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ANALYSIS OF THREE-DIMENSIONAL MAGNETIC FIELD IN THE INTERIOR OF RING SHAPED SECTOR

ABSTRACT *The subject of the paper is analytical solution of common boundary value problem for Laplace equation in the interior of sector of torus with rectangular cross-section using Fourier method of separation of variables.*

Keywords: *magnetic field analysis, boundary problem, Laplace's equations*

1. PROBLEM DEFINITION

Fourier variable separation method is one of the basic analytical method of solution of mathematical physics boundary problem in linear medium [2]. In terms of it the accurate development (in formula form) can be derived.

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In electromechanics problems the cross-section of electrical machine's active zone usually can be divided into three-dimensional elementary regions shaped as torus ring sectors with rectangular cross-sections (Fig. 1). Elementary regions will be considered to be irreducible discrete geometric forms within the bounds of which the medium is homogeneous (magnetic permeability within the bounds of elementary region is invariable and can variate discontinuously only on the bounds of the region).

If the Dirichlet's problem's solution for the elementary region is obtained the magnetizing force components (normal and tangential) on its bounds can be determined. Then the Dirichlet's problem for the electrical machine's active zone can be solved using magnetic field's boundary conditions for the aggregate amount of elementary regions [1].

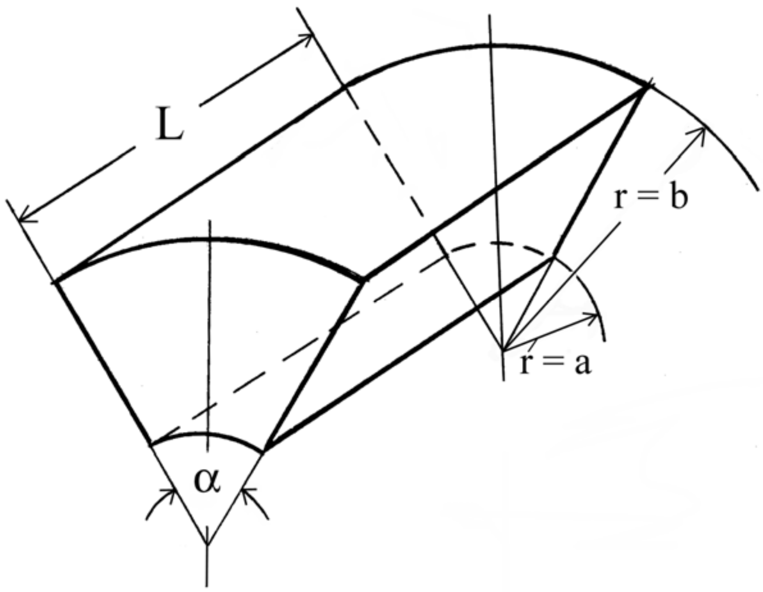

Fig. 1. Three-dimensional elementary region

2. THE SOLUTION OF LAPLACE'S EQUATION FOR THE SECTOR OF TORUS RING

The general solution of Laplace's equation for the sector of torus ring as on Figure 1 can be written as

$$U(r,\varphi,z) = U_1(r,\varphi,z) + U_2(r,\varphi,z) + U_3(r,\varphi,z)\,, \tag{1}$$

where $U_j(r,\varphi,z)\,(j\ =\ 1,\ 2,\ 3)$ are partial solutions which equal specified values on opposite boundary surfaces and equal zero on the other four boundaries:

$$U_1(r,\varphi,z)\big|_{z=0} = f_1(r,\varphi);U_1(r,\varphi,z)\big|_{z=l} = f_2(r,\varphi). \tag{2}$$

$$U_1(r,\varphi,z)\big|_{r=a} = 0;\ U_1(r,\varphi,z)\big|_{\varphi=0} = 0;\ \ U_1(r,\varphi,z)\big|_{r=b} = 0;\ U_1(r,\varphi,z)\big|_{\varphi=\alpha} = 0;$$

$$U_2(r,\varphi,z)\big|_{r=a} = g_1(\varphi,z);\ U_2(r,\varphi,z)\big|_{r=b} = g_2(\varphi,z); \tag{3}$$

