Paper

Geometrical representation of a monochromatic electromagnetic wave using the tangential vector approach

Laura Carrea and Gerd Wanielik

Abstract — The aim of this work is to develop a coherent polarimetric model and to find a geometrical description of a monochromatic wave. The spinor form of the electrical field, its links to the coherency matrix and the Poincare' sphere are introduced with the aim to obtain a geometrical representation of the spinor. It consists, from the "polarization point of view", on the polarization vector and a tangential plane to the Poincare' sphere where it is possible to visualize the zero phase.

Keywords — polarimetric, coherent model, Poincare' sphere.

1. Introduction

Pulse radar has a very narrow band, so, to describe the state of the signal, it is possible to consider one single pulse like a monochromatic electromagnetic wave, which is completely polarized [1, 2]. A very useful representation of the electrical field is its spinor form which contains the complete information even the zero phase[1]. The aim of this work is to develop a coherent polarimetric description which has a geometrical representation.

2. Spinors and quadrivectors – the coherency matrix

The two-component complex field of the Jones representation may be treated as a spinor η^A:

$$\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} = \begin{pmatrix}E_x\\ E_y\end{pmatrix} = \begin{pmatrix}a_x e^{i\delta_x}\\ a_y e^{i\delta_y}\end{pmatrix}, \tag{1}$$

where a_x, a_y are the amplitudes and δ_x, δ_y are the phases of the phasor representation of a RF signal.
A quadrivector $x^\mu = (x^0, x^1, x^2, x^3)$ may be regarded as a Hermitian second-rank spinor. The spin matrix X [3]:

$$X = x^0 + (\vec{x}\cdot\vec{\sigma}) = \left\| \begin{matrix} x^0+x^4 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^4 \end{matrix} \right\| = \\ = \left\| \begin{matrix} X^{1\dot{1}} & X^{1\dot{2}} \\ X^{2\dot{1}} & X^{2\dot{2}} \end{matrix} \right\| \tag{2}$$

is transformed like a second rank spinor namely the coefficients in the law for the transformation of the components of the spin matrix $X^{A\dot{V}}$ are identical with the coefficient in the law for the transformation of the second rank spinor $\chi^A \xi^{\dot{V}}$ (the dots are used for the conjugate complex, not transpose). In more compact form:

$$X^{A\dot{V}} = \left[x^0 + (\vec{x}\cdot\vec{\sigma})\right]^{A\dot{V}} = x^\mu \sigma_\mu^{A\dot{V}} \quad (\mu = 0,\, 1,\, 2,\, 3), \tag{3}$$

where σ_0 is the unit matrix and σ_1, σ_2, σ_3 are the Pauli matrices:

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3 \tag{4}$$

and cyclic permutations. In this way a geometric representation of the spinor η^A which the spinor form of the Jones vector, is possible. Then, if $X^{A\dot{V}}$ is calculated as

$$X^{A\dot{V}} = \eta^A \left(\overline{\eta}\right)^{\dot{V}} \tag{5}$$

it results: $X^{1\dot{1}} = E_x E_x^*$, $X^{1\dot{2}} = E_x E_y^*$, $X^{2\dot{1}} = E_y E_x^*$, $X^{2\dot{2}} = E_y E_y^*$, which are the components of the coherency matrix J [4] (where E_i^* is the conjugate complex of the complex number E_i).
The correspondent 4-vector x^μ is obtained from the Eq. (3) and from Eq. (5):

$$\left\| \begin{matrix} x^0+x^1 & x^2-ix^3 \\ x^2+ix^3 & x^0-x^1 \end{matrix} \right\| = \left\| \begin{matrix} \eta^1\overline{\eta}^{\dot{1}} & \eta^1\overline{\eta}^{\dot{2}} \\ \eta^2\overline{\eta}^{\dot{1}} & \eta^2\overline{\eta}^{\dot{2}} \end{matrix} \right\|, \tag{6}$$

where the cyclic permutation: $\sigma_1 \to \sigma_2$, $\sigma_2 \to \sigma_3$, $\sigma_3 \to \sigma_1$ is considered. Substituting the components of the Jones vector, the components of the Stokes vector are found:

$$x^0 = \frac{1}{2}g^0, \quad x^1 = \frac{1}{2}g^1, \quad x^2 = \frac{1}{2}g^2, \quad x^3 = \frac{1}{2}g^3. \tag{7}$$

