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TWO THEOREMS ABOUT ELECTROMAGNETIC FORCE IN ACTIVATED ANISOTROPIC REGIONS

DWA TWIERDZENIA O SIŁACH ELEKTROMAGNETYCZNYCH WE WZBUDZONYCH ŚRODOWISKACH ANIZOTROPOWYCH

Abstract The paper has dealt with two problems of calculation of electromagnetic force/torque. The first one is for magnetically anisotropic and conductive region. It has been presented sufficient condition for surface-integral representation of electromagnetic force/torque in conductive and anisotropic region. The second approach deals with the problem of independence of force/torque calculated value from shape of integral-surface. The second theorem gives the sufficient condition for this independence for Maxwell stress tensor method is applied.

1. Electromagnetic field forces

The electromagnetic field forces [1, 2] theoretical analysis is still vital problem [3, 4, 5, 6]. For evaluation of the electromagnetic force density is applied the following relation

$$\vec{f} = \vec{f}_L + \vec{N} + \vec{M} \tag{1}$$

which is proved [12]. The proof bases on Lorentz force density formula, Maxwells equations, and assumptions that displacement current (Poynting force [3]) and magnetic polarization (hysteresis force) can be neglected. In Eqn (1) $\vec{f}_L$ denotes Lorentz force density, $\vec{N}$ is the so-called nonhomegenity component [12] (it appears in nonhomogeneous regions), and

$$\vec{M} = \tfrac{1}{2}(\nu_{vu} - \nu_{uv})B_v \mathrm{grad}(B_u) \tag{2}$$

is the anisotropy component for magnetic field. Total electromagnetic force/torque can be calculated by the following equation

$$\vec{f} = -\vec{i}_u \mathrm{div}_u(\vec{\sigma}_u) - \vec{\Delta} \tag{3}$$

where Maxwell's stress appears [12].

2. The first theorem – surface-integral representation for magnetically anisotropic and conductive region

The first theorem considers the equivalence between both volume and surface integrals representations for total electromagnetic force /torque. This problem could be called as *force/torque surface-integral representation problem*.
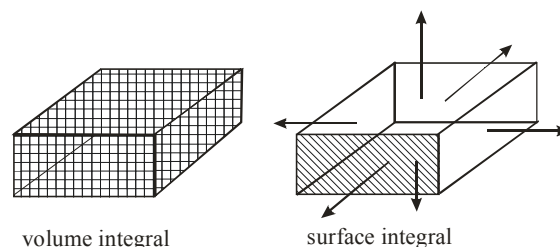

Fig. 1. *Equivalence of volume and surface integrals*

This problem is analogous to the surface-integral representation of total electric charge placed in finite volume due to Gaussian law. The next analogous problem appears for Ampere law. The contour-integral representation of total currents passing through the surface is considered. It is known [1, 2, 4, 5, 6, 11] that the surface-integral representation for electromagnetic field forces can be introduced for electromagnetic field regions if the Maxwell stress tensor is symmetrical. The symmetry of the Maxwell stress tensor is guaranteed for isotropic media, only [1, 2]. The main interest is whether for some anisotropic media the surface-integral of force/torque representation is tenable. The answer is positive under an extended condition. The theorem of surface integral representation is satisfied for either isotropic or anisotropic media (normal anisotropy). For isotropic medium the reluctivity (permeability) matrix is diagonal and all pivot values are equal to each other. The normal anisotropy is stated for the media which reluctivity (permeability) matrix is symmetrical one. There are also media for those the reluctivity (permeability) matrix is

asymmetrical (the so-called active media [8]). It will be proved that

for magnetic field where the following conditions are satisfied

- no nonhomogeneous force(no reluctance force),

- no hysteresis phenomenon appears,

the force calculated by Lorentz's (volume integral) and Maxwell (surface integral) methods lead to the same result for magnetically anisotropic region if reluctivity matrix is symmetrical

i.e. for u ≠ v it is satisfied

$$\nu_{vu} = \nu_{uv} \qquad (4)$$

If the condition (4) is satisfied thus the anisotropy component vanishes $M_u = 0$ - due to Eqn (2). It should be pointed out that if the condition (4) is not satisfied

$$\nu_{vu} \neq \nu_{uv} \qquad (5)$$

the surface-integral representation is not possible, generally. The mathematical proof of this theorem is based on Eqn (1). Indeed, the assumptions specified above and Eqn (1) for u^{th} force component lead to the relation as follows

$$f_u = f_{Lu} + M_u = -\text{div}_u\left(\vec{\sigma}_u\right) - \Delta_u \qquad (6)$$