$$U_2(r,\varphi,z)\big|_{z=0} = 0;\ U_2(r,\varphi,z)\big|_{z=l} = 0;\ \ U_2(r,\varphi,z)\big|_{\varphi=0} = 0;\ U_2(r,\varphi,z)\big|_{\varphi=\alpha} = 0;$$

$$U_3(r,\varphi,z)\big|_{\varphi=0} = h_1(r,z);\ U_3(r,\varphi,z)\big|_{\varphi=\alpha} = h_2(r,z). \tag{4}$$

$$U_3(r,\varphi,z)\big|_{r=a} = 0;\ U_3(r,\varphi,z)\big|_{r=b} = 0;\ U_3(r,\varphi,z)\big|_{z=0} = 0;\ U_3(r,\varphi,z)\big|_{z=l} = 0.$$

To solve the first problem the partial solutions of Laplace's equation must be found. It can be represented as:

$$U_1(r,\varphi,z) = \vartheta(r,\varphi)Z(z) \tag{5}$$

(it is significant that the variable that was separated is that one on the boundaries of which heterogeneous (non-zero) boundary conditions are specified).

Substituting (5) into three-dimensional Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{6}$$

and separating the variables, we will get:

$$\frac{\Delta_2 \vartheta}{\vartheta(r,\varphi)} \equiv -\frac{z''}{Z(z)} = -\lambda\,, \tag{7}$$

where

$$\Delta_2 \vartheta = \frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial \vartheta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vartheta}{\partial \varphi^2} \tag{8}$$

Expression (8) is two-dimensional Laplacian for two variables (r, φ).

From (7) it is resulted that

$$\left.\begin{array}{l} \Delta_2 \vartheta + \lambda \vartheta = 0\,,\ a < r < b\,,\ 0 < \varphi < \alpha\,, \\ \vartheta_1\big|_{r=a} = \vartheta_1\big|_{r=b} = \vartheta_1\big|_{\varphi=0} = \vartheta_1\big|_{\varphi=\alpha} = 0\,,\ \vartheta(r,\varphi) \not\equiv 0 \end{array}\right\} \tag{9}$$

and

$$Z''(z) - \lambda Z(z) = 0\,,\ 0 < z < l \tag{10}$$

Expression (9) is the Sturm-Liouville problem for the ring sector.

Representing the solution of this problem as

$$\vartheta = R(r)\Phi(\varphi)\,, \tag{11}$$

after substituting (11) into (9) and separating the variables we will get:

$$\frac{r^2 R'' + r R'}{R} + \lambda r^2 \equiv -\frac{\Phi''}{\Phi} = \upsilon\,. \tag{12}$$

Two common differential equations with zero boundary conditions follow from (12):

$$\left.\begin{array}{l} r^2 R'' + rR' + (\lambda r^2 - \upsilon)R = 0\,,\ a < r < b\,, \\ R(r)\big|_{r=a} = 0;\ R(r)\big|_{r=b} = 0; \end{array}\right\} \tag{13}$$

$$\left.\begin{array}{l} \Phi'' + \upsilon\Phi = 0\,,\ 0 < \varphi < \alpha\,, \\ \Phi(\varphi)\big|_{\varphi=0} = 0;\ \Phi(\varphi)\big|_{\varphi=\alpha} = 0\,. \end{array}\right\} \tag{14}$$

For the solution (14) it is valid that:

$$\Phi(\varphi) = \Phi_n(\varphi) = \sin\sqrt{\upsilon_n}\varphi\,, \tag{15}$$

where

$$\upsilon = \upsilon_n = \left(\frac{\pi\, n}{\alpha}\right)^2,\ n = 1,2,3,\ldots\infty\,.$$

Bessel differential equation (13) has the solution

$$R_n(r) = C_1 J_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, r) + C_2 N_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, r), \tag{16}$$

where $J_n(x)$, $N_n(x)$ – Bessel functions of the first and the second kind correspondingly, constants C_1 and C_2 can be found using zero boundary conditions (13)

$$C_1 J_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, a) + C_2 N_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, a) = 0\,, \tag{17}$$

$$C_1 J_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, b) + C_2 N_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, b) = 0\,. \tag{18}$$

