Dariusz Spałek
Politechnika Śląska, Gliwice

SYNCHRONOUS TORQUE OF INDUCTION MOTOR WITH SOLID AND MAGNETICALLY ANISOTROPIC ROTOR

MOMENT SYNCHRONICZNY SILNIKA INDUKCYJNEGO O LITYM I MAGNETYCZNIE ANIZOTROPOWYM WIRNIKU

Abstract: The paper deals with problem of synchronous state of work for induction motor with solid rotor. The have been considered different kinds of magnetic anisotropy for the solid rotor. The paper presents analytical solution for synchronous state of work for induction motor. The electromagnetic field distribution is evaluated with the help of separation variable method. The electromagnetic torque is calculated by means of Maxwell stress tensor method, material component and with the help of coenergy function for different rotor magnetic anisotropy cases at synchronous state of work. It has been pointed out that for magnetic rotor may appear electromagnetic torque for certain case of rotor anisotropy.

1. Introduction

The paper deals with synchronous state of work for induction motor with solid rotor which is magnetically anisotropic. The analyzes have been carried basing on both physical and analytical way of phenomena description. The development of material technology today leads to electrical machine configuration in which magnetic anisotropy cannot be neglected. The electromagnetic torque for induction motor at synchronous state of work can be developed if rotor is made of magnetic materials with asymmetrical reluctivity - the so-called active or forced anisotropy.

2. Physical phenomenon

The magnetic features of rotor influence significantly on the electromagnetic torque developed by electrical machine. The features are important especially for either synchronous motor either induction motor at synchronous state of work. The synchronous work of a machine lasts continuously if the rotor magnetic field is exerted if either
- coils are supplied, or
- rotor is shaped, or
- are used permanent magnets.

The third method can be described mathematically by constitutive relation for rotor in the general form of

$$\begin{bmatrix} H_r \\ H_\alpha \end{bmatrix} = \begin{bmatrix} \nu_{rr} & \nu_{r\alpha} \\ \nu_{\alpha r} & \nu_{\alpha\alpha} \end{bmatrix} \begin{bmatrix} B_r \\ B_\alpha \end{bmatrix} - \begin{bmatrix} \Delta I_r \\ \Delta I_\alpha \end{bmatrix}. \quad (1)$$

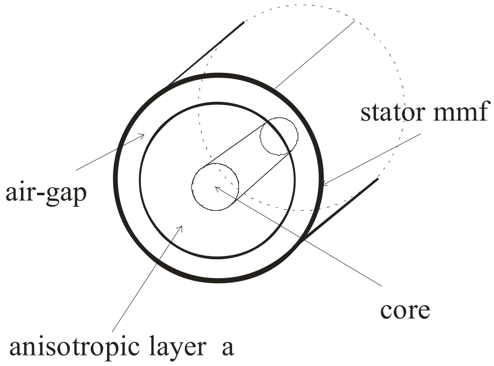

Fig. 1. Induction motor with solid rotor

The above written equation is denoted for two-dimensional cylindrical co-ordinate system that is most appropriate for cylindrical electrical machines. ΔI_r, ΔI_α denote components of residual magnetization vector. The reluctivity matrix can be **asymmetrical** one for the so-called active material.
The relation (1) is mathematically equivalent to the well-known physical one [1]

$$\begin{bmatrix} H_r \\ H_\alpha \end{bmatrix} = \begin{bmatrix} \nu_0 & 0 \\ 0 & \nu_0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\alpha \end{bmatrix} - \begin{bmatrix} I_r \\ I_\alpha \end{bmatrix}. \quad (2)$$

Let us consider magnetic region (non-magnetized permanently) for which reluctivity matrix is **asymmetrical** *($\nu_{\alpha r} \neq \nu_{r\alpha}$)* in the form of

$$\begin{bmatrix} H_r \\ H_\alpha \end{bmatrix} = \begin{bmatrix} \nu_{rr} & \nu_{r\alpha} \\ \nu_{\alpha r} & \nu_{\alpha\alpha} \end{bmatrix} \begin{bmatrix} B_r \\ B_\alpha \end{bmatrix}. \quad (3)$$

The Eqn 3 can be rewritten in the form of

$$\begin{bmatrix} H_r \\ H_\alpha \end{bmatrix} = \begin{bmatrix} \nu_{rr} & \nu_{r\alpha} \\ \nu_{r\alpha} & \nu_{\alpha\alpha} \end{bmatrix} \begin{bmatrix} B_r \\ B_\alpha \end{bmatrix} + \begin{bmatrix} 0 \\ (\nu_{\alpha r} - \nu_{r\alpha})B_r \end{bmatrix} \quad (4)$$

which contains the **symmetrical** reluctivity matrix and non-zero residual vector. The Eqns (3) and (4) enable one to state that

the region with asymmetrical reluctivity matrix (or permeability matrix) is equivalent to active region with symmetrical reluctivity matrix.