For properly chosen co-ordinate system it can be set $\Delta_u = 0$ (e.g. for Cartesian coordinate system always $\Delta_u = 0$, for cylindrical system $\Delta_\alpha = 0$), thus according to (3) is can be written

$$L_u f_u = L_u f_{Lu} = -\text{div}\left(\vec{\sigma}_u\right) \qquad (7)$$

where L_u means lame coefficient. Applying Gaussian theorem is obtained the formula

$$\int_V L_u f_u dV = -\int_S \vec{\sigma}_u d\vec{S} \qquad (8)$$

that proves the thesis. This means that for regions with symmetric reluctivity matrix (either isotropic or normal anisotropic region) the Lorentz force/torque can be presented by surface-integral of the Maxwell stress tensor. In other words, the surface-integral representation is possible for media with symmetric reluctivity matrix. The above statement, however it applies only to a small range group of anisotropic materials, is important from theoretical point of view. In order to explain this theorem in physical way electromechanical converter with solid

cylindrical rotor has been considered. An example confirms this thesis for asymmetrical stress tensor for simple electromechanical converter containing anisotropic (paragraph 5). The analysis of electromagnetic field has been provided with the help of variable separation method. This way of analysis is chosen for giving precise insight into torque calculations.

3. The second theorem – surface-integral representation for magnetically anisotropic and nonconductive region

The surface-integral representation of electromagnetic force/torque it is important problem from both theoretical and computational point of view. Let us regard electromagnetic force/torque that is exerted in finite region R (e.g. electromechanical converter moving part - rotor, carriage) can be evaluated by surface integral over the surface S (e.g. either S_1 or $S_2 \neq S_1$). The surface S must surround the whole finite region R. If the outer region R_{out} does not involve electromagnetic force/torque thus the surface S (e.g. S_1 or $S_2 \neq S_1$) can be placed differently in the gap fulfilling condition that it must surrounds whole exerting force region R. Exemplary, for electromechanical converter such as rotating electric machine the electromagnetic torque does not depend on the radius of surface S placed in the gap region – Fig. 2. The independence of the force/torque surface integral results from magnetic feature of the gap – it is the air-gap (i.e. the vacuum gap).
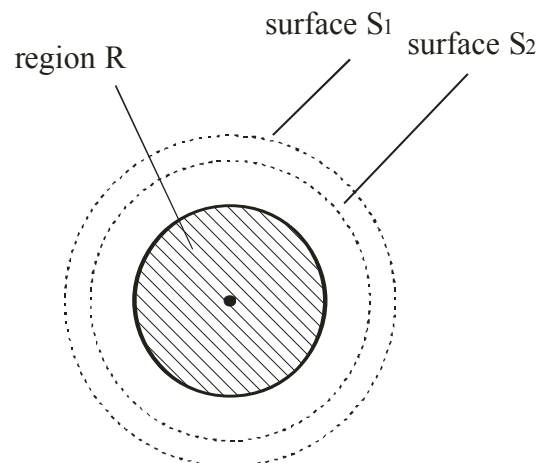

Fig. 2. The electromagnetic torque evaluation by surface integral – cross-section

The problem affects to electrostatic while calculating total electric charge with the help of Gauss law. The outer surface (Gaussian sur-

face) must be spread so as to surround the whole charge independently of its shape, but the surrounding region must not contain any charge. The gap which surrounds the active region R (e.g. rotor) could be filled with ferrofluid [9, 10]. As a consequence, the outer region R_{out} (e.g. the gap) could be either isotropic or anisotropic. The second theorem affects the condition for possibility of surface-integral representation for different magnetic features of surrounding region R_{out} (e.g. the gap). Let us consider problem of calculation of electromagnetic force/torque acting on region R. The region R affecting electromagnetic force/torque is included in volume V. The volume V can be presented as a sum of

$$V = R \oplus (V / R) \tag{9}$$

where the $V/R = R_{out}$ denotes the region which surrounds active region R. The integrals that lead to total electromagnetic force and torque values exerted in region R

$$\vec{F}_V = \int_V \vec{f} dV = \int_R \vec{f} dV + \int_{R_{out}} \vec{f} dV \tag{10a}$$