The non-zero solution of combined equations (17), (18) with unknown C_1 and C_2 is possible if its determinant is zero

$$J_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, a)\, N_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, b) - N_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, a)\, J_{\sqrt{\upsilon_n}}(\sqrt{\lambda}\, b) = 0\,. \tag{19}$$

In this equation (it is called dispersing [2]) the characteristic constants $\lambda = \lambda_n^{(k)},\ k = 1, 2, 3,...$ can be determined which are the k-th roots of equation (19) on every fixed $n = 1, 2, 3,....$

From the first eqation (17) we have

$$C_2 = -C_1 \frac{J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}\, a)}{N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k0}}\, a)}\,. \tag{20}$$

Choosing

$$C_1 = N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k0}}\, a)\,, \tag{21}$$

(becouse of degeneracy of combined equations (17), (18) one of the constants can be determined arbitrarily) we will get from (20)

$$C_2 = -J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}\, a)\,. \tag{22}$$

After substituting (21) and (22) into (16) we will get

$$R_{nk}(r) = J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}r)N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}a) - N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}r)J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}a). \qquad (23)$$

Considering (19) for the solution (16) it is also valid

$$R_{nk}(r) = \frac{J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}a)}{J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}b)} * \left[J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}r)N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}b) - N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}r)J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}b)\right]. \qquad (24)$$

It is convenient to represent the common solution of equation (10) considering equality $\lambda = \lambda_n^{(k)}$ as

$$Z(z) = Z_{nk}(z) = A_{nk}\frac{\operatorname{sh}\sqrt{\lambda_n^{(k)}}z}{\operatorname{sh}\sqrt{\lambda_n^{(k)}}l} + B_{nk}\frac{\operatorname{sh}\sqrt{\lambda_n^{(k)}}(l-z)}{\operatorname{sh}\sqrt{\lambda_n^{(k)}}l}, \qquad (25)$$

where A_{nk}, B_{nk} – constants.

Thus partial solutions of Laplace's equation for the first standard problem take the form

$$U_{nk}(r,\varphi,z) = R_{nk}(r)\Phi_n(\varphi)Z_{nk}(z). \qquad (26)$$

Substituting (14), (23) and (25) into (26) and adding up partial solutions we can find the common solution of the first standard problem:

$$U_1(r,\varphi,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}[J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}r)N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}a) - N_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}r) *$$

$$* J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}a)][A_{nk}\frac{\operatorname{sh}\sqrt{\lambda_n^{(k)}}z}{\operatorname{sh}\sqrt{\lambda_n^{(k)}}l} + B_{nk}\frac{\operatorname{sh}\sqrt{\lambda_n^{(k)}}(l-z)}{\operatorname{sh}\sqrt{\lambda_n^{(k)}}l}]\sin\frac{\pi n}{\alpha}\varphi, \qquad (27)$$

from which follows

$$U_1(r,\varphi,z)\big|_{z=0} = f_1(r,\varphi) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}R_{nk}(r)B_{nk}\sin\frac{\pi n}{\alpha}\varphi, \qquad (28)$$

$$U_1(r,\varphi,z)\Big|_{z=l} = f_2(r,\varphi) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} R_{nk}(r) A_{nk} \sin\frac{\pi n}{\alpha}\varphi \,. \tag{29}$$

Expressions (28), (29) are the two-fold Fourier series with coefficients A_{nk}, B_{nk}

$$B_{nk} = \frac{2}{\alpha \left\| R_{nk}(r) \right\|^2} \int_a^b \int_0^\alpha [\, f_1(r,\varphi) R_{nk}(r) \sin\frac{\pi n}{\alpha}\varphi] r \partial r \partial\varphi \,, \tag{30}$$

$$A_{nk} = \frac{2}{\alpha \left\| R_{nk}(r) \right\|^2} \int_a^b \int_0^\alpha [\, f_2(r,\varphi) R_{nk}(r) \sin\frac{\pi n}{\alpha}\varphi] r \partial r \partial\varphi \,, \tag{31}$$

where

$\left\| R_{nk}(r) \right\|^2 = \int R_{nk}^2(r) r dr$ – the norm $R_{nk}(r)$ squared. According to [2] we have

$$\left\| R_{nk}(r) \right\|^2 = \frac{2}{\pi^2} \frac{1}{\lambda_n^{(k)}} \frac{J^2_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}a) - J^2_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}b)}{J^2_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}b)} \,. \tag{32}$$