Hence it can be deduced that the permanent magnets placed on the electrical motor rotor may be equivalent to magnetic material with asymmetrical reluctivity matrix. However, there is one difference: permanent magnets are autonomous energy sources in spite of anisotropic region that is rather like controlled source of magnetic energy (controlled by B_r).

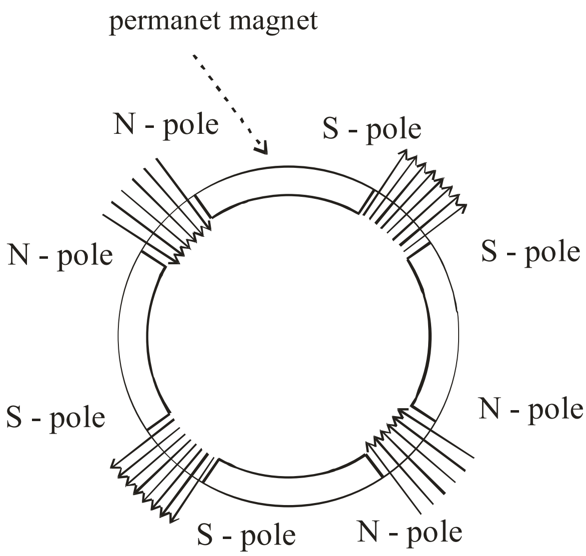

Fig. 2. Electrical machine with permanent magnets

3. Solid rotor induction motor model

The model of induction motor with solid rotor can be analysed under the following assumptions:

- electric displacement current vanishes (due to the small supply frequency),
- stator windings exert the sinusoidal *p* pair-pole mmf

$$\Theta_s(\alpha) = \Theta_s\cos(p\alpha\text{-}2\pi ft+\alpha_o), \quad (5)$$

where Θ_s stands for the magnitude of mmf, α is the position angle, f means the stator current frequency,

- there exists the cylindrical anisotropy of the magnetic reluctivities for machine rotor, thus the reluctivity matrix is built of four coefficients

$$\nu_{rr} = \nu_r \qquad \nu_{\alpha\alpha} = \nu_\alpha \qquad \nu_{r\alpha} \qquad \nu_{\alpha r}$$

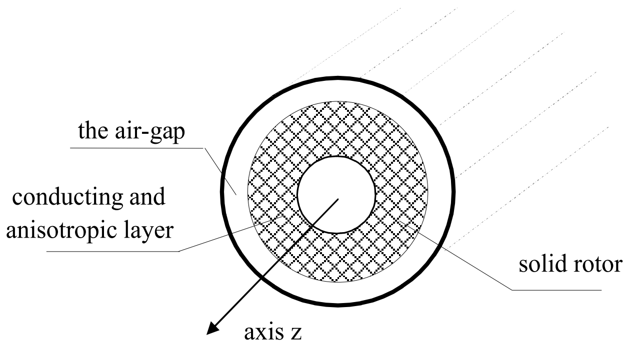

Fig. 3. Electrical machine with solid rotor

and all of them may differ,

- conductivity of machine rotor is γ (isotropic parameter),
- rotor is homogeneous,
- hysteresis phenomenon can be neglect,
- the air-gap reluctivity is equal to the vacuum reluctivity ν_o,
- Maxwell's stress tensor components of electric field can be omitted in comparison to the magnetic components

$$D_r E_r << H_r B_r, \ldots\ldots \quad (6)$$

For the model described above the electromagnetic field distribution can be calculated in analytical way.

4. Electromagnetic field distribution

The assumed symmetry of the both motor and source enable to carry two-dimensional analysis. The magnetic flux density in terms of an axial component of magnetic potential A_z (z axis is rotor axis) can be presented as follows

$$\vec{B} = \vec{i}_r \frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \vec{i}_\alpha \frac{\partial A_z}{\partial r}. \quad (7)$$