and

$$\vec{T}_V = \int_V (\vec{r} \times \vec{f}) dV = \int_R (\vec{r} \times \vec{f}) dV + \int_{R_{out}} (\vec{r} \times \vec{f}) dV \tag{10b}$$

respectively. The total electromagnetic force and torque are affected in region R, thus

$$\vec{F}_V = \vec{F}_e + \int_{R_{out}} \vec{f} dV = \vec{F}_e + \Delta \vec{F}_{out} \tag{11a}$$

and

$$\vec{T}_V = \vec{T}_e + \int_{R_{out}} (\vec{r} \times \vec{f}) dV = \vec{T}_e + \Delta \vec{T}_{out} \tag{11b}$$

where $\vec{F}_V$, $\vec{T}_V$ are force and torque calculated over the volume V; $\vec{F}_e$, $\vec{T}_e$ are total electromagnetic force and torque; $\Delta \vec{F}_{out}$, $\Delta \vec{T}_{out}$ denote the residual integrals over the outer region R_{out}. Now, the theoretical problem can be formulated precisely. Whether the electromagnetic force/torque arising in region R can be calculated by surface-integral over the volume constant ? Practical meaning of this question is whether the electromagnetic force/torque can be calculated by surface-integral over different surfaces which includes the experiencing force/torque region R. If the answer is positive the total elec-

tromagnetic force/torque can be calculated for any surface which surrounds R (e.g. for different boundary surfaces S_1 and S_2 as shown in Fig. 3).
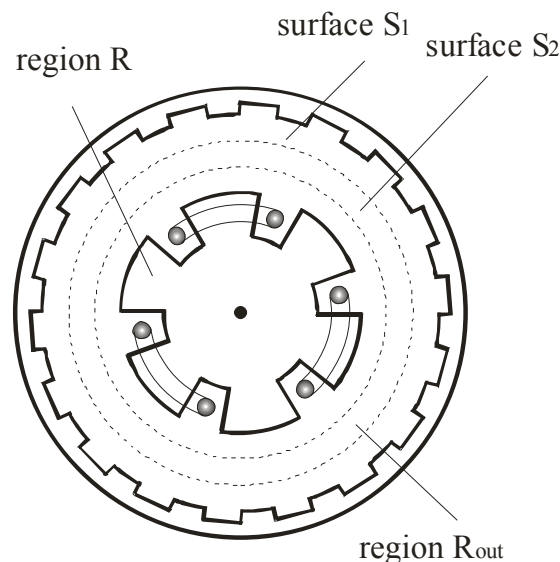

Fig. 3. The electromagnetic torque does not depend on radius of integral-surface in air-gap

The solution of the problem results from Eqns (1), (2). If for the region R_{out} is assumed, that: it is nonconductive, frequency is small (no displacement current), region is homogeneous, and there is no hysteresis phenomenon, hence only anisotropic component can give a contribution to residual integrals in Eqns (11a,b). In other words, only magnetic anisotropy of the gap do not affect on electromagnetic force/torque, because the anisotropy component can be different from zero if for u ≠ v it is

$$\nu_{vu\delta} = \nu_{uv\delta} \tag{12}$$

Now, it could be formulated *the second theorem*
The electromagnetic force/ torque value calculated with the help of
either volume integral or Maxwell stress tensor surface integral
does not depend on surface position in the gap if the condition (12) is satisfied.

4. Electromagnetic field

In order to confirm the theoretical results described above, an example of electromechanical converter with solid conductive rotor is considered Fig. 4.
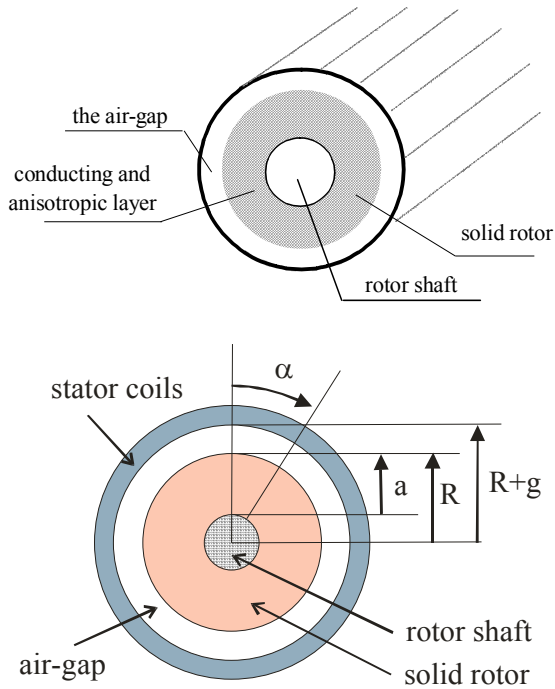
Fig. 4. Electromechanical converter with solid rotor - a), its cross section -b)

The numerical errors are omitted by applying analytical solution of electromagnetic field distribution and electromagnetic torque calculation.