The solution of the second standard problem is represented in the same form as for the first problem:

$$U_2(r,\varphi,z) = \vartheta(r,\varphi) Z(z) \,, \tag{33}$$

but substituting (33) into three-dimensional Laplace's equation (6) and separating variables we will changes the sign of the 3-rd member of the equality

$$\frac{\Delta_2 \vartheta}{\vartheta(r,\varphi)} \equiv -\frac{z''}{Z(z)} = \lambda$$

and we will get, as in case with the first standard problem, two new equations:

$$\left.\begin{aligned} &Z''(z) + \lambda Z(z) = 0 \,,\ 0 < z < l \,, \\ &Z(0) = Z(l) = 0 \,, \end{aligned}\right\} \tag{34}$$

and

$$\left.\begin{aligned} &\Delta_2 \vartheta - \lambda \vartheta = 0\,,\ a < r < b\,,\ 0 < \varphi < \alpha\,, \\ &\vartheta_1\big|_{r=a} = \vartheta_1\big|_{r=b} = \vartheta_1\big|_{\varphi=0} = \vartheta_1\big|_{\varphi=\alpha} = 0\,,\ \vartheta(r,\varphi) \not\equiv 0\,. \end{aligned}\right\} \tag{35}$$

The first one is the Sturm-Liouville problem for the line segment. Its characteristic constants and eigen-functions looks like

$$\lambda = \lambda_k = \left(\frac{\pi k}{l}\right)^2,\ Z = Z_k(z) = \sin\frac{\pi k}{l} z\,,\ k = 1,2,\ldots,\infty \tag{36}$$

Partial solutions of equation (35) can be found using variable separation method

$$\vartheta(r,\varphi) = R(r)\Phi(\varphi) \tag{37}$$

Substituting (37) into (35) and separating variables we will get

$$\frac{r^2 R'' + rR' - \lambda r^2 R}{R} \equiv -\frac{\Phi''}{\Phi} = \upsilon\,.$$

From this the problem for determining $\Phi(\varphi)$ folows:

$$\left.\begin{aligned} &\Phi'' + \upsilon\Phi = 0\,;\ 0 < \varphi < \alpha\,; \\ &\Phi(\varphi)\big|_{\varphi=0} = 0\,;\ \Phi(\varphi)\big|_{\varphi=\alpha} = 0 \end{aligned}\right\} \tag{38}$$

and the problem for determining $R(r)$ folows:

$$r^2 R'' + rR' - (\lambda r^2 + \upsilon)R = 0\,,\ a > r > b\,. \tag{39}$$

The problem (39) has its characteristic constants and eigen-functions, which looks like

$$\upsilon = \upsilon_n = \left(\frac{\pi n}{\alpha}\right)^2,\ \Phi(\varphi) = \Phi_n(\varphi) = \sin\sqrt{\upsilon_n}\,\varphi\,,\ n = 1,2,3,\ldots\infty\,. \tag{40}$$

The equation (39) is Bessel equation with pure imaginary argument. Its common solution looks like [3]

$$R(r) = C_1 I_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} r) + C_2 K_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} r) , \tag{41}$$

where $I_{\sqrt{\upsilon_n}}(x), K_{\sqrt{\upsilon_n}}(x)$ are modified Bessel functions, Infeld function and Macdonald function correspondingly;

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2 , \; k = 1,2,...,\infty .$$

It is convenient to represent expression (41) as

$$R(r) = {}^1R_{nk}(r) = C_{nk} \frac{I_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} r) - I_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} a)}{I_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} b) - I_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} a)} + D_{nk} \frac{K_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} b) - K_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} r)}{K_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} b) - K_{\sqrt{\upsilon_n}}(\sqrt{\lambda_k} a)} . \tag{42}$$

The common solution of the second standard problem will take the form

$$U_2(r,\varphi,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} {}^1R_{nk}(r) \sin(\frac{\pi k}{l} z) \sin(\frac{\pi n}{\alpha} \varphi) . \tag{43}$$

