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RELUCTANCE MOTOR MATHEMATICAL MODEL IN NATURAL REFERENCE FRAME USING HAMILTONIAN EQUATIONS – SIMULATIONAL ANALYSIS

MODEL MATEMATYCZNY SILNIKA RELUKTANCYJNEGO W NATURALNYM UKŁADZIE ODNIESIENIA WYKORZYSTUJĄCY RÓWNANIA HAMILTONA – BADANIA SYMULACYJNE

Abstract. In the paper a novel mathematical model of electromechanical actuator is presented. It is based on application of Hamiltonian equations in the description of electromechanical energy conversion. It results in employment of flux linkages as state variables in the state space equations. A 3-phase wye connected stator winding without a neutral wire is considered in detail as the most important connection schema in practical applications. The procedure can be generalised to any number of phases and arbitrary connection schema. Topology-based approach is used in the model implementation. Eddy currents and hysteresis phenomenon are neglected in formulation of the model to enable application of Hamiltonian description. Simulation analysis is performed using data from FEM model of reluctance cageless motor.

1. Introduction

In the paper a novel mathematical model of electromechanical actuator is presented. It is based on application of Hamiltonian equations in the description of electromechanical energy conversion which results in employment of flux linkages as state variables in the state space equations [12][13]. For simplicity of description only a 3-phase wye connected stator winding without a neutral wire is considered in detail. Though the simplest, this connection scheme is the most important from practical viewpoint [2]. The procedure can be extended to any number of phases and an arbitrary connection schema. In formulation of the model eddy currents and hysteresis phenomenon were neglected to enable application of state-space description [12].

The model is formulated in natural (phase) reference frame. The circuit voltage equations are established using Kirchhoff's equations with flux linkages as state variables which is a result of Hamiltonian approach. The most important problem arising in this approach is evaluation of currents as function of flux linkages in multidimensional variable space [3] [13]. The proposed solution is based on triangulation of databases (obtained using FEM [4]) and application of local linear (affine) homeomorphism between variable spaces [6] (of currents and flux linkages). In algebraic topology this approach is defined as simplical approximation [1].

2. An electromechanical actuator model using Hamiltonian equations

Application of Hamiltonian equations in analysis of electromechanical actuators is very uncommon although it is equivalent to Lagrange equations [14][13][12][11]. In case of electromechanical actuators potential advantage of Hamiltonian description is the canonical form of these equations [13]. The state of the system can be uniquely described using the so-called Hamiltonian H [13]:

$$H = E_{mag} + E_k + E_p \tag{1}$$

where E_{mag} – magnetic field energy, E_k – kinetic energy of mechanical part of the system, E_p – potential energy of the system. When a machine with m insulated phase windings $A, B,\ldots, M$ is considered the magnetic field energy can be described by the following formula [13]:

$$E_{mag} = \sum_{j=A}^{M} \int_{0}^{\psi_j} i_j\left(\varphi, \psi_A, \ldots, \psi_j, 0, \ldots, 0\right) d\psi_j \tag{2}$$

where φ – angular position of the rotor (Fig.2), $\mathbf{\Psi}_{ph} = \left[\psi_A, \ldots, \psi_j, \ldots, \psi_M\right]^T$ - vector of phase flux linkages, $\mathbf{i}_{ph} = \left[i_A, \ldots, i_j, \ldots, i_M\right]^T$ - vector of phase currents.