The analysis of electromagnetic field has been provided with the help of variable separation method [8]. This way of analysis is chosen for giving precise insight into electromagnetic torque calculations, and may be treated as a benchmark test task [12, 13, 14]. For cylindrical converter (r, α, z) the z-component of magnetic vector potential is

$$\vec{A} = \vec{A}_z = A_z \vec{i}_z = A \vec{i}_z \tag{13}$$

The accuracy of such assumptions for magnetic vector potential symmetry results from simplified construction of mechanical converter - Fig. 4. The magnetic flux density by means of vector magnetic potential is as follows

$$\vec{B} = \frac{\vec{i}_r}{r} \frac{\partial A}{\partial \alpha} - \vec{i}_\alpha \frac{\partial A}{\partial r} \tag{14}$$

where for cylindrical coordinate system $(x_1 = r, x_2 = \alpha, x_3 = z)$. The model of induction motor with solid rotor can be analyzed under the following assumptions:

- electric displacement current vanishes (due to the small supply frequency),
- stator windings exert the sinusoidal p pair-pole mmf

$$\Theta_s(\alpha) = \Theta_s \cos(p\alpha - 2\pi ft + \alpha_o) \tag{15}$$

where Θ_s stands for the magnitude of mmf, α is the position angle, f means the stator current frequency, there exists the cylindrical anisotropy of the magnetic reluctivities for machine rotor, thus the reluctivity matrix is built of four coefficients $\nu_{rr} = \nu_r$, $\nu_{\alpha\alpha} = \nu_\alpha$, $\nu_{r\alpha}$, $\nu_{\alpha r}$, and all of them can be different, conductivity of rotor is γ (isotropic parameter), rotor is homogeneous, hysteresis phenomenon does not appear.

5. Electromagnetic torque calculation - example

The assumed symmetry of the electromechanical converter enables to carry twodimensional analysis. Magnetic field strength components are

$$H_r = \nu_{rr} B_r + \nu_{r\alpha} B_\alpha \tag{16}$$

$$H_\alpha = \nu_{\alpha r} B_r + \nu_{\alpha\alpha} B_\alpha \tag{17}$$

the electric field strength is of the form of

$$\vec{E} = -i\omega \vec{A} \tag{18}$$

where A is magnetic field vector potential for steady-state, ω rotor current angular speed, $\vec{E}$ means electric field strength. Basing on Maxwell's equation

$$\text{curl}(\vec{H}) = \gamma \vec{E} \tag{19}$$

the Eqns (16) - (19) lead to equation

$$\frac{\nu_\alpha}{r} \frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) - \frac{\nu_{r\alpha} + \nu_{\alpha r}}{r^2} \frac{\partial^2 A}{\partial r \partial \alpha} + \frac{\nu_r}{r^2} \frac{\partial^2 A}{\partial \alpha^2} = i\omega\gamma A \tag{20}$$

The Eqn (20) is solved by the separation of the variables in the following form

$$A = Z(r)S(\alpha) = ZS, \tag{21}$$

with the steady-state function $S(\alpha)$ for monoharmonic field in the form of:

$$S(\alpha) = \exp(-ip\alpha) \tag{22}$$

The variable separation for functions $Z(r)$ and $S(\alpha)$ leads to the differential equation

$$\frac{d^2 Z}{dr^2} + \frac{[1 - 2c]}{r}\frac{dZ}{dr} - \left[\frac{\nu_r p^2}{\nu_\alpha r^2} + \beta^2\right]Z = 0 \tag{23}$$

where $c = -i\dfrac{\nu_{r\alpha} + \nu_{\alpha r}}{2\nu_\alpha}$, $\quad p_B = \sqrt{c^2 + p^2 \nu_{rr}/\nu_{\alpha\alpha}}$

with solution given in Table 1.