For a monochromatic wave, (g^0, g^1, g^2, g^3) is a real null 4-vector

$$(g^0)^2 - (g^1)^2 - (g^2)^2 - (g^3)^2 = 0 \Rightarrow (x^0)^2 - (x^1)^2 + \\ -(x^2)^2 - (x^3)^2 = 0. \tag{8}$$

All the directions of the 4-vectors x^μ in the Minkowski space-time for which the components satisfy (8) are null directions and they build the null cone [5]. The space of the null directions can be represented in the Euclidean space by the intersections of the null cone with the hyper planes $x^0 = const$ and so $g^0 = const$ (with the same intensity of the electrical field, because $g^0 = I$). If the $const = \pm 1$, the intersection is a sphere which can be regarded as a Riemann sphere of an Argand plane, which is the Poincare'

[1] D. H. O. Bebbington, "Analytical foundations of polarimetry: I" – to be published.

sphere. But in general for any value of the constant, unless $g^0 = 0$, we get from the relation (8):

$$\left(\frac{g^1}{g^0}\right)^2 + \left(\frac{g^2}{g^0}\right)^2 + \left(\frac{g^3}{g^0}\right)^2 = 1 \qquad (9)$$

and we can define

$$p^1 = \frac{g^1}{g^0}, \quad p^2 = \frac{g^2}{g^0}, \quad p^3 = \frac{g^3}{g^0} \qquad (10)$$

which are the components of the polarization vector. The equation of the Poincare' sphere is in general:

$$(p^1)^2 + (p^2)^2 + (p^3)^2 = 1. \qquad (11)$$

The exterior of the sphere represents space-like directions namely unpolarized or partially polarized light.
Multiplying the spinor η^A by a complex number $p = \lambda e^{i\theta}$ (λ and θ real) the 4-vector x^μ is stretched of λ^2 but is unchanged in direction (cfr. (5)), namely it is independent from the choice of the angle θ. The 4-vector is uniquely defined by the spinor but to a 4-vector correspond a lot of spinors, which differ by the multiplicative factor $e^{i\theta}$.
On the other side we want find a coherent description of a monochromatic wave, which contains the so-called "zero phase" $\alpha = \delta_x$ $(0 < \alpha < 2\pi)$. In order to do this, we look at the spinor in its polarization vector form [6]:

$$\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} = \sqrt{\frac{I}{2}}\, e^{i\alpha}\begin{pmatrix}(1+p^1)^{1/2}\\ (1+p^1)^{-1/2}(p^2+ip^3)\end{pmatrix}. \qquad (12)$$

This form of the spinor contains explicitily the zero phase and, as we have stated below, the corresponding 4-vector (g^0, g^1, g^2, g^3), is unaffected by the choice of the angle α.

3. The tangential plane and the angle α

Let us consider the spinor mate [3] ξ^B of η^A:

$$\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} = \sqrt{\frac{I}{2}}\, e^{-i\alpha}\begin{pmatrix}-(1+p^1)^{-1/2}(p^2-ip^3)\\ (1+p^1)^{1/2}\end{pmatrix}. \qquad (13)$$

The spinor and the spinor mate so defined satisfy the condition:

$$\eta_A \xi^A = I. \qquad (14)$$

They build a basis normalized to I and if we consider $I = 1$ the two spinor build a basis normalized to 1. The spinor and the spinor mate are linked by the equations:

$$\eta^A\xi^B - \xi^A\eta^B = \varepsilon^{AB} \quad (A, B = 1, 2), \qquad (15)$$

where ε^{AB} is an antisymmetric symbol such that: $\varepsilon^{12} = \varepsilon_{12} = 1$, $\varepsilon^{AB} = -\varepsilon^{BA}$. The spinor and the spinor mate constitute a spinor basis.