When a number of w holonomic constraints are imposed on the system the number of degrees of freedom can be reduced [14][12]. The state of the system is then described by N+1 independent canonical equations where $N = m–w$:

$$\frac{d}{dt}\lambda_j + \frac{\partial H}{\partial q_j} - Q_j = 0 \tag{3a}$$

$j \in \{1,...,N+1\}$, with additional N+1 equations defining relationship between generalised momenta and generalised velocities:

$$\frac{d}{dt}q_j = \frac{\partial H}{\partial \lambda_j} \tag{3b}$$

where $\boldsymbol{\lambda} = [K, \psi_1,...,\psi_j,...,\psi_N]^T = [K, \boldsymbol{\Psi}^T]$, K – angular momentum, $\boldsymbol{\Psi}$ - vector of generalised flux linkages, $\mathbf{q} = [\varphi, q_1,...,q_j,...,q_N]^T = [\varphi, \mathbf{q}_e^T]$, $\mathbf{q}_e$ – vector of generalised electrical coordinates of the system (charges), Q_j – j-th component of generalised non-potential force acting in the system, $\dot{\mathbf{q}} = [\omega, i_1,...,i_j,...,i_N]^T = [\omega, \mathbf{i}^T]$, ω – angular velocity, $\mathbf{i}$ – vector of generalised electrical velocities (currents).

When there are no potential energy elements in the system the Hamiltonian can be rewritten in a simpler form:

$$H(\varphi, K, \boldsymbol{\Psi}) = E_{mag}(\varphi, \boldsymbol{\Psi}) + \frac{K^2}{2J} \tag{4}$$

with magnetic field energy definition for an arbitrary variation of the flux linkages (Fig.10) [12]:

$$E_{mag}(\varphi, \boldsymbol{\Psi}) = \int_{\mathbf{0}}^{\boldsymbol{\Psi}} \mathbf{i} \cdot d\boldsymbol{\Psi} \Bigg|_{\varphi = const} \tag{5}$$

In the matrix form the equation set (3) can be written separately for electric and mechanical degrees of freedom:

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}t} = \mathbf{u} - \mathbf{R}\mathbf{i}(\varphi, \boldsymbol{\Psi}) \tag{6a}$$

$$\frac{\mathrm{d}K}{\mathrm{d}t} = T_e - T_m \tag{6b}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{K}{J} \tag{6c}$$

$$T_e = -\frac{\partial E_{mag}(\varphi, \boldsymbol{\Psi})}{\partial \varphi} \Bigg|_{\boldsymbol{\Psi} = \mathbf{const}} \tag{6d}$$

where $\mathbf{i}(\varphi, \boldsymbol{\Psi})$ is the function defining currents as function of rotor angular position φ and generalised electric momenta $\boldsymbol{\Psi}$ (flux linkages); $\mathbf{u}$ – generalised external electric forces (voltages); $\mathbf{R}$ – resistance matrix; T_e – electromagnetic torque; T_m – mechanical load torque. Equation set (6) can be presented in a form of block diagram shown in Fig.1.

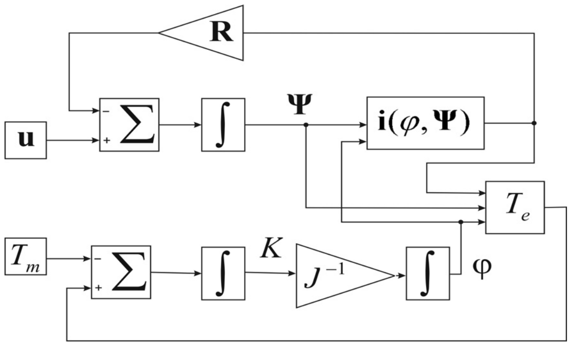


Fig.1. Block diagram of an electromechanical actuator mathematical model using Hamiltonian equations

3. Application of Kirchhoff's equations

Standard connection schemas for a 3-phase (m=3) reluctance motor are shown in Fig.2 [2][7]. For each of them the Hamiltonian equation (6a) can be obtained using Kirchhoff's equations.

Therefore, for each schema its voltage equation can be written in two forms (Eq.7):

- Kirchhoff's form – using phase variable description; denoted by (K.),
- Hamiltonian form – using generalised variable description; denoted by (H.).