The magnetic field strength components for the anisotropic region can be shown in the general form of

$$H_r = \nu_{rr}B_r + \nu_{r\alpha}B_\alpha, \quad (8)$$

$$H_\alpha = \nu_{\alpha r}B_r + \nu_{\alpha\alpha}B_\alpha, \quad (9)$$

the electric field strength is of the form of

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}. \quad (10)$$

where A = magnetic field vector potential, $\vec{E}$ means electric field strength.
Combining the Maxwell's equation

$$\mathrm{curl}(\vec{H}) = \gamma\vec{E}\,, \qquad (11)$$

with Eqns (8), (9), (10) in cylindrical co-ordinates system leads to differential equation:

$$\frac{\nu_\alpha}{r}\frac{\partial}{\partial r}(r\frac{\partial A_z}{\partial r}) - \frac{\nu_{r\alpha}+\nu_{\alpha r}}{r}\frac{\partial^2 A_z}{\partial r\partial\alpha} + \frac{\nu_r}{r^2}\frac{\partial^2 A_z}{\partial\alpha^2} = \gamma\frac{\partial A_z}{\partial t} \qquad (12)$$

The partial time derivative of A_z as a multiplication of the operand iω and the complex magne-tic potential A at the steady state could be represented as follows

$$\frac{\partial A}{\partial t} \;\Rightarrow\; i\omega A\,, \qquad (13)$$

where *ω = 0* for rotor currents at synchronous state of work.
Eqn (13) leads either to the differential equation for the rotor:

$$\frac{\nu_\alpha}{r}\frac{\partial}{\partial r}(r\frac{\partial A}{\partial r}) - \frac{\nu_{r\alpha}+\nu_{\alpha r}}{r^2}\frac{\partial^2 A}{\partial r\partial\alpha} + \frac{\nu_r}{r^2}\frac{\partial^2 A}{\partial\alpha^2} = 0\,, \qquad (14)$$

and to Laplace's equation for the air-gap:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial A}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 A}{\partial\alpha^2} = 0\,. \qquad (15)$$

Both Eqns (14) and (15) will be solved by the separation of the variables in the following form

$$A = Z(r)S(\alpha) = ZS. \qquad (16)$$

At the steady-state function S(α) has the given below form:

$$S(\alpha) = \exp(-ip\alpha)\,. \qquad (17)$$

Rearranging the Eqn (14) with respect to the given above separation scheme for the functions Z(r) and S(α) gives the differential equation for the layer

$$\frac{d^2 Z}{dr^2} + \frac{[1-2c]}{r}\frac{dZ}{dr} - \left[\frac{\nu_r p^2}{\nu_\alpha r^2} + \beta^2\right]Z = 0\,, \qquad (18)$$

where $c = -i\frac{\nu_{r\alpha}+\nu_{\alpha r}}{2\nu_\alpha}$, $p_B = \sqrt{c^2 + p^2\nu_{rr}/\nu_{\alpha\alpha}}$.

Analogously, for the air-gap the Eqn (15) has the form given below:

$$\frac{r}{Z}\frac{d}{dr}(r\frac{dZ}{dr}) = p^2\,. \qquad (19)$$

The solutions of the Eqns (18) and (19) are presented in Table.1.

Table 1. Solution of the differential equations

Region	anisotropic rotor	air-gap
Solutions	$Z(z) = a_a f_1(z) + b_a f_2(z)$ $f_1(z) = z^{c+p_B}$ $f_2(z) = z^{c-p_B}$	$Z(r) = a_\delta r^p + b_\delta r^{-p}$
constans	a_a, b_a	a_δ, b_δ

The four unknown constants a_a, b_a, a_δ, b_δ can be evaluated by formulating the boundary conditions. The values of these constants are presented in the Table 2. The constant values are presented in Table 3, and the dimensions are shown in Fig. 4.

Table 2. Boundary conditions for magnetic field

	Boundary condition
Stator mmf	$\nu_o B_{\delta\alpha} = -\frac{1}{R_s}\frac{\partial\Theta}{\partial\alpha}$
Rotor outer surface r = R	$B_{\delta r} = B_{ar}$ $\nu_o B_{\delta\alpha} = \nu_\alpha B_{a\alpha} + \nu_{\alpha r} B_{ar}$,
Inner layer surface r = R-a = R_a	$\nu_\alpha B_\alpha + \nu_{\alpha r} B_r = 0$