As we have stated below that $X^{A\dot{V}}$ is transformed like a second rank spinor $\chi^A\overline{\xi}^{\dot{V}}$, we can calculate the component of $Q^{A\dot{V}}$ using the spinor mate:

$$Q^{A\dot{V}} = \eta^A\left(\overline{\xi}\right)^{\dot{V}}. \qquad (16)$$

Now multiplying the spinor η^A by a complex number $\rho = \lambda e^{i\theta}$, the vector is stretched but also it depends on the choice of the angle θ and in particular it depends on 2θ. The calculation of $Q^{A\dot{V}}$ gives:

$$Q^{1\dot{1}} = -\tfrac{I}{2}e^{2i\alpha}(p^2+ip^3), \qquad Q^{1\dot{2}} = \tfrac{I}{2}e^{2i\alpha}(1+p^1),$$

$$Q^{2\dot{1}} = -\tfrac{I}{2}e^{2i\alpha}(1+p^1)^{-1}(p^2+ip^3)^2,\ Q^{2\dot{2}} = \tfrac{I}{2}e^{2i\alpha}(p^2+ip^3) \qquad (17)$$

which corresponds to a complex point. Infact, by the Eq. (3) the components of the corresponding 4-vector q^μ are:

$$q^0 = 0,$$
$$q^1 = -\frac{I}{2}e^{2i\alpha}(p^2+ip^3),$$
$$q^2 = \frac{I}{4}e^{2i\alpha}\,\frac{(1+p^1)^2-(p^2+ip^3)^2}{1+p^1},$$
$$q^3 = -\frac{I}{4i}\,\frac{(p^2+ip^3)-(1+p^1)^2}{1+p^1}. \qquad (18)$$

If the real and imaginary parts are separated, the two real 4-vectors have components $q^\mu_R = (0, \vec{q}_R)$ and $q^\mu_I = (0, \vec{q}_I)$ which are:

$$q^0_R = 0,$$
$$q^1_R = I(-p^2\cos 2\alpha + p^3\sin 2\alpha),$$
$$q^2_R = I\left(\frac{p^1(1+p^1)+(p^3)^2}{1+p^1}\cos 2\alpha + \frac{p^2p^3}{1+p^1}\sin 2\alpha\right),$$
$$q^3_R = I\left(-\frac{p^2p^3}{1+p^1}\cos 2\alpha - \frac{p^1(1+p^1)+(p^2)^2}{1+p^1}\sin 2\alpha\right). \qquad (19)$$

$$q^0_I = 0,$$
$$q^1_I = I(-p^2\sin 2\alpha - p^3\cos 2\alpha),$$
$$q^2_I = I\left(\frac{p^1(1+p^1)+(p^3)^2}{1+p^1}\sin 2\alpha - \frac{p^2p^3}{1+p^1}\cos 2\alpha\right),$$
$$q^3_I = I\left(-\frac{p^2p^3}{1+p^1}\sin 2\alpha + \frac{p^1(1+p^1)+(p^2)^2}{1+p^1}\cos 2\alpha\right). \qquad (20)$$

The 4-vector q^μ is space-like and in particular of magnitude equal to I. The 4-vector $p^\mu(1, p^1, p^2, p^3)$, $q^\mu_R(0, \vec{q}_R)$, $q^\mu_1(0, \vec{q}_I)$ are orthogonal in the sense:

$$p^\mu(q_R)_\mu = 0, \quad p^\mu(q_I)_\mu = 0, \quad (q_I)^\mu(q_R)_\mu = 0. \qquad (21)$$

And it is easy to see that even $\vec{g} = (g^1, g^2, g^3)$, $\vec{q}_R = (q_R^1, q_R^2, q_R^3)$ and $\vec{q}_I = (q_I^1, q_I^2, q_I^3)$ are orthogonal and of modul equal to I in the Euclidean space. So the vectors $\vec{q}_R$ and $\vec{q}_I$ provide basis vectors ($I = 1$) in the two-dimensional space which is the tangential plane at the point $\vec{p}$ on the Poincare' sphere. When the angle α varies, the vectors $\vec{q}_R$ and $\vec{q}_I$ rotate in the tangential plane.

The aim is now to visualize the angle α and to find a reference for $\alpha = 0$. For the horizontal polarization $\vec{p}_H = (1, 0, 0)$ and for $\alpha = 0$, $\vec{q}_R$ is the tangential vector to the equatorial great circle. If α increases, $\vec{q}_R$ rotates in the tangential plane clockwise through an angle of 2α. Keeping $\alpha = 0$, the fact that the point $\vec{p}_H$ moves into the point $\vec{p}$ corresponds to a rotation applied to the spinor η^A. This means a change of the basis, which means different $\vec{q}_{R(\alpha=0)}$ and $\vec{q}_{I(\alpha=0)}$. The rotation matrix, which preserves the angle α and which moves the point $\vec{p}_H$ to the point $\vec{p}$ is:

$$R = \left(\frac{1}{1+|\rho|^2}\right)^{-1/2} \begin{pmatrix} 1 & -|\rho| e^{-i\delta} \\ |\rho| e^{i\delta} & 1 \end{pmatrix}, \qquad (22)$$

where $\rho = \frac{E_y}{E_x} = |\rho| e^{i\delta}$ ($\delta = \delta_y - \delta_x$, cfr. (1)) is the polarization ratio. This is a rotation around the axis $\vec{n}(0, -\sin\delta, \cos\delta)$ through an angle such that $\cos\theta = \frac{1-|\rho|^2}{1+|\rho|^2} = p^1$. The rotation (22) preserves the angle between the directions but not the direction, so the vector $\vec{q}_{R(\alpha=0)}$ changes its direction. The direction r (cfr. Fig. 1), obtained by the intersection of the great circle through $\vec{p}_H$ and $\vec{p}$, forms with the vector $\vec{q}_{R(\alpha=0)}$ an angle δ and with the vector $\vec{q}_R$ the angle $2\alpha + \delta$. It is very important to find a reference for $\alpha = 0$ because $\vec{q}_R$ forms an angle δ with the direction r but δ is different for every point on the sphere. To solve this problem, let us consider $\vec{p}$ and $\vec{q}_R$ and $\vec{q}_I$ for any α, consider the correspondent spinor, apply the rotation which preserves the angle α and move the vector $\vec{p}$ in the point $\vec{p}_H$ to obtain the vector $\vec{q}_{RH}$ and the angle 2α is the angle between $\vec{q}_{RH}$ and $\vec{q}_{RH(\alpha=0)}$.

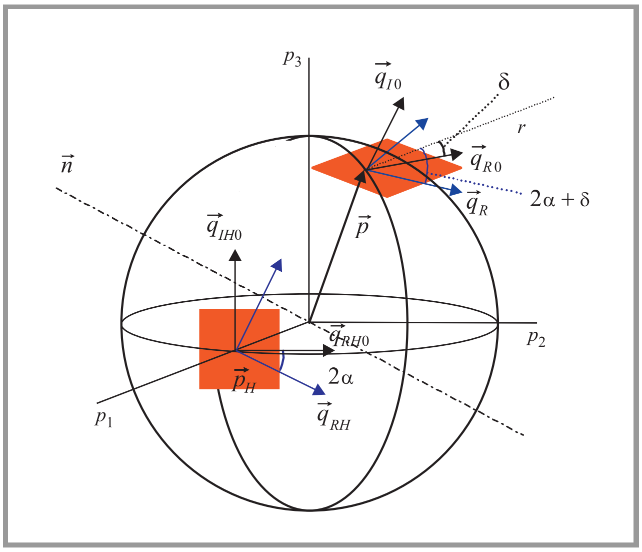

Fig. 1. The Poincare' sphere and the tangential planes in the point $\vec{p}_H$ and in the point $\vec{p}$.

The spinor and the spinor mate constitute a spinor basis. It is easy to see that the correspondent 4-vectors (cfr. Eq. (6)) fix on the Poincare' sphere two antipodal points $((p^1, P^2, p^3)$ and $(-p^1, -p^2, -p^3))$ which are the basis states of polarization [7]. If the corresponding $\vec{q}_R$ and $\vec{q}_I$ vectors are calculated, the result is:

$$\vec{p} \to \vec{q}_R, \vec{q}_I \qquad -\vec{p} \to -\vec{q}_R, \vec{q}_I\,. \qquad (23)$$

With the help of the spinor, the change of basis is easy because it corresponds to a unitary transformation of the spinor which corresponds to a rotation in the three dimensional space. Infact the group of two-dimensional special unitary transformations (with unit determinants), which preserve the invariants, are homomorphic to the three-dimensional rotation group [5]. The general form of the spin rotation matrix is:

$$R = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3)\,, \qquad (24)$$

where θ is the angle of rotation, (n_1, n_2, n_3) are the components of the axis $\vec{n}$ of rotation in the Euclidean space and σ_i are the Pauli matrices. The transformation law of a spinor is:

$$\eta \to \eta' = R\eta\,. \qquad (25)$$

It is possible to show that the rotation spin matrix is a unitary matrix and its determinant is necessarily unity.

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Laura Carrea