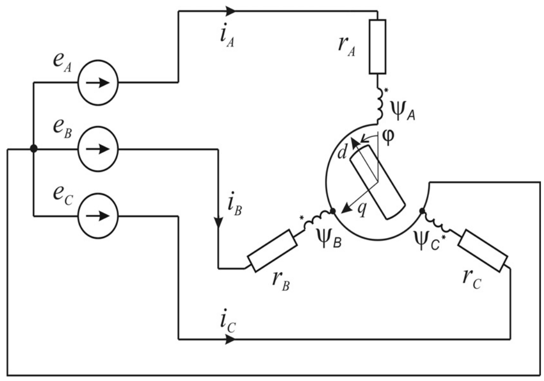
In case of the first two connection schemas (wye with neutral wire, delta) number of degrees of freedom is unchanged and resistance matrices are diagonal, equal to the phase resistance matrix. The generalised variables in case of wye connection with neutral wire are the same like phase variables (Eq.7a, Fig.2a). In case of the delta connection generalised external electric forces u_1, u_2, u_3 are line-to-line voltages e_{AC}, e_{BA}, e_{CB} while all the other generalised variables are phase variables (Eq.7b, Fig.2b).

The most interesting case from practical point of view is wye without neutral wire (Eq.7c, Fig.2c). Application of Hamiltonian approach (Eq.6a) shows that:

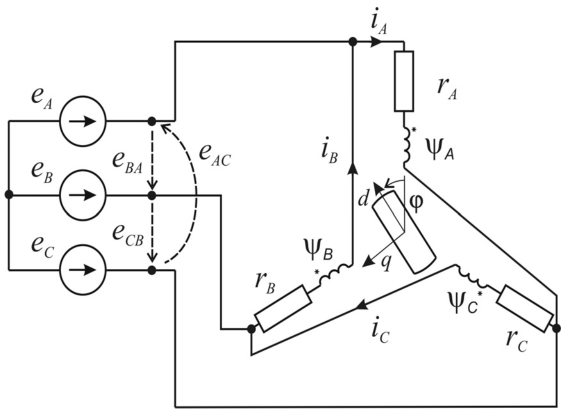
- flux linkages ψ_{AC}, ψ_{BC} – line-to-line flux linkages being linear combinations of phase flux linkages $\psi_{AC} = \psi_A - \psi_C$, $\psi_{BC} = \psi_B - \psi_C$ are generalised electric momenta ψ_1, ψ_2,

- voltages e_{AC} ,e_{BC} – line-to-line voltages being linear combinations of phase voltages $e_{AC}=e_A-e_C$, $e_{BC}=e_B-e_C$ are generalised external electric forces u_1 ,u_2 ,
- currents i_A, i_B – being simultaneously loop and phase currents are generalised electric velocities i_1, i_2,
- resistance matrix is symmetric but non-diagonal.

a)



b)



c)

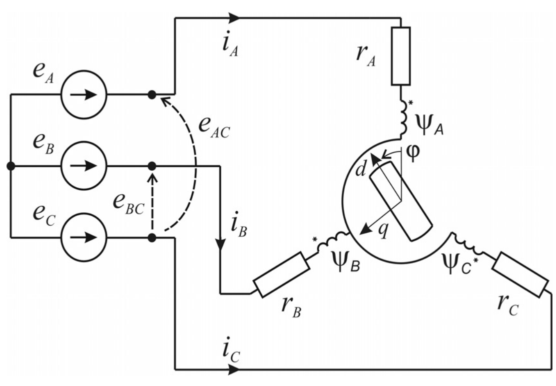


Fig.2. Schematic representations of the reluctance motor for different connection schemas: a) wye with neutral wire, b) delta,c) wye without neutral wire

The forms voltage equations are:

- for wye with neutral wire (w=0, N=m)

$$\text{(K.)}\quad \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi_A \\ \psi_B \\ \psi_C \end{bmatrix} + \begin{bmatrix} r_A & 0 & 0 \\ 0 & r_B & 0 \\ 0 & 0 & r_C \end{bmatrix}\begin{bmatrix} i_A \\ i_B \\ i_C \end{bmatrix}$$

$$\text{(H.)}\quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} + \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \tag{7a}$$