For heterogeneous boundary conditions from (42) and (43) we have

$$U_2(r,\varphi,z)\big|_{r=a} = g_1(\varphi,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} D_{nk} \sin(\frac{\pi k}{l} z) \sin(\frac{\pi n}{\alpha} \varphi) , \tag{44}$$

$$U_2(r,\varphi,z)\big|_{r=b} = g_2(\varphi,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} C_{nk} \sin(\frac{\pi k}{l} z) \sin(\frac{\pi n}{\alpha} \varphi) . \tag{45}$$

Expressions (44), (45) are the two-fold Fourier series with coefficients

$$D_{nk} = \frac{4}{\alpha l} \int_0^{\alpha}\int_0^{l} g_1(\varphi,z) \sin(\frac{\pi k}{l} z) \sin(\frac{\pi n}{\alpha} \varphi) \partial z \partial \varphi \tag{46}$$

and

$$C_{nk} = \frac{4}{\alpha l} \int_0^{\alpha}\int_0^{l} g_2(\varphi,z) \sin(\frac{\pi k}{l} z) \sin(\frac{\pi n}{\alpha} \varphi) \partial z \partial \varphi . \tag{47}$$

It can be seen from the common solution (43) that homogeneous boundary conditions are satisfied

$$U_2(r,\varphi,z)\big|_{\varphi=0} = U_2(r,\varphi,z)\big|_{\varphi=\alpha} = U_2(r,\varphi,z)\big|_{z=0} = U_2(r,\varphi,z)\big|_{z=l} = 0\,.$$

The third standard problem has heterogeneous boundary conditions

$$U_3(r,\varphi,z)\big|_{\varphi=0} = h_1(r,z)\,,\ U_3(r,\varphi,z)\big|_{\varphi=\alpha} = h_2(r,z)$$

On the bounds of variable φ. That is why we will start the separation of variables with formula

$$U_3(r,\varphi,z) = \vartheta(r,z)\,\Phi(\varphi)\,. \tag{48}$$

Substituting (41) into three-dimensional Laplace's equation (6) and separating variables we will get

$$\left.\begin{aligned} &\frac{\Delta_2\vartheta}{\vartheta(r,z)} \equiv -\frac{\Phi''}{\Phi(\varphi)} = -\lambda\,, \\ &\text{where } \Delta_2\vartheta = r^2\frac{\partial^2\vartheta}{\partial r^2} + r\frac{\partial\vartheta}{\partial r} + r^2\frac{\partial^2\vartheta}{\partial z^2}\,. \end{aligned}\right\} \tag{49}$$

From (48) follows

$$\begin{gathered} \Delta_2\vartheta + \lambda\vartheta = 0\,,\ a<r<b\,,\ 0<z<l\,, \\ \vartheta\big|_{r=a} = \vartheta\big|_{r=b} = \vartheta\big|_{z=0} = \vartheta\big|_{z=l} = 0\,,\ \vartheta(r,z)\not\equiv 0\,. \end{gathered} \tag{50}$$

and

$$\Phi'' - \lambda\Phi = 0\,;\ 0<\varphi<\alpha\,. \tag{51}$$

The solution of the last equation can be represented as

$$\Phi(\varphi) = E\frac{\operatorname{sh}\sqrt{\lambda}\varphi}{\operatorname{sh}\sqrt{\lambda}\alpha} + F\frac{\operatorname{sh}\sqrt{\lambda}(\alpha-\varphi)}{\operatorname{sh}\sqrt{\lambda}\alpha}\,, \tag{52}$$

where E and F – constants.

Equation (50) can ba considered as the Sturm-Liouville problem.
Considering the solution of this problem in the form

$$\vartheta(r,z) = R(r)Z(z)\,, \tag{50}$$

we will get after substitutiong (53) into (50) and separating variables

$$\frac{r^2 R''(r) + rR'(r)}{r^2 R(r)} + \frac{\lambda}{r^2} \equiv -\frac{Z''(z)}{Z(z)} = \mu\,. \tag{54}$$

Two common differential equations follow from the expression above:

$$r^2 R'' + rR' + (-\mu\, r^2 + \lambda)R = 0,\;\; a < r < b,\;\; R(r)\big|_{r=a} = 0\,,\; R(r)\big|_{r=b} = 0 \tag{55}$$

and

$$Z''(z) + \mu\, Z(z) = 0\,,\; 0 < z < l\,,\; Z(0) = Z(l) = 0\,. \tag{56}$$

The last one has the solution looks like

$$Z = Z_n(z) = \sin\sqrt{\mu_n}\, z\,, \tag{57}$$

where $\mu = \mu_n = \left(\frac{\pi\, n}{l}\right)^2,\; n = 1,2,...,\infty\,.$

The first equation, which is Bessel equation, differ in structure from analogous (13) with change of signs of items in the round brackets.

