fig. 3b in [1]):

$$v_L(ON) = v_G - v_O, \qquad (8)$$

$$i_G(ON) = i_L \qquad (9)$$

- *OFF* (Fig. 3c in [1]):

$$v_L(OFF) = -v_O, \qquad (10)$$

$$i_G(OFF) = 0. \qquad (11)$$

Applying formula (5) to v_L and i_G given by (8) – (11), one obtains:

$$v_{LS} = d_A \cdot v_G - v_O, \qquad (12)$$

$$i_{GS} = d_A \cdot i_L. \qquad (13)$$

The averaged, large signal model of *BUCK* consists of Equations (6), (7), (12), (13).

3.1.2. BOOST converter

The equations independent of the switches state for ideal *BOOST* shown in Figure 7a) in [1], true also for averaged values, are Equation (6) and:

$$i_C = C \cdot \frac{dv_O}{dt}. \qquad (14)$$

The dependencies of the variables of group *B* on variables of group *A*:
- *ON* (Fig. 7b in [1]):

$$v_L(ON) = v_G, \qquad (15)$$
$$i_C(ON) = -G \cdot v_O. \qquad (16)$$

- *OFF* (Fig. 7c in [1]):

$$v_L(OFF) = v_G - v_O, \qquad (17)$$

$$i_C(OFF) = i_L - G \cdot v_O. \qquad (18)$$

Applying formula (5) to v_L and i_C one obtains:

$$v_{LS} = v_G - (1 - d_A) \cdot v_O, \qquad (19)$$

$$i_{CS} = (1 - d_A) \cdot i_L - G \cdot v_O. \qquad (20)$$

The large-signal, averaged model of ideal *BOOST* in *CCM* is expressed by Equations (6), (14), (19), (20).

3.2. Steady-state and small-signal dependencies

Steady-state and small-signal averaged characteristics may be obtained from large-signal models in similar way as in the standard approach discussed in the first part of the paper, [1].

In the case of *BUCK* and *BOOST* converters, each variable in the large-signal, averaged model is expressed as a sum of steady-state and small-signal terms, for example

$$v_G = V_G + v_g(t), \tag{21}$$

$$v_O = V_O + v_o(t), \tag{22}$$

$$i_L = I_L + i_l(t). \tag{23}$$

Similar description, corresponding to Equation (65) in [1] is applied to the duty ratio d_A.

After introducing the above expressions into large signal averaged equations of converter, one can obtain the steady-state and small-signal models of converter in the same way as in standard approach (state-space averaging or averaged switch models, see [1]). For example of ideal *BUCK* in *CCM*, the steady-state equations are:

$$I_G = D_A \cdot I_L, \tag{24}$$

$$V_O = D_A \cdot V_G, \tag{25}$$

$$I_L = G \cdot V_O. \tag{26}$$

Small-signal dependencies (in the case of simple nonlinearities are obtained by neglecting the products of small-signal quantities) for *BUCK* in *CCM* are:

$$i_g = D_A \cdot i_l + d_a \cdot I_L, \tag{27}$$

$$L \cdot \frac{di_l}{dt} = D_A \cdot v_g + V_G \cdot d_a - v_o. \tag{28}$$

$$i_l = G \cdot v_o + C \cdot \frac{dv_o}{dt}. \tag{29}$$

The corresponding description in *s*-domain is:

$$I_g(s) = D_A \cdot I_l(s) + I_L \cdot \Theta(s), \tag{30}$$

$$sL \cdot I_l(s) = D_A \cdot V_g(s) + V_G \cdot \Theta(s) - V_o(s), \tag{31}$$

$$I_l(s) = G \cdot V_o(s) + s \cdot C \cdot V_o(s), \tag{32}$$

where θ is the *s*-domain representation of small signal term of the duty ratio d_A.

The averaged models of ideal *BUCK* in *CCM*: large-signal (Eqns. 6, 7, 12, 13), steady-state (Eqns. 24, 25, 26) and small-signal (Eqns. 30, 31, 32) derived above, are identical to corresponding models obtained by state-space averaging or with the use of averaged models of a switch pair. The same is observed for other ideal converters in *CCM*. As a result, the small-signal transmittances H_g, H_d defined in [1] (Eqns. 74, 75) for ideal converters in *CCM*, for example obtained from the above Equations (30)-(32) for *BUCK*, are expressed by formulas

identical with (76), (77) in [1]. Similarly, the transmittances input-to-output H_g and control-to-output H_d derived from the Equations (6), (14), (19), (20) for ideal *BOOST* converter in *CCM* are expressed by equations:

$$H_g = \frac{1 - D_A}{LC \cdot s^2 + G \cdot L \cdot s + (1 - D_A)^2},$$ (33)

$$H_d = \frac{I_L \cdot L \cdot s - (1 - D_A) \cdot V_O}{LC \cdot s^2 + G \cdot L \cdot s + (1 - D_A)^2},$$ (34)

identical with the formulas derived by switch averaging technique [2, 3].

Other characteristics of converters may be derived from the large-signal averaged models, for example input and output small-signal impedances or inductor current response on the changes of input voltage or duty ratio. They are not discussed here for ideal converters working in *CCM* because the presented description leads in this case to the same results as the switch averaging approach.