- for delta (w=0, N=m)

$$\text{(K.)}\quad \begin{bmatrix} e_{AC} \\ e_{BA} \\ e_{CB} \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi_A \\ \psi_B \\ \psi_C \end{bmatrix} + \begin{bmatrix} r_A & 0 & 0 \\ 0 & r_B & 0 \\ 0 & 0 & r_C \end{bmatrix}\begin{bmatrix} i_A \\ i_B \\ i_C \end{bmatrix}$$

$$\text{(H.)}\quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} + \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \tag{7b}$$

- for wye without neutral wire (w=1, N=m-1=2)

$$\text{(K.)}\quad \begin{bmatrix} e_{AC} \\ e_{BC} \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi_{AC} \\ \psi_{BC} \end{bmatrix} + \begin{bmatrix} r_A + r_C & r_C \\ r_C & r_B + r_C \end{bmatrix}\begin{bmatrix} i_A \\ i_B \end{bmatrix}$$

$$\text{(H.)}\quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{7c}$$

The 3-phase wye connected stator winding without neutral wire will be considered in detail in the further description.

4. Algorithm for modelling of current-flux linkage spatial characteristics

Evaluation of multidimensional function $\mathbf{i}(\varphi,\mathbf{\Psi})$ in Eq.6a is the most important task in application of Hamiltonian description to electromechanical actuators [12][13].

In previous papers it was achieved with application of uniform databases along with appropriate approximation [3][4]. It employed „griddata3" numerical procedure [9]. The drawback of this method is creation of large number of **NaN** (Not-a-Number) entries in databases [3].

In the paper a different methodology is proposed. It employs newly introduced TriRep/DelaunayTri structures in Matlab'2009 [9]. Those entities enable application of topology-based evaluation methodology using local piece-wise linear (affine) homeomorphism between current and flux linkage spaces [6]. In algebraic topology such transformation is defined as simplical approximation [1].

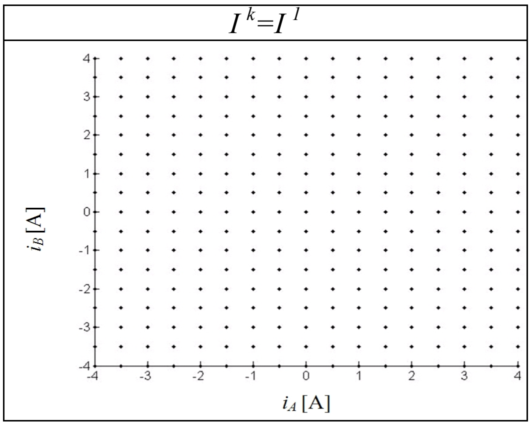


Fig.3. Current point set for $\varphi = \varphi_1$

Numerical data for the primary databases used in description (Fig.3) were obtained with the Finite Element Method for a real reluctance cageless motor [4][5][10]. The structure of the databases is related to the evaluation methodology which can be performed either using FEM or measurement [4][8]. In order to define the databases the following sets are used:

- the position set $\Theta = \{\varphi_1, \ldots, \varphi_k, \ldots, \varphi_K\}$ composed of K points where $\varphi_1 = 0$ [rad], $\varphi_K = \varphi_1 + 2\pi/p$ to account for symmetry, p – number of pole pairs. Their choice depends on assumed precision of the model (e.g. if the model accounts only for the main space harmonic or also for slotting effects etc. [13]),
- for each angle $\varphi_k \in \Theta$ a current set I^k composed of $P(k)$ points is defined $I^k = \{\mathbf{i}_1, \ldots \mathbf{i}_p, \ldots, \mathbf{i}_{P(k)}\}$ where $\mathbf{i}_p \in \mathbf{E}^N$ (Fig.3). The I^k set belongs to current subspace $\boldsymbol{RI}^k \subset \mathbf{E}^N$. Choice of I^k depends on range of saturation and assumed maximum current of the device.