Table 1. Solution of the differential equations

anisotropic rotor	the gap
$Z(z) = a_a f_1(z) + b_a f_2(z)$	$Z(z) = a_\delta f_1(z) + b_\delta f_2(z)$
$f_1(z) = z^c I_{p_B}(z)$	$f_1(z) = z^{c+p_B}$,
$f_2(z) = z^c K_{p_B}(z)$	$f_2(z) = z^{c-p_B}$

The four unknown constants a_a, b_a, a_δ, b_δ can be evaluated by formulating the boundary conditions gap. The magnetic flux density components are equal to

$$\begin{cases} B_{ar} = \dfrac{p}{r}\{a_a f_1(r) + b_a f_2(r)\}\exp(-ip\alpha - i\tfrac{\pi}{2}) \\ B_{a\alpha} = -\dfrac{d}{dr}\{a_a f_1(r) + b_a f_2(r)\}\exp(-ip\alpha) \end{cases} \quad (20)$$

The magnetic field distribution leads to electromagnetic torque values.

The electromagnetic torque T_e equals to

$$T_e = \int_{\partial V} r H_\alpha B_r dS \quad (24)$$

where l is rotor length, r is the radius of surface placed in the gap. The total electromagnetic torque can be calculated also by means coenergy W_c as follows

$$T_e = \left.\frac{\partial W_C}{\partial \alpha}\right|_{j=const} = \int_V \left(\vec{j}\frac{\partial \vec{A}}{\partial \alpha} + \vec{B}\frac{\partial \vec{H}}{\partial \alpha}\right) dV \quad (25)$$

The both Maxwell and coenergy methods give the same results always. The material (anisotropy) torque component is equal to

$$T_{eM} = \int_V \tfrac{1}{2}(\nu_{r\alpha} - \nu_{\alpha r}) B_r \frac{\partial B_\alpha}{r\partial \alpha} dV \quad (26)$$

and it does not vanish for asymmetrical anisotropy medium $\nu_{r\alpha} \neq \nu_{\alpha r}$.

Exemplary, it is considered cylindrical motor for it $\gamma = 50\cdot 10^6$ S/m (rotor conductivity), a = 0.05 m (conductive rotor layer width), R = 0.1 m (rotor outer radius), l = 0.25 m (rotor length), g = 0.0015 m (the gap width), Θ_1 = 560 A (magnetomotive force first harmonic), p = 2 (pair pole number), n = 20 s^{-1} (rotor speed), $\nu_{rr} = \nu_r = \nu_0/3$ (radial reluctivity), $\nu_{\alpha\alpha} = \nu_\alpha = \nu_0/2$ (tangential reluctivity) and different anisotropy reluctivities $\nu_{r\alpha}$, $\nu_{\alpha r}$ (Table 2).

The Table 2 and the Figs 5a,b confirm that if condition (4) is satisfied the first theorem thesis is fulfilled. However the cases c) and d) show that if condition (4) is not satisfied the first theorem thesis can not be applied.

Table 2. Examples for torques evaluation for first theorem presentation

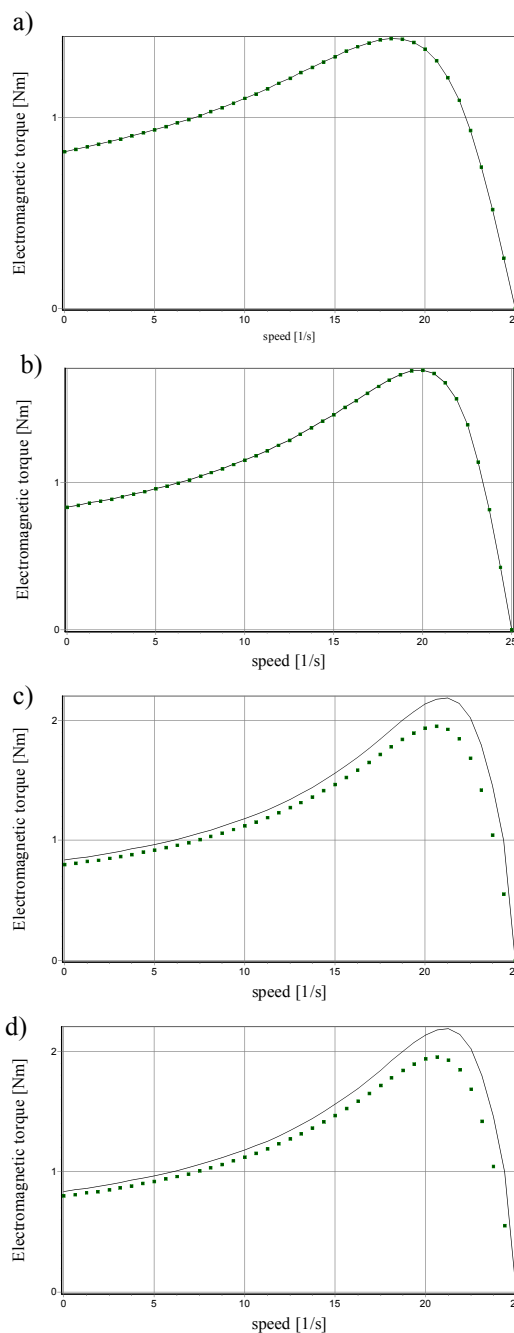|	magnetic reluctivities	First theorem
a)	$\nu_{r\alpha} = 0.0\nu_0$ $\nu_{\alpha r} = 0\nu_0$	Eqn (4) true - Fig 5a.
b)	$\nu_{r\alpha} = 0.2\nu_0$ $\nu_{\alpha r} = 0.2\nu_0$	Eqn (4) true - Fig 5b
c)	$\nu_{r\alpha} = 0.2\nu_0$ $\nu_{\alpha r} = 0.3\nu_0$	Eqn (4) false - Fig 5c
d)	$\nu_{r\alpha} = 0.3\nu_0$ $\nu_{\alpha r} = 0.2\nu_0$	Eqn (4) false - Fig 5d