After introducing designation $\lambda = -\upsilon^2$ expression (55) can be represented as

$$r^2 R'' + rR' - (\mu\, r^2 + \upsilon^2)R = 0,\;\; a < r < b,\;\; R(r)\big|_{r=a} = 0\,,\; R(r)\big|_{r=b} = 0\,.$$

This is modified Bessel equation. Its solution looks like [4]

$$R_n(r) = C_1 I_\upsilon(\sqrt{\mu_n}\; r) + C_2 K_\upsilon(\sqrt{\mu_n}\; r),$$

where constants C_1 and C_2 are determined in the same conditions as in the first standard problem.

As a result we will get

$$R_{nk}(r) = {}^2R_{nk}(r) = I_{\upsilon_n^{(k)}}(\sqrt{\mu_n}\, r)\, K_{\upsilon_n^{(k)}}(\sqrt{\mu_n}\, a) - K_{\upsilon_n^{(k)}}(\sqrt{\mu_n}\, r)\, I_{\upsilon_n^{(k)}}(\sqrt{\mu_n}\, a), \tag{58}$$

where $\upsilon = \upsilon_n^{(k)}$ - the *k*-th root of dispersion equation

$$I_\upsilon(\sqrt{\mu_n}\, a)\, K_\upsilon(\sqrt{\mu_n}\, b) - I_\upsilon(\sqrt{\mu_n}\, b)\, K_\upsilon(\sqrt{\mu_n}\, a) = 0. \tag{59}$$

As a result the solution of the third standard problem considering equation $\lambda_{(k)}^n = -\upsilon_n^{(k)}$ will be represented as

$$U_3(r,\varphi,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} {}^2R_{nk}(r)\left[E_{nk}\frac{sh\sqrt{\lambda_{(k)}^n}\varphi}{sh\sqrt{\lambda_{(k)}^n}\alpha} + F_{nk}\frac{sh\sqrt{\lambda_{(k)}^n}(\alpha-\varphi)}{sh\sqrt{\lambda_{(k)}^n}\alpha}\right]\sin\sqrt{\mu_n}z. \tag{60}$$

From (60) we have

$$U_3(r,\varphi,z)\big|_{\varphi=0} = h_1(r,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} {}^2R_{nk}(r) F_{nk}\sin\sqrt{\mu_n}z, \tag{61}$$

$$U_3(r,\varphi,z)\big|_{\varphi=\alpha} = h_2(r,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} {}^2R_{nk}(r) E_{nk}\sin\sqrt{\mu_n}z. \tag{62}$$

From expressions (61), (62) follows

$$F_{nk} = \frac{4}{l\left\|{}^2R_{nk}(r)\right\|^2}\int_a^b\int_0^l h_1(r,z)\,{}^2R_{nk}(r)\sin\sqrt{\mu_n}z\, r\partial r\partial z, \tag{63}$$

$$E_{nk} = \frac{4}{l\left\|{}^2R_{nk}(r)\right\|^2}\int_a^b\int_0^l h_2(r,z)\,{}^2R_{nk}(r)\sin\sqrt{\mu_n}z\, r\partial r\partial z, \tag{64}$$

where the squared norm $\left\|{}^2R_{nk}(r)\right\|^2$ is determined using formula (32), in which $J_{\sqrt{\upsilon_n}}(\sqrt{\lambda_n^{(k)}}a)$ must be replaced with $I_{\upsilon_n^{(k)}}(\sqrt{\mu_n}\, a)$ and $\sqrt{\lambda_n^{(k)}}$ must be replaced with $\sqrt{\mu_n}$.

3. ANALYSIS OF MAGNETIC FIELD IN THE TORUS SECTOR

The components of magnetic intensity can be determined using formulas

$$H_r = -\frac{\partial U}{\partial r};\ H_\varphi = -\frac{\partial U}{r\partial \varphi};\ H_z = -\frac{\partial U}{\partial z}. \qquad (65)$$

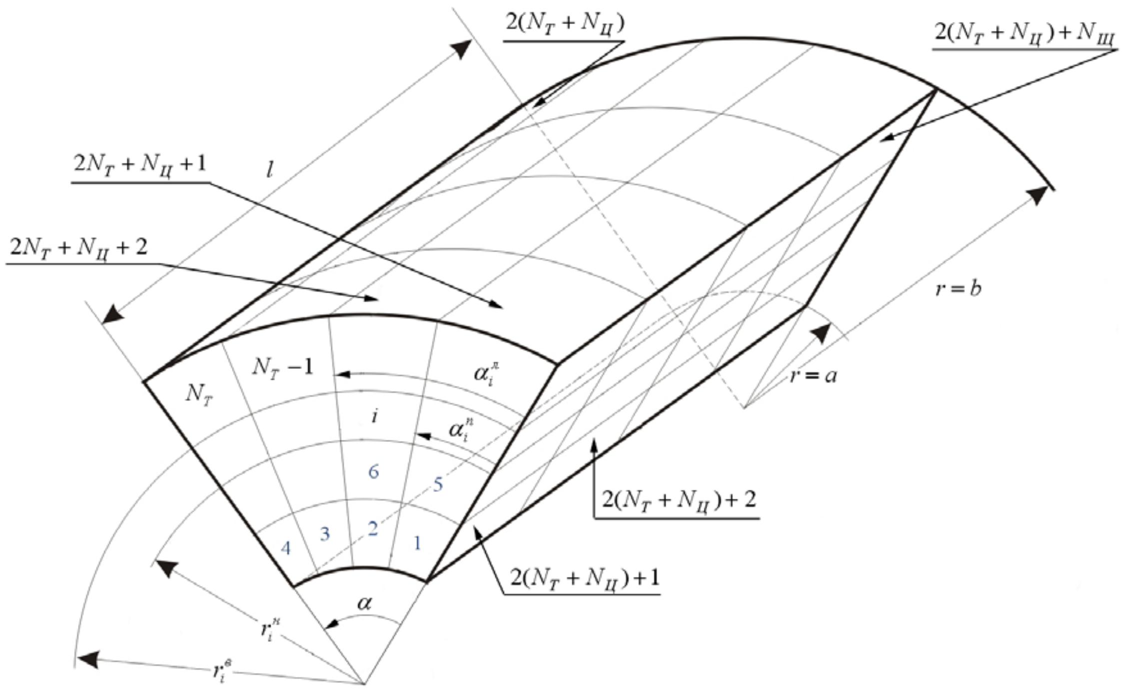

Fig. 2. Fragmentation of elementary region's external surface into local areas in the bounds of which the scalar magnetic potential is considered to be constant

Using piecewise-benched approximation of scalar magnetic potential functions on external surfaces of elementary regions (Fig.2) (for example, $f_1(r,\varphi)$ and $f_2(r,\varphi)$ which are assigned on the front and on the end) we can represent constants in two-fold Fourier series as linear combination of local areas' magnetic potential. For example, from formulas (30) and (31) follows

$$B_{nk} = \sum_{i=1}^{N_T} \alpha_i U_i\ ,\ A_{nk} = \sum_{i=N_T+1}^{2N_T} \alpha_{i-N_T} U_i\ , \qquad (66)$$

where U_i is the i-th local area's magnetic potential;

$$\alpha_i = \frac{2}{\alpha \left\| R_{nk} \right\|^2} \int\limits_{r_i^{í}}^{r_i^{â}} \int\limits_{\alpha_i^{ï}}^{\alpha_i^{ë}} R_{nk}(r) \sin(\frac{\pi n}{\alpha} \varphi) r dr d\varphi ,$$

$r_i^{í}, r_i^{â}$ – radii of correspondingly lower and upper arc bounds of the i-th local area on the face;

$\alpha_i^{ï}, \alpha_i^{ë}$ – angular data of correspondingly right and left radial bounds of the i-th local area on the face.

Considering magnetic intensity's dispensing on the external surface of elementary region is also piecewise-constant, in according with formula (65) we can determine relation between local values of magnetic intensity normal components and scalar magnetic potentials in the vector-matrix form:

$$\mathbf{H}_n = \mathbf{g}\,\mathbf{U} . \tag{67}$$

The structure of square matrix $\mathbf{g}$ which dimension is $2(N_T + N_{Ö} + N_{Ù})$ looks like

$[g]_1^1$	$[g]_2^1$	$[g]_3^1$	$[g]_4^1$	$[g]_5^1$	$[g]_6^1$
$[g]_1^2$	$[g]_2^2$	$[g]_3^2$	$[g]_4^2$	$[g]_5^2$	$[g]_6^2$
$[g]_1^3$	$[g]_2^3$	$[g]_3^3$	$[g]_4^3$	$[g]_5^3$	$[g]_6^3$
$[g]_1^4$	$[g]_2^4$	$[g]_3^4$	$[g]_4^4$	$[g]_5^4$	$[g]_6^4$
$[g]_1^5$	$[g]_2^5$	$[g]_3^5$	$[g]_4^5$	$[g]_5^5$	$[g]_6^5$
$[g]_1^6$	$[g]_2^6$	$[g]_3^6$	$[g]_4^6$	$[g]_5^6$	$[g]_6^6$

For example, constituent of matrix $[g]_1^1$ looks like

$$g_{ji} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \alpha_i \frac{\mathrm{ch}\sqrt{\lambda_n^{(k)}}\, l}{\mathrm{sh}\sqrt{\lambda_n^{(k)}}\, l} \sin \frac{\pi n}{\alpha} \tilde{\varphi}_j \ , \ i, j = 1, 2, ..., N_T .$$

Formula (65) allows to determine vector-matrix expression for local area's magnetic intensity tangential components

$$\mathbf{H}_\tau = \mathbf{h}\,\mathbf{U} .$$

Equation (67), which express the relation between magnetic intensity components and scalar magnetic potentials, is basic for numerical definition of magnetic field on the whole accounting area, which consist of different torus sectors with summarized amount of i. If ${}^{i}U_j$ is the *j*-th scalar magnetic potential component of the *i*-th sector (elementary region), then under continuous numbering of all elementary regions' sighting points we will have

$${}^{i}U_j = U_s\,,\ \mathrm{s} = 1, 2, \ldots, Q.$$

For the point with number *q* $(q \in s)$, coincides with two sighting points of two elementary regions numbered i and *k* is valid

$${}^{i}B_{nq} = {}^{k}B_{nq}\,. \tag{68}$$

Extending equation (68) to all similar sighting points we will get the system of linear algebraic equations relative to unknown vector $\mathbf{U} = [U_1 \quad U_2 \quad \ldots \quad U_N]^T$ which looks like

$$\mathbf{A}\,\mathbf{U} = \mathbf{F}\,, \tag{69}$$

where $\mathbf{A}$ is a square matrix with dimension Q;
non-zero components of vector $\mathbf{F}$ are represented with magnetic field sources: residual magnetization $M_{r\alpha}$ of a magnet on the line of magnetization α; scalar magnetic potential jump values U_s in the sighting points, throw which current magnetic sheet of electrical machine winding; values of additional vortex areas' magnetic intensity (in this areas current density is non-zero) [1].

LITERATURE

1. Afanasev A.A., Vorobyev A.N.: *For calculation of parallel-sided magnetic fields in non-linear mediums*, Proceedings of the Academy of Sciences Power engineering and Transport, 1992, Vol. 22
2. Bogolubov A.N., Kravtsov V.V.: *Mathematical physics problems*, Moscow State University Publishing House, Moscow, 1998
3. Kamke E.: *Ordinary differential equation's reference book*, Science, Moscow, 1971
4. Zaytsev V.F., Polyanin A.D.: *Linear ordinary differential equation's reference book*, Factorial, Moscow, 1997

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TRÓJWYMIAROWA ANALIZA POLA MAGNETYCZNEGO WE WNĘTRZU PIERŚCIENIA

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STRESZCZENIE *Zaprezentowano rozwiązanie analityczne wyznaczenia wartości brzegowych dla równania Laplace'a we wnętrzu pierścienia. Zastosowano metodę Fouriera z rozdzieleniem zmiennych.*