Dimension N of **E**uclidian space $\mathbf{E}^N$ is equal to the number of generalised (independent) variables which is N=2 in the presented example.

5. Methodology for φ = const

Applying the FEM calculation at an arbitrary angle $\varphi_k \in \Theta$ one obtains two sets:

- a flux linkage point set $\Psi^k = \{\mathbf{\Psi}_1, \ldots \mathbf{\Psi}_p, \ldots, \mathbf{\Psi}_{P(k)}\}$, $\mathbf{\Psi}_p \in \mathbf{E}^N$, (Fig.4),
- an electromechanical torque set $T_e(k) = \{T_{e,1}, \ldots T_{e,p}, \ldots, T_{e,P(k)}\}$.

The Ψ^k set belongs to flux linkage subspace $\boldsymbol{R\Psi}^k \subset \mathbf{E}^N$.

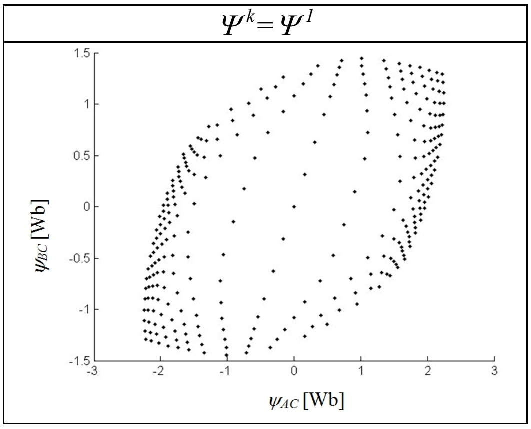


Fig.4. Flux linkage point set for $\varphi = \varphi_1$

As eddy currents and hysteresis phenomenon are neglected in the model, thus the function relationship f^k between the above mentioned subspaces f^k: $\boldsymbol{RI}^k \xrightarrow{on} \boldsymbol{R\Psi}^k$ is a homeomorphism [1]. For such a function a local linear approximation f_A^k can be defined [6]:

- its domain is $\boldsymbol{RI}^k$ defined with the help of triangulation T^k, which consists of N-simplexes $\mathbf{\Delta}_{I^k,j}$ which are triangles for analysed case (N=2, Fig.5) ,
- $f_A^k(\mathbf{i}_p) = \mathbf{\Psi}_p$,
- for an arbitrary j-th N-simplex $\mathbf{\Delta}_{I^k,j} \in T^k$ function $f_A^k(\mathbf{i})$ is affine on $\mathbf{\Delta}_{I^k,j}$ [1][6].

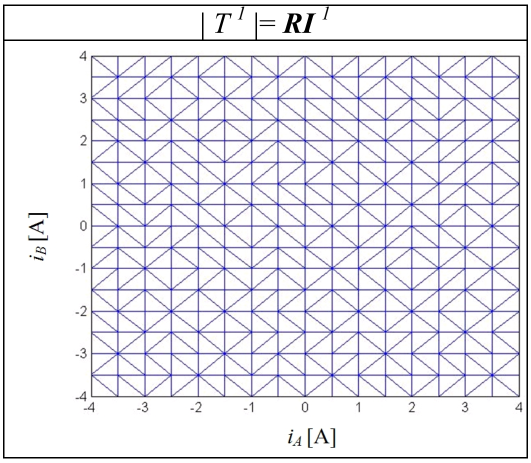


Fig.5. Current triangulation for $\varphi = \varphi_1$

For a homeomorphism f^k there exists an inverse function f^{-k}: $\boldsymbol{R\Psi}^k \xrightarrow{on} \boldsymbol{RI}^k$ [6]. It enables a definition of its local linear approximation f_A^{-k} which has the following properties (Fig.7):

- its domain is $\boldsymbol{R\Psi}^k$ defined with the help of triangulation T^{-k} (Fig.6),
- $f_A^{-k}(\mathbf{\Psi}_p) = \mathbf{i}_p$,

- for an arbitrary j-th N-simplex $\Delta_{\Psi^k,j} \in T^{-k}$ function $f_A^{-k}(\mathbf{i})$ is affine on $\Delta_{\Psi^k,j}$.