Fig. 5. Electromagnetic torque vs. speed (Maxwell method - line, Lorentz method - points)

Examples for the second theorem are presented for the same data of cylindrical solid rotor motor and for introduced the anisotropic magnetically gap as specified in the Table 3.

Table 3. Examples for torques evaluation for second theorem presentation

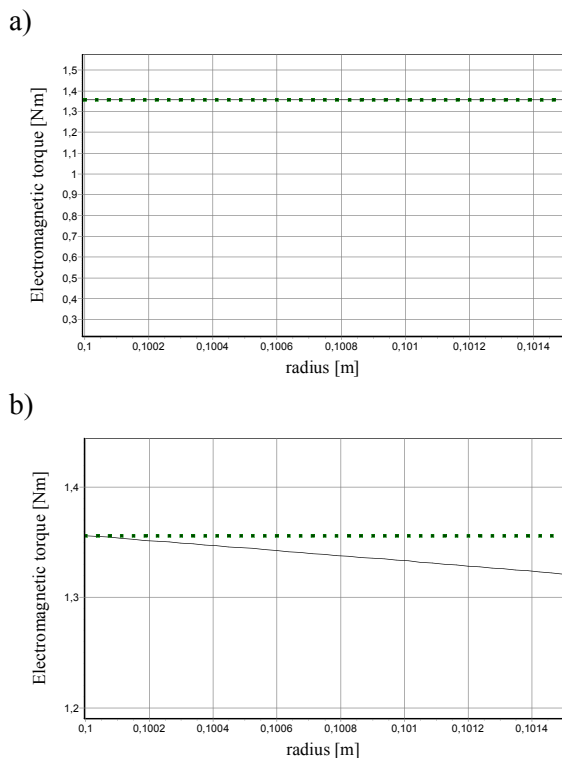|	magnetic reluctivities	First theorem
a)	$\nu_{r\alpha\delta} = 0.0\nu_0 =$ $\nu_{\alpha r\delta} = 0\nu_0$	Eqn (12) is satisfied - Fig 6a.
b)	$\nu_{r\alpha\delta} = 0.5\nu_0 =$ $\nu_{\alpha r\delta} = 0.0\nu_0$	Eqn (12) is NOT satisfied - Fig 6b

a)

b)

Fig. 6. Electromagnetic torque vs. radius for the gap region (Maxwell method - line, Lorentz method - points)

The Figs 6a,b present the electromagnetic torque value obtained by Maxwell's method for different radius of integration surface (cylindrical surface). The radius changes from r = R (outer solid rotor surface) up to R+g (inner stator surface). The Table 3 and the Fig. 6a confirm that if condition (12) is satisfied the second theorem thesis is fulfilled. In opposite, in the case b) is shown that if condition (12) is not satisfied the second theorem thesis can not be applied.

6. Conclusions

There have been presented two theorems stated in the main text proved by author in 1997 and 2007 about electromagnetic force/torque in anisotropic medium.

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