4. Ideal converters in DCM

4.1. Large-signal description

4.1.1. Introduction

The large-signal averaged equations describing the ideal *BUCK, BOOST* and *BUCK-BOOST* in discontinuous conduction mode (*DCM*) are derived here, according to separation of variables approach. As it is discussed in Sections 2.3 and 3.3 of paper [1], the *OFF* state in *DCM* consists of two subintervals, *OFF1* and *OFF2*. In the *OFF2* state, the inductor current and consequently the inductor voltage v_L are zero. Therefore, there are three subintervals of switching period in *DCM* corresponding to *ON, OFF1* and *OFF2* states of the duration $d_A \cdot T_S$, $d_B \cdot T_S$ and $(1 - d_A - d_B) \cdot T_S$ respectively. The simplified inductor current waveform in *DCM* for these converters are depicted in Figure 2. It is evident, that the inequality is true:

$$i_L(ON) \neq i_L(OFF) \neq i_{LS,}$$ (35)

therefore the inductor current belongs to the group *B* in *DCM*, according to classification given in Section 2. Group *A* consists only of input and output voltages v_G and v_O. In the large-signal averaged model of converter in *DCM*, apart from the equations true for each subinterval, the dependencies of currents and voltages belonging to the group *B* on the input and output voltages (group *A*) should be specified. The formula for averaging a given quantity $w(t)$ in *DCM* is:

$$w_S = d_A \cdot w(ON) + d_B \cdot w(OFF1) + (1 - d_A - d_B) \cdot w(OFF2).$$ (36)

4.1.2 BUCK converter

The following equations, identical with (6) and (7) are valid for each subinterval (and also for average values):

$$v_L = L \cdot \frac{di_L}{dt},$$ (37)

$$i_L = G \cdot v_O + C\frac{dv_O}{dt}.$$ (38)

For deriving additional dependencies, the inductor voltage for consecutive states should be found. According to Figures 3 b), c) and 4 in [1], the voltage v_L values for *ON, OFF1* and *OFF2* state are: $v_G - v_O;\ -v_O$ and 0 respectively. The slope of the dependence of inductor current on time, shown graphically in Figure 2 is v_L/L. By comparing the changes of inductor current in *ON* and *OFF1* states, one obtains:

$$\frac{v_G - v_O}{L} \cdot d_A \cdot T_S = \frac{v_O}{L} \cdot d_B \cdot T_S.$$ (39)

As a consequence we have:

$$d_B = d_A \cdot \frac{v_G - v_O}{v_O}.$$ (40)

The average value v_{LS} of inductor voltage over the whole switching period is obtained by substituting the values of v_L for *ON, OFF1, and OFF2* states into (36) and using (40). As a result, one obtains the average value of inductor voltage in *DCM*:

$$v_{LS} = 0.$$ (41)

The average value of inductor current may be expressed as:

$$i_{LS} = \frac{S_L}{T_S},$$ (42)

where S_L is the area of a triangle under the graph of inductor current waveform (Fig. 2). The result for *BUCK* is:

$$i_{LS} = \frac{d_A^2 \cdot T_S}{2L} \cdot \frac{v_G}{v_O} \cdot (v_G - v_O).$$ (43)

The input current i_G is nonzero only in *ON* subinterval:

$$i_G(ON) = i_L(ON) = \frac{1}{2} \frac{v_G - v_O}{L} \cdot d_A \cdot T_S.$$ (44)

The average of input current is then:

$$i_{GS} = d_A \cdot i_G (ON) = \frac{d_A^2 \cdot T_S}{2L} \cdot (v_G - v_O). \tag{45}$$

The large-signal averaged model of ideal *BUCK* converter in *DCM* consists of Equations (37) with (41), (38), (43) and (45).

4.1.3. Ideal BOOST and BUCK-BOOST converters

The derivation of large-signal averaged models for ideal *BOOST* and *BUCK-BOOST* converters is, in part, similar to that of *BUCK* converter. Equation (37) is true for *BOOST* and *BUCK-BOOST*, as well as for *BUCK*. The dependence of diode current i_D on output voltage v_O should be used for *BOOST* instead of Eqn. (38) (see Fig. 7 in [1]).

$$i_D = G \cdot v_O + C \cdot \frac{dv_O}{dt}. \tag{46}$$

The expression for i_D in *BUCK-BOOST* is similar to (46):

$$i_D = -G \cdot v_O - C \cdot \frac{dv_O}{dt}. \tag{47}$$

The dependencies (46), (47) are valid for instantaneous values and for averaged values as well.

The slope of time-dependence of inductor current is of course v_L/L. The values of v_L for *ON* and *OFF1* states are taken from Figures 7 and 11 b) and c) in [1] and are as follows:
- for *BOOST*:

$$v_L(ON) = v_G; \quad v_L(OFF1) = v_G - v_O, \tag{48}$$

- for *BUCK-BOOST*:

$$v_L(ON) = v_G; \quad v_L(OFF1) = v_O. \tag{49}$$

By comparing the changes of inductor current in *ON* and *OFF1* states one obtains for *BOOST*:

$$d_B(BOOST) = \frac{v_G}{v_O - v_G} \cdot d_A \tag{50}$$

and for *BUCK-BOOST:*

$$d_B(BUCK - BOOST) = -\frac{v_G}{v_O} \cdot d_A. \tag{51}$$

Calculations of the average value v_{LS} of inductor voltage from Equation (36), using (48), (50) for *BOOST* and (49), (51) for *BUCK-BOOST* result, for both converters, in the same relations as (41) i.e.:

$$v_{LS} = 0. \tag{52}$$

The derivation of averaged inductor current i_L, input current i_{GS}, and diode current i_D leads to:

• *BOOST*:

$$i_{GS} = i_{LS} = \frac{d_A^2 \cdot T_S}{2L} \cdot \frac{v_G \cdot v_O}{v_O - v_G}, \tag{53}$$

$$i_{DS} = \frac{d_A^2 \cdot T_S}{2L} \cdot \frac{v_G^2}{v_O - v_G}, \tag{54}$$

• *BUCK-BOOST*:

$$i_{DS} = -\frac{d_A^2 \cdot T_S}{2L} \cdot \frac{v_G^2}{v_O}, \tag{55}$$

$$i_{GS} = \frac{d_A^2 \cdot T_S}{2L} \cdot v_G, \tag{56}$$

$$i_{LS} = \frac{d_A^2 \cdot T_S}{2L} \cdot \frac{v_G}{v_O} \cdot (v_O - v_G). \tag{57}$$

As it is mentioned in Section 3.3. of [1], the controversies involved with application of averaged switch modeling to description of converters in *DCM* concern the dynamic characteristics, which, according to [2, 4, 5] are of the second order whereas, according to state-space averaging approach, should be of the first order. The equation set presented in subsection 4.1.2 leads also to the first order dynamics because of the condition (41) (together with 37) for *BUCK*. The same is true for *BOOST* and *BUCK-BOOST*, according to Equation (52) and (37). The large-signal averaged model of ideal *BOOST* converter in *DCM* consists of Equations (37), (46), (53) and (54). For *BUCK-BOOST* it consists of Equations (37), (47), (55) and (56). The expressions for i_{LS} for *BOOST* (53) and *BUCK-BOOST* (57) may be treated as additional equations. They are not necessary for finding typical small-signal transmittances.

4.2. Steady-state dependencies and small-signal transmittances

Steady-state and small-signal dependencies may be obtained from large-signal averaged models. The derivation of these dependencies is presented here for ideal *BUCK* converter only. From (38) for steady-state we obtain:

$$I_L = G \cdot V_O. \tag{58}$$

By substituting (58) into (43) rewritten for steady-state quantities, one obtains:

$$G \cdot V_O^2 = G_Z \cdot D_A^2 \cdot (V_G^2 - V_G \cdot V_O), \tag{59}$$

where:

$$G_Z = \frac{T_S}{2L}. \tag{60}$$

Solving (59) we get the steady-state output-to-input voltage ratio:

$$M_V = \frac{V_O}{V_G} = \frac{1}{2} \cdot D_A \cdot R \cdot G_Z \cdot \left(\sqrt{D_A^2 + 4G / G_Z} - D_A \right). \tag{61}$$

The above dependence is known in the literature (for example may be found in [2]). The formulas for steady-state output voltage dependence on V_G which may be derived for *BOOST* and *BUCK-BOOST* converters from the equations given in subsection 4.1.3, are also consisted with the formulas presented in [2].

The steady-state value of input current can be obtained immediately from (45). Using additionally Equation (61), the quiescent value of input current dependence on input voltage can be expressed in the form:

$$I_G = G_{IN} \cdot V_G, \tag{62}$$

where the steady-state input conductance is:

$$G_{IN} = 2\beta \cdot G \cdot (1 + \beta - \sqrt{\beta^2 + 2\beta}) \tag{63}$$

and

$$\beta = \frac{D_A^2 \cdot T_S}{4L \cdot G} = \frac{D_A^2}{2} \cdot G_Z \cdot R. \tag{64}$$

The small signal model of ideal *BUCK* converter in *DCM* may be obtained from the large signal description expressed by Equations (38), (43) and (45) using similar procedure, as described above in Section 3.2 and in [1], Section 4. The Equation (43), with the use of (60) is rewritten in the form:

$$i_L = G_Z \cdot d_A^2 \cdot \left(\frac{v_G^2}{v_O} - v_G \right). \tag{65}$$

It is a functional dependence:

$$i_L = f(d_A, v_G, v_O). \tag{66}$$

Its small-signal equivalent is expressed in the form:

$$I_l = \alpha_{i1} \cdot \theta + \alpha_{i2} \cdot V_g + \alpha_{i3} \cdot V_o, \tag{67}$$

where coefficients α_i are partial derivatives of the dependence (66) calculated at quiescent point (V_G, V_O, D_A).

$$\alpha_{i1} = \frac{\partial f}{\partial d_A} = 2G_Z \cdot V_G \cdot D_A \cdot \left(\frac{V_G}{V_O} - 1 \right), \tag{68}$$

$$\alpha_{i2} = \frac{\partial f}{\partial v_G} = G_Z \cdot D_A^2 \cdot \left(\frac{2V_G}{V_O} - 1 \right), \tag{69}$$

$$\alpha_{i3} = \frac{\partial f}{\partial v_O} = -G_Z \cdot D_A^2 \cdot \frac{V_G^2}{V_O^2}. \tag{70}$$

Equation (38) is linear, therefore its small-signal equivalent in *s*-domain is:

$$I_l = (G + s \cdot C) \cdot V_o. \tag{71}$$

From Equations (67)-(70) one obtains:

$$I_l = 2G_Z \cdot V_G \cdot D_A \cdot (M_I - 1) \cdot \theta + G_Z \cdot D_A^2 \cdot (2M_I - 1) \cdot V_g - G_Z \cdot D_A^2 \cdot M_I^2 \cdot V_o. \tag{72}$$