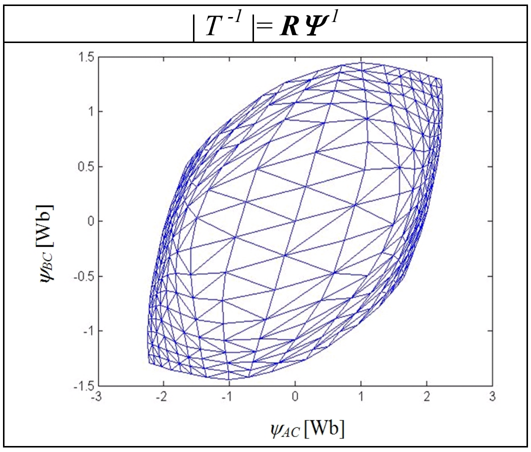


Fig.6. Flux linkage triangulation for $\varphi = \varphi_1$

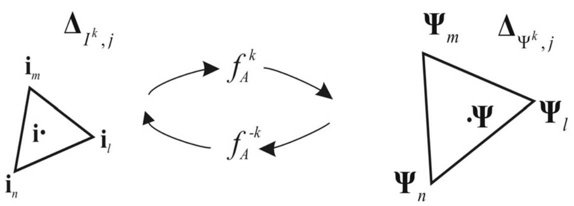


Fig.7. Local invertible affine transformations on simplexes (maps)

Exemplary data are shown in the Figs.8÷10.