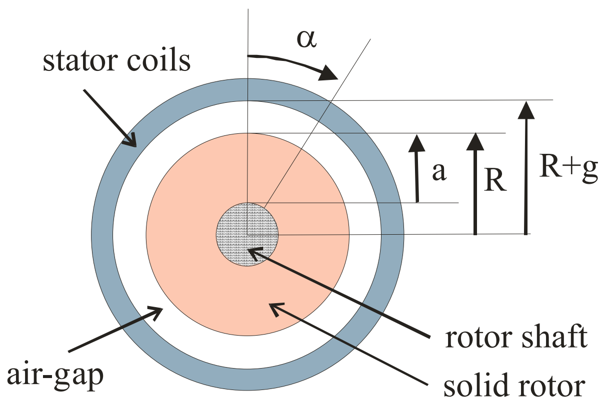

Fig. 4. Dimensions for induction motor

Basing on the magnetic field potential distribution both the magnetic flux density components and the electromagnetic torque components can be evaluated analytically. For the anisotropic rotor the magnetic flux density components are equal to (the index a indicates the solution for the rotor anisotropic region)

$$\begin{cases} B_{ar} = \frac{p}{r}\{a_a f_1(r) + b_a f_2(r)\}\exp(-ip\alpha - i\frac{\pi}{2}) \\ B_{a\alpha} = -\frac{d}{dr}\{a_a f_1(r) + b_a f_2(r)\}\exp(-ip\alpha) \end{cases} \quad (20)$$

where $p_B = p\sqrt{\frac{\nu_r}{\nu_\alpha} + c^2}$.

For the air-gap the magnetic flux density components are equal to (index δ indicates the solution in the air-gap):

$$\begin{cases} B_{\delta r} = \frac{p}{r}\{a_\delta r^p + b_\delta r^{-p}\}\exp(-ip\alpha - i\frac{\pi}{2}) \\ B_{\delta\alpha} = -p\{a_\delta r^{p-1} - b_\delta r^{-p-1}\}\exp(-ip\alpha) \end{cases} . \quad (21)$$

Knowing the magnetic field distribution the electromagnetic torque can be evaluated.

Table 3. The constants for the magnetic field solutions

Constants for solutions of Eqns (14), (15)
$a_a = \Theta\nu_0^{-1}\{UR_s^{p} - WR_s^{-p}\}^{-1}$, $U = 0.5(R^{-p+1}P + R^{-p}Q)$, $W = 0.5(-R^{p+1}P + R^{p}Q)$.
$b_a = -a_a S$
$a_\delta = a_a U$
$b_\delta = a_a W$
$P = \frac{\beta\nu_\alpha}{p\nu_o}\{f'_1(R) - Sf'_2(R)\} + \frac{\nu_{\alpha r}Qw}{p\nu_o}$, $Q = f_1(R) - Sf_2(R)$, $S = \frac{\nu_a f_1'(R_a) + \nu_{ar} f_1(R_a) w_a}{\nu_a f'_2(R_a) + \nu_{ar} f_2(R_a) w_a}$.
$Z_\delta(r) = a_\delta r^p + b_\delta r^{-p}$, $Z_a(r) = a_a f_1(r) + b_a f_2(r)$

5. Electromagnetic torque

Basing on the magnetic field distribution electromagnetic torque has been calculated by Maxwell's stress tensor method [1, 2, 3] T_e, material component [6, 7] T_{eM} and coenergy function [3, 4]. For the motor at synchronous speed the Lorentz forces do not appear, thus

$$T_{eL} = 0 , \quad (22)$$

thus

$$T_e = T_{eL} + T_{eM} = T_{eM} . \quad (23)$$

The electromagnetic torque T_e equals to

$$T_e = \nu_o \int_{\partial V} r B_r B_\alpha dS , \quad (24)$$

where l is rotor length, r is the radius of surface situated in the air-gap.
The material/anisotropy torque is equal to

$$T_{eM} = \int_V \tfrac{1}{2}(\nu_{r\alpha} - \nu_{\alpha r}) B_r \frac{\partial B_\alpha}{r\partial\alpha} dV . \quad (25)$$

and disappears if the region is either isotropic or anisotropic while $\nu_{r\alpha} = \nu_{\alpha r}$.

The total electromagnetic torque can be calculated also by means coenergy W_c as follows

$$T_e = \left.\frac{\partial W_C}{\partial\alpha}\right|_{j=const} = \int_V \left(\vec{j}\frac{\partial\vec{A}}{\partial\alpha} + \vec{B}\frac{\partial\vec{H}}{\partial\alpha}\right) dV . \quad (26)$$