Substituting small-signal inductor current given by (71) into (72) we have:

$$(G_O + sC) \cdot V_o + G_Z \cdot D_A^2 \cdot M_I^2 \cdot V_o = 2G_Z \cdot V_G \cdot D_A \cdot (M_I - 1) \cdot \theta + G_Z \cdot D_A^2 \cdot M_I^2 \cdot V_o, \tag{73}$$

where:

$$M_I = \frac{V_G}{V_O}. \tag{74}$$

From the above equations, the input-to-output and control-to-output transmittances are obtained:

$$H_{gD} = \frac{G_Z \cdot D_A^2 \cdot (2M_I - 1)}{s \cdot C + G + G_Z \cdot D_A^2 \cdot M_I^2}. \tag{75}$$

$$H_{dD} = \frac{2 \cdot G_Z \cdot V_G \cdot D_A \cdot (M_I - 1)}{s \cdot C + G + G_Z \cdot D_A^2 \cdot M_I^2}. \tag{76}$$

The above transmittances are of the first-order, as it was earlier mentioned. Other small-signal characteristics may be easily obtained from Equations (71), (72), or with the additional use of Equation (45). As an example, the small-signal input characteristics may be considered. The small-signal term of input current can be expressed in the form:

$$I_g = \alpha_{g1} \cdot \theta + \alpha_{g2} \cdot V_g + \alpha_{g3} \cdot V_o. \tag{77}$$

The coefficients α_g are partial derivatives of the dependence (45):

$$\alpha_{g1} = \frac{\partial i_{GS}}{\partial d_A} = G_Z \cdot 2 \cdot D_A \cdot (V_G - V_O), \tag{78}$$

$$\alpha_{g2} = \frac{\partial i_{GS}}{\partial v_G} = G_Z \cdot D_A^2, \tag{79}$$

$$\alpha_{g3} = \frac{\partial i_{GS}}{\partial v_O} = -G_Z \cdot D_A^2. \tag{80}$$

Additional dependence for V_o is of the form:

$$V_o = V_g \cdot H_{gD} + \theta \cdot H_{dD}, \tag{81}$$

where H_{gD} and H_{dD} transmittances are expressed by (75) and (76). From (77) and (81) we can obtain the small-signal input current dependence on input voltage and duty ratio in the form:

$$I_g = Y_{in} \cdot V_g + I_\theta \cdot \theta. \tag{82}$$

In Equation (82) the input admittance is defined as:

$$Y_{in} = \frac{I_g}{V_g}\bigg|_{\theta=0} \tag{83}$$

and control-dependent input current term is:

$$I_\theta = \frac{I_g}{\theta}\bigg|_{V_g=0}. \tag{84}$$

The above characteristics, after simple manipulations can be presented in the form:

$$Y_{in} = \alpha_2 + \alpha_3 \cdot H_{gD}, \tag{85}$$

$$I_\theta = \alpha_1 + \alpha_3 \cdot H_{dD}. \tag{86}$$

Using earlier dependencies, the input characteristics of ideal *BUCK* converter in *DCM* may be expressed as:

$$Y_{in} = G_Z \cdot D_A^2 \cdot \frac{sC + G + G_Z \cdot D_A^2 \cdot (M_I - 1)^2}{sC + G + G_Z \cdot D_A^2 \cdot M_I^2}, \tag{87}$$

$$I_\theta = 2G_Z \cdot D_A \cdot (V_G - V_O) \cdot \frac{sC + G + G_Z \cdot D_A^2 \cdot M_I(M_I - 1)}{sC + G + G_Z \cdot D_A^2 \cdot M_I^2}. \tag{88}$$

Similar procedures may be easily applied to other converters and their small-signal transmittances may be found. The procedure of derivation of small-signal transmittances for ideal converters in *DCM* and the resulting formulas differ from that presented in literature (for example [2, 4-6]).

5. Converters with parasitic resistances

5.1. Introduction

The parasitic resistances of inductors, capacitors and semiconductor switches may strongly influence the features of real converters, in particular their efficiency and dynamic characteristics. In addition, as it is discussed in Section 3.4 of paper [1], the assumptions used in derivation of converter characteristics based on the concept of averaged switch models [2-4], may not be fulfilled in the presence of parasitic effects. The small-signal transmittances of converters presented in the literature usually account for only a part of parasitic resistances (as for example in application notes [7-9]).

The models of inductor, capacitor and semiconductor switches with the parasitic resistances used in this section correspond to Figure 14 in [1], Section 3.4. The considerations of this section are addressed to *BUCK* and *BOOST* converters with parasitic elements. In subsections 5.2 and 5.3, the characteristics of converters working in *CCM* are considered: large signal and next – steady-state and small-signal. Subsections 5.4 and 5.5. are devoted to characteristics of converters in *DCM*.

5.2. Large signal description of nonideal *BUCK* and *BOOST* in *CCM*

5.2.1. BUCK converter

The equivalent circuit of *BUCK* converter, including parasitic resistances is depicted in Figure 3. Symbols *K* and *D* denote the ideal parts of semiconductor switches. It should be pointed out, that in modern converters the synchronous pair of switches i.e. two transistors switched alternately are usually used.
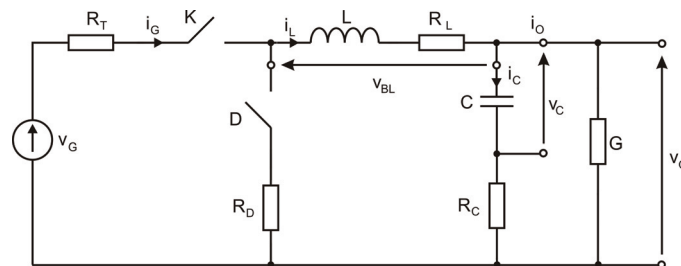

Fig. 3. *BUCK* converter including parasitic resistances

Some equations of converter are valid in each subintervals, namely:

$$v_{BL} = L \cdot \frac{di_L}{dt} + R_L \cdot i_L, \tag{89}$$

$$v_C = v_O - R_C \cdot i_C, \tag{90}$$

$$i_C = i_L - G \cdot v_O. \tag{91}$$

From the description of ideal capacitance and Equation (90) it results:

$$i_C = C \cdot \frac{dv_C}{dt} = C \cdot \left(\frac{dv_O}{dt} - R_C \cdot \frac{di_C}{dt} \right). \tag{92}$$

After introducing (91) into (92) we get:

$$i_L + C \cdot R_C \cdot \frac{di_L}{dt} = G \cdot v_O + C \cdot \left(1 + R_C \cdot G\right) \cdot \frac{dv_O}{dt}. \tag{93}$$

Apart from Equations (89), (93), valid in each state of converter, the dependencies of variables of group *B* on variables of group *A*, including inductor current and input and output voltages, should be written. The equivalent circuits for states *ON* and *OFF* are shown in Figures 4 a) and b) respectively.

From Figure 4 a) we get:

$$v_{BL}(ON) = v_G - v_O - R_T \cdot i_L, \tag{94}$$

$$i_G(ON) = i_L. \tag{95}$$

From Figure 4 b) it is obtained:

$$v_{BL}(OFF) = -v_O - R_D \cdot i_L, \tag{96}$$

$$i_G(OFF) = 0, \tag{97}$$

After applying a formula (5) for averaging, one obtains:

$$v_{BLS} = d_A \cdot v_G - v_O - i_L \left[d_A \cdot R_T + \left(1 - d_A\right) \cdot R_D \right], \tag{98}$$

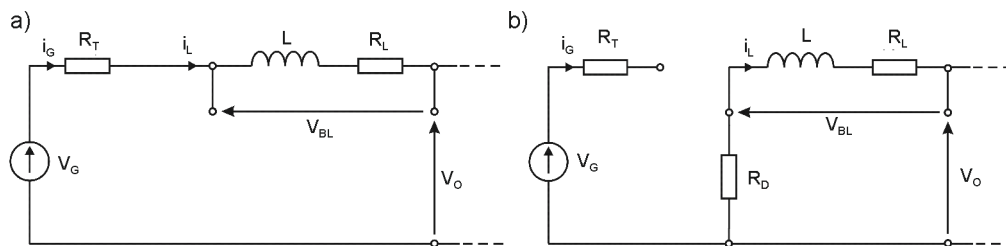$$i_{GS} = d_A \cdot i_L. \tag{99}$$

Fig. 4. Equivalent circuits of *BUCK* converter with parasitic resistances for *ON* (a) and *OFF* (b) subintervals. The output part of converter is not shown

The large-signal averaged model of *BUCK* converter including parasitic resistances consists of Equations (89), (93), (98), (99). It may be compared with the large signal averaged model obtained by the switch averaging approach, presented in [1], Section 3.4. It is observed, that the equation for v_{BLS} obtained on switch averaging (Eqn. (58) in Section 3.4, of [1]) differs

from Equation (98). It is a result of informalities in deriving the averaged model based on switch averaging technique, as discussed in [1], Section 3.4. On the other hand, Equation (98) is identical with the Equation (58') in [1]. Equation (58') in [1] has been obtained after introducing the equivalent resistances R_{TS} and R_{DS} according to "energy conservation rule" discussed in [3], (Chapter 10). The way of introducing these resistances seems to be problematic. There is no necessity to dissipate the same power in artificial resistances of equivalent circuit as in resistances in original circuit. It may be observed on the example of Thevenin or Norton equivalent circuits, often used in circuit analysis. Nevertheless, Equation (58') in [1], obtained this way, is in accordance with Equation (98) based on the presented approach.

5.2.2. BOOST converter in CCM

Similar derivation may be performed for *BOOST* converter in *CCM,* with parasitic resistances. The general scheme and equivalent circuits for *ON* and *OFF* subintervals are presented in Figure 5.
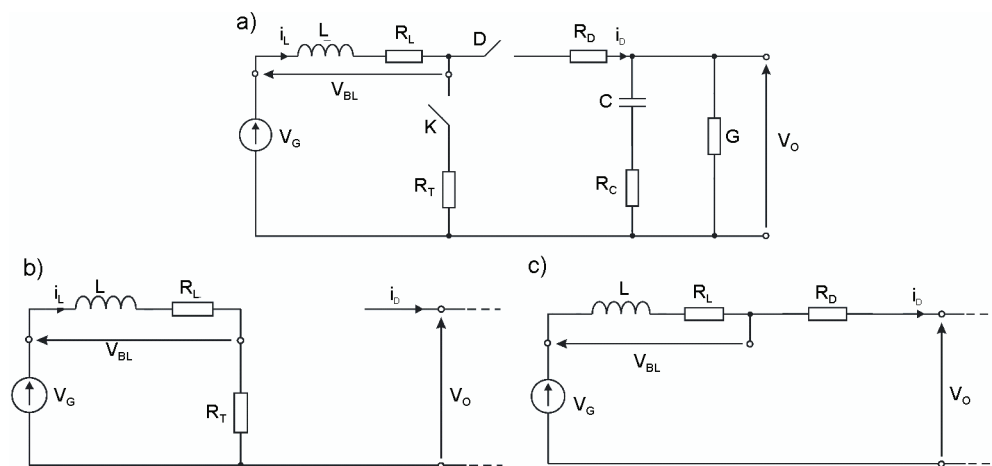

Fig. 5. *BOOST* converter with parasitic resistances (a) and equivalent circuits for *ON* (b) and *OFF* (c).
The output subcircuit is not shown in (b) and (c)

Equations valid for each subinterval are Equation (89) and:

$$i_D + C \cdot R_C \cdot \frac{di_D}{dt} = G \cdot v_O + C \cdot (1 + R_C \cdot G) \cdot \frac{dv_O}{dt}. \tag{100}$$

The voltage v_{BL} and current i_D belong to group B and should be expressed as functions of group A variables. After procedure of averaging it is obtained:

$$v_{BLS} = v_G - (1 - d_A) \cdot v_O - i_L [d_A \cdot R_T + (1 - d_A) \cdot R_D] \tag{101}$$

$$i_{DS} = (1 - d_A) \cdot i_L. \tag{102}$$

The large-signal averaged model of *BOOST* converter with parasitic resistances in *CCM* consists of Equations (89), (100), (101), (102).

5.3. Steady-state and small-signal description of *BUCK* and *BOOST* in *CCM,* with parasitic resistances included

Steady-state and small signal characteristics are obtained in the way discussed earlier. For *BUCK* with parasitics, in *CCM* one have to start with Equations (89), (93), (98), (99). The steady-state equations are obtained in the form:

$$I_L = G \cdot V_O, \tag{103}$$

$$V_O = D_A \cdot V_G - R_Z \cdot I_L, \tag{104}$$

where:

$$R_Z = D_A \cdot (R_T - R_D) + R_D + R_L, \tag{105}$$

$$I_G = D_A \cdot I_L. \tag{106}$$

From (103) and (104) we have additionally:

$$V_O = \frac{D_A \cdot V_G}{1 + R_Z \cdot G}. \tag{107}$$

Small-signal dependencies in *s*-domain may be expressed in the form:

$$(1 + s \cdot C \cdot R_C) \cdot I_l = (G + s \cdot C_Z) \cdot V_o, \tag{108}$$

$$(R_Z + s \cdot L) \cdot I_l = D_A \cdot V_g - V_o + [V_G - I_L(R_T - R_D)] \cdot \theta, \tag{109}$$

where:

$$C_Z = C \cdot (1 + R_C \cdot G). \tag{110}$$

The small-signal transmittances input-to-output H_g and control-to-output H_d (as defined in [1], Equations (74), (75)) are:
- *BUCK – CCM*

$$H_{gp} = \frac{D_A \cdot (1 + \cdot s \cdot C \cdot R_C)}{s^2 \cdot L \cdot C_Z + s \cdot (G \cdot L + R_Z \cdot C_Z + C \cdot R_C) + 1 + G \cdot R_Z}, \tag{111}$$

$$H_{dp} = \frac{[V_G - I_L \cdot (R_T - R_D)] \cdot (1 + \cdot s \cdot C \cdot R_C)}{s^2 \cdot L \cdot C_Z + s \cdot (G \cdot L + R_Z \cdot C_Z + C \cdot R_C) + 1 + G \cdot R_Z}. \tag{112}$$

Additional subscript "p" denotes "with parasitic resistances".
Steady-state dependencies for *BOOST* in *CCM* are:

$$V_O = \frac{(1 - D_A) \cdot V_G}{(1 - D_A)^2 + R_Z \cdot G}, \tag{113}$$

$$I_G = I_L = \frac{G \cdot V_O}{1 - D_A}. \tag{114}$$

Small-signal transmittances obtained from large-signal model (89), (100), (101), (102) are:

- *BOOST – CCM*

$$H_{gp} = \frac{(1 - D_A) \cdot (1 + \cdot s \cdot C \cdot R_C)}{s^2 \cdot L \cdot C_Z + s \cdot [G \cdot L + R_Z \cdot C_Z + (1 - D_A)^2 \cdot C \cdot R_C] + (1 - D_A)^2 + G \cdot R_Z}, \tag{115}$$

$$H_{dp} = \frac{-s^2 \cdot L \cdot C \cdot R_C \cdot I_L + s(V_A \cdot C \cdot R_C - L \cdot I_L - C \cdot R_C \cdot R_Z \cdot I_L) + V_A - I_L \cdot R_Z}{s^2 \cdot L \cdot C_Z + s \cdot [G \cdot L + R_Z \cdot C_Z + (1 - D_A)^2 \cdot C \cdot R_C] + (1 - D_A)^2 + G \cdot R_Z}, \tag{116}$$

where:

$$V_A = (1 - D_A) \cdot \left(V_O - I_L \cdot \left(R_T - R_D \right) \right). \tag{117}$$

Other small-signal characteristics can be easily obtained starting from large-signal averaged models, for example input characteristics Y_{in} and I_θ defined by Equations (83) and (84) may be found for nonideal *BUCK* and *BOOST* converters in *CCM* on expressions (99) and (102).

5.4. *BUCK* converter in *DCM* with parasitic resistances – large signal description

Model for discontinuous conduction mode (*DCM*) including parasitic resistances is presented here for *BUCK* converter only. The equivalent circuits of this converter in *ON* and *OFF1* subintervals correspond to Figures 4 a) and b). The inductor voltage in *ON* state is:

$$v_L(ON) = v_G - v_O - i_L(ON) \cdot (R_T + R_L). \tag{118}$$

The changes of inductor current in *ON* are expressed as:

$$i_L(t)\big|_{ON} = \frac{v_L(ON)}{L} \cdot t; \quad 0 \le t \le d_A \cdot T_S \tag{119}$$

and its average for *ON* is:

$$i_L(ON) = \frac{i_{LM}}{2} = \frac{v_L(ON)}{2 \cdot L} \cdot d_A \cdot T_S. \tag{120}$$

From (118) and (120):

$$i_L(ON) = \frac{(v_G - v_O) \cdot d_A \cdot T_S}{2 \cdot L + (R_T + R_L) \cdot d_A \cdot T_S}. \tag{121}$$

For *OFF1* one obtains:

$$v_L(OFF1) = -v_O - i_L(OFF1) \cdot (R_D + R_L), \tag{122}$$

$$i_L(OFF1) = \frac{i_{LM}}{2} = \frac{v_L(OFF1)}{2 \cdot L} \cdot d_B \cdot T_S \tag{123}$$

and, as a result:

$$i_L(OFF1) = \frac{d_B \cdot T_S \cdot v_O}{2 \cdot L - d_B \cdot T_S \cdot (R_D + R_L)}. \tag{124}$$

As it is seen in Figure 2, the local average values $i_L(ON)$ and $i_L(OFF1)$ are equal, therefore they are denoted by i_{LP}. In further considerations it is assumed for simplicity that parasitic resistances of diode and transistor are equal (true for synchronous converter) and equivalent resistance R_P is introduced:

$$R_P = R_T + R_L = R_D + R_L. \tag{125}$$

From (121) and (124) we obtain:

$$d_B = \frac{2L \cdot d_A(v_G - v_O)}{2L \cdot v_O + d_A \cdot T_S \cdot R_P \cdot v_G}. \tag{126}$$

The average value of inductor voltage is:

$$v_{LS} = v_L(ON) \cdot d_A + v_L(OFF1) \cdot d_B. \tag{127}$$

By introducing Equations (118)-(126) into (127) one obtains:

$$v_{LS} = d_A \cdot R_P \cdot v_G \cdot \frac{d_A \cdot T_S \cdot (v_G - v_O) - i_{LP}(d_A \cdot T_S \cdot R_P + 2L)}{2 \cdot L \cdot v_0 + v_G \cdot d_A \cdot T_S \cdot R_P}. \tag{128}$$

By substituting expression (121) for i_{LP} (equal to $i_L(ON)$) into (128) one obtains

$$v_{LS} = 0. \tag{129}$$

It is an important result, which shows that in *DCM*, the converter with parasitic resistances is a first order system, similarly as an ideal converter.

The average value of inductor current is found in similar way as for ideal converter:

$$i_{LS} = \frac{i_{LM}}{2} \cdot (d_A + d_B). \tag{130}$$

The result is:

$$i_{LS} = \frac{(v_G - v_O) \cdot v_G \cdot d_A^2 \cdot T_S}{2L \cdot v_O + v_G \cdot d_A \cdot T_S \cdot R_P}. \tag{131}$$

The additional equations for *BUCK* in *DCM* are

$$v_{BLS} = R_L \cdot i_{LS}, \tag{132}$$

$$i_{GS} = d_A \cdot i_L(ON) = \frac{(v_G - v_O) \cdot d_A^2 \cdot T_S}{2L + d_A \cdot T_S \cdot R_P}.$$ (133)

From (129) we have:

$$\frac{di_{LS}}{dt} = 0.$$ (129')

Therefore, using (93), we obtain:

$$i_{LS} = G \cdot v_O + C(1 + R_C \cdot G) \cdot \frac{dv_O}{dt}.$$ (134)

The large-signal averaged model of *BUCK* converter in *DCM*, with parasitic resistances is expressed by Equations (129), (131), (133), (134).

5.5. Steady-state and small-signal description of *BUCK* in *DCM* with parasitic resistances

From the large-signal model, the steady-state and small-signal characteristics are derived in a standard way. Steady-state dependencies are:

$$I_L = \frac{(V_G - V_O) \cdot V_G \cdot D_A^2 \cdot T_S}{2L \cdot V_O + V_G \cdot D_A \cdot T_S \cdot R_P},$$ (135)

$$I_G = \frac{(V_G - V_O) \cdot D_A^2 \cdot T_S}{2L + D_A \cdot T_S \cdot R_P},$$ (136)

$$I_L = G \cdot V_O.$$ (137)

From (135) and (137) it is obtained:

$$G \cdot V_O \cdot (2L \cdot V_O + V_G \cdot D_A \cdot T_S \cdot R_P) = (V_G - V_O) \cdot V_G \cdot D_A^2 \cdot T_S.$$ (138)

It may be rewritten as:

$$2L \cdot M_V^2 + \left(\frac{D_A^2 \cdot T_S}{G} + D_A \cdot T_S \cdot R_P \right) \cdot M_V - \frac{D_A^2 \cdot T_S}{G} = 0,$$ (139)

where M_V is the steady-state output-to-input voltage ratio. By solving Equation (139) we get:

$$M_V = \frac{V_O}{V_G} = \frac{1}{2} \cdot D_A \cdot R \cdot G_Z \cdot \left(\sqrt{(D_A + R_P \cdot G)^2 + 4G/G_Z} - D_A - R_P \cdot G \right).$$ (140)

Small-signal equivalent of Equation (134) is:

$$I_l = V_o \cdot (G + sC_Z),$$ (141)

where C_Z is given by (110). Further dependencies are obtained from (131) similarly as in Section 4.2, see Equations (66), (67). The partial derivatives of inductor current, in present case, are:

$$\gamma_D = \frac{\partial i_L}{\partial d_A} = \frac{D_A \cdot V_G (V_G - V_O) \cdot T_S}{M^2} \cdot (4L \cdot V_O + V_G \cdot D_A \cdot T_S \cdot R_P), \qquad (142)$$

$$\gamma_G = \frac{\partial i_L}{\partial v_G} = \frac{D_A^2 \cdot T_S}{M^2} \cdot (4L \cdot V_G \cdot V_O + V_G^2 \cdot D_A \cdot T_S \cdot R_P - 2L \cdot V_O^2), \qquad (143)$$

$$\gamma_O = \frac{\partial i_L}{\partial v_O} = -\frac{D_A^2 \cdot T_S}{M^2} \cdot V_G^2 (2L + D_A \cdot T_S \cdot R_P), \qquad (144)$$

where:

$$M = 2L \cdot V_O + V_G \cdot D_A \cdot T_S \cdot R_P. \qquad (145)$$

The resulting expression for *s*-domain, small signal representation of inductor current is:

$$I_l = \gamma_D \cdot \theta + \gamma_G \cdot V_g + \gamma_O \cdot V_o. \qquad (146)$$

From the standard definitions of small-signal input-to-output and control-to-output transmittances, using equations (141) and (146), one obtains:

$$H_{gDP} = \left. \frac{V_o}{V_g} \right|_{\theta=0} = \frac{\gamma_G}{s \cdot C_Z + G - \gamma_O}, \qquad (147)$$

$$H_{dDP} = \left. \frac{V_o}{\theta} \right|_{V_g=0} = \frac{\gamma_D}{s \cdot C_Z + G - \gamma_O}. \qquad (148)$$

The quantities γ in accordance with their definitions, are real numbers, therefore the above transmittances have a single pole and the converter in *DCM*, with parasitic resistances is a first-order system. It should be mentioned however, that the converter model with parasitic series resistances of components is still an approximation and cannot be considered as fully accurate. It is possible to derive more accurate models by including for example the parasitic inductances of capacitors and connection paths on printed circuit board etc. Such effects may be responsible for more involved features of averaged models, especially in high frequency range.

Other transmittances (for example input admittance or dependencies of small-signal inductor current on input voltage and duty ratio) can be easily derived from the above equations. The procedures of finding the characteristics of other converters (for example *BOOST* or *BUCK-BOOST*) with parasitic resistances, working in *DCM* are similar.

6. Conclusions

The presented papers, including Part I [1] have three main purposes. The first is the presentation of existing methods of deriving the averaged models of power stages of switch-mode power converters applicable to designing the control circuits. The second is to point out the

sources of inaccuracies of the averaged models of converters, in particular in *DCM* mode, or converters with parasitic resistances, obtained by the known method of switch averaging. The third is to propose a procedure of creating averaged models by separation of variables, free of the limitations of the known methods.

The accurate model of the power stage of converter is necessary for proper design of the control circuit of converter. Many types of control circuits in the form of integrated circuits are offered by manufacturers but in many cases, the specific design is needed, to fulfill the demands of special applications. There is a great amount of converter kinds, with many modifications and it is impossible to find the proper controller "off the shelf" to each type or modification. The use of digital controllers gives the possibility of precise design of controller characteristics for given application and has to be based on the accurate model of the power stage.

The original procedure of deriving the averaged models of power converters is presented in the second part of the paper series. The proposed method is general and is applicable in particular to converters operated in discontinuous conduction mode or converters with parasitic resistances. For these cases, the equations describing the large-signal, steady-state and small-signal averaged models, derived with the use of the separation of variables approach, are presented. It is shown for example, that the averaged models of simple converters (*BUCK, BOOST, BUCK-BOOST*) working in *DCM* are of the first order (both: in ideal case and in the presence of parasitic resistances).

The paper is devoted mainly to presentation of the new procedure of deriving the averaged models so, the final results are in the form of equations describing various kinds of these models. There are no numerical examples or discussions of the quantitative differences between the different models, nor the measurement results are presented.

The method of deriving the averaged models are presented in application to the simplest, basic converters, mainly to *BUCK* (step-down) converter and in part, to *BOOST* and *BUCK-BOOST* converters. The presented approach can be easily applied to other converter types. On the other hand, the additional averaged characteristics of converters may be similarly obtained.

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