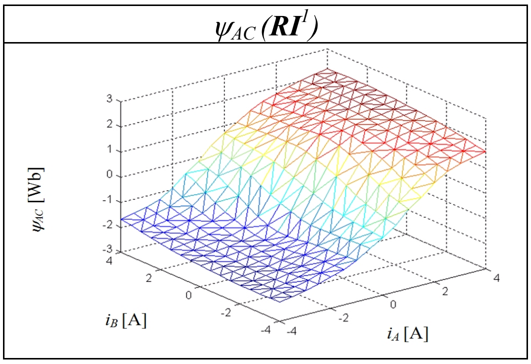


Fig.8. Flux linkage ψ_{AC} on triangulation

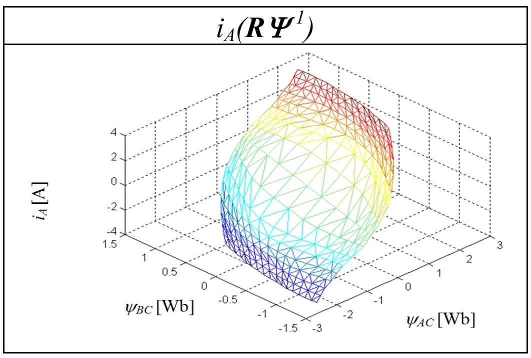


Fig.9. Current i_A on triangulation

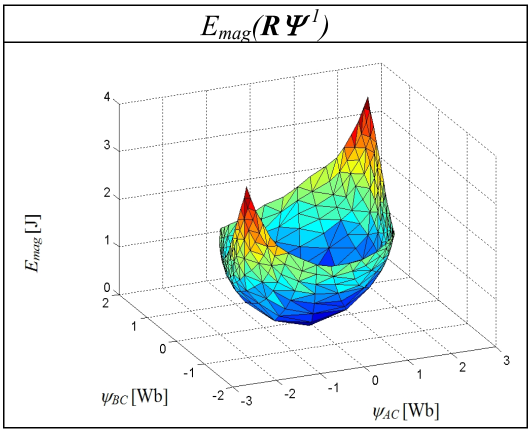


Fig.10. Energy E_{mag} on triangulation

6. Exemplary results for φ = const

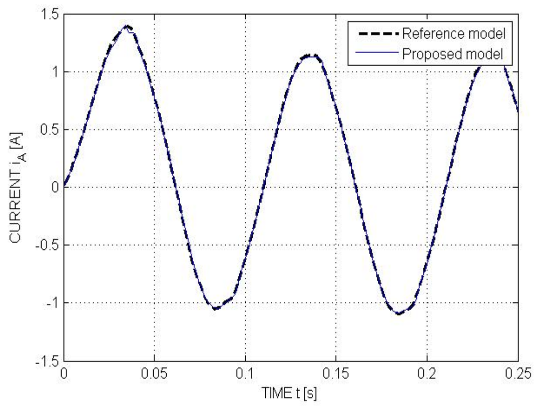
Numerical simulations were performed for 3-phase symmetrical phase voltages of magnitude U_{ph}=30/60 [V], frequency f=10 [Hz], $\varphi(t)$=0°, for initial conditions: $\psi_{AC}(0)=\psi_{BC}(0)=0$ [Wb]. Remaining model parameters were: r_f=13 [Ω] – phase resistance, T_m = 0 [N·m]. J =0.01 [kg·m^2],. Simulation time in SIMULINK is 0.25[s]. Results are shown in Fig.11,12. Reference model used for comparison is a uniform database model [3].

7. Conclusions

Results in Fig.11,12 show that compatibility of results both for current and torque evaluation is very good. It is especially important in case of electromagnetic torque as values for proposed model (with triangulated databases) are internally evaluated using algorithm based on Eq.6d while values for reference model (with uniform databases [3][4]) were obtained using approximation of FEM results.

Simultaneously the proposed model overcame the main drawback of the reference model which was limited active region [3]. It often resulted in Loss of Stability due to overflow of the database range (LofS., Fig.11).

a)



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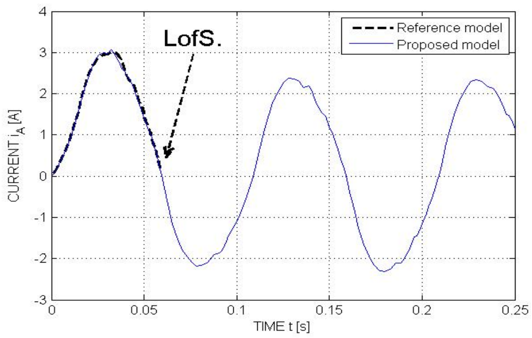
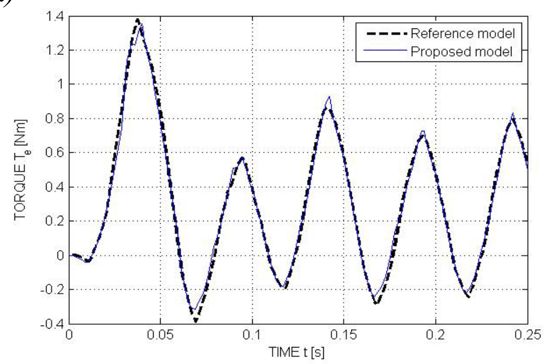


Fig.11. Current i_A plots: a) 30 [V], b) 60 [V]

a)



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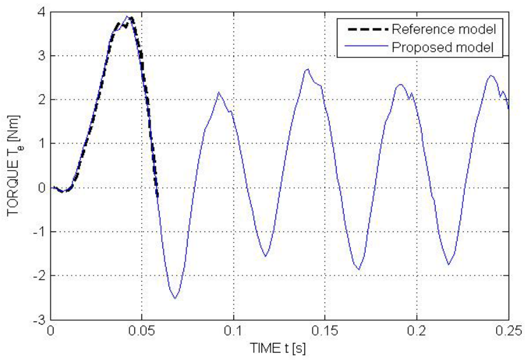

Fig.12. Torque T_e plots: a) 30 [V], b) 60 [V]

During conference results for *ω=const, ω=var* will be presented.

8. References

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