The both Maxwell and coenergy methods give the same results always.
Exemplary, for model of cylindrical motor data $\gamma = 56\cdot10^6$ S/m (rotor conductivity), a = 0.07 m (conductive rotor layer width), R = 0.1 m (rotor outer radius), l = 0.25 m (rotor length), g = 0.0015m (air-gap width), Θ_1 = 750 A (magnetomotive force first harmonic), p = 2 (pair pole number), s = jω = 0 (for rotor currents at synchronous state of work) $\nu_{rr} = \nu_r = \nu_o/30$ (radial reluctivity), $\nu_{\alpha\alpha} = \nu_\alpha = \nu_o/35$ (tangential reluctivity) and different anisotropy reluctivities $\nu_{r\alpha} = (0, \nu_r)$, $\nu_{\alpha r} = 0.5\ \nu_r$ torques values have been obtained with all three method as shown in Fig 5.
As shown in Fig 5 - for induction motor with round solid rotor which is magnetically anisotropic - appears electromagnetic torque

$$T_e \neq 0$$

that enable synchronous state of work if

$$\nu_{r\alpha} \neq \nu_{\alpha r} .$$

The electromagnetic torque does not appear at synchronous state of work for rotor that is anisotropic if

$$\nu_{r\alpha} = \nu_{\alpha r} \neq 0$$

the so called normal anisotropy (e.g. crystal structure of the material).
It should be stated that the practical importance of the effect seems to be nowadays slightly. However, a persistent progress of the material

technology may arrange in the future the possibility for its practical use.

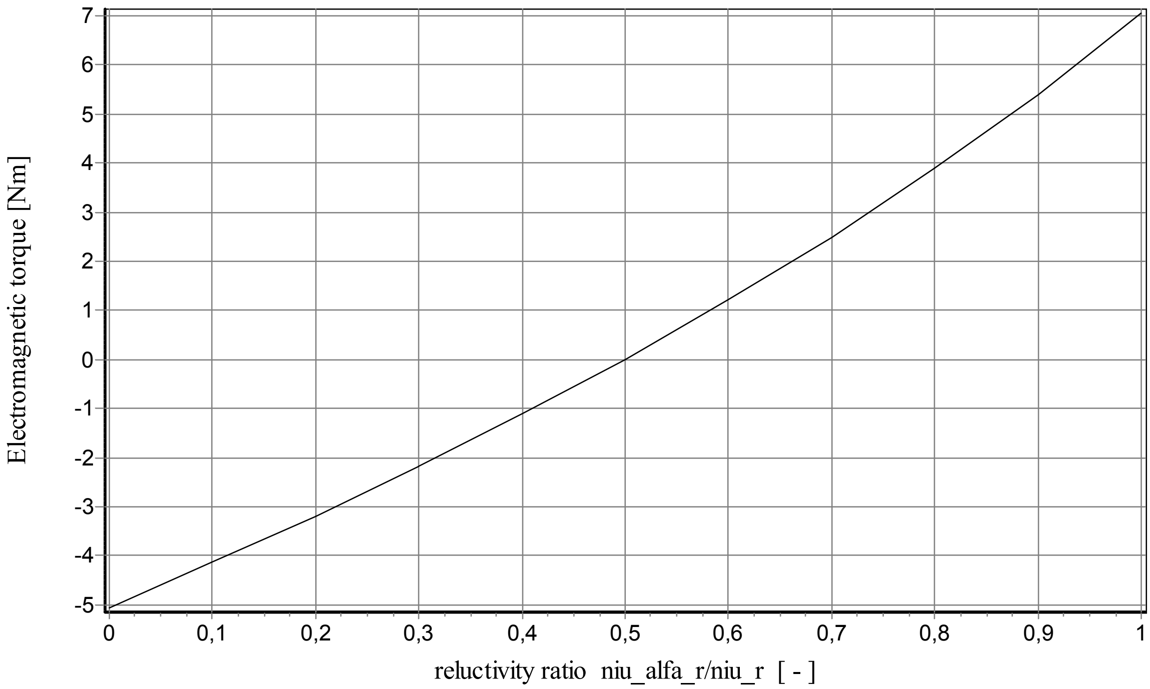

Fig. 5. Electromagnetic torque at synchronous state of work vs. ratio $\nu_{\alpha r}/\nu_r$

6. Conclusions

The analyses have brought out to the issue, as follows

The electromagnetic torque can be exerted not only by inhomogeneous rotor and magnets (hysteresis phenomenon), but also by magnetic anisotropy features of motor rotor.

For the presented cylindrical induction motor with solid anisotropic rotor the electromagnetic torque has been calculated by means of:

- Maxwell's stress tensor method
- material component,
- coenergy function.

All three methods give the same results.

7. References

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Author

Dr hab. inż. Dariusz Spałek, prof. PŚl.
e-mail: Dariusz.Spalek@polsl.pl
Politechnika Śląska,
Wydział Elektryczny
Instytut Elektrotechniki Przemysłowej i Informatyki
ul. Akademicka 10, 44-100 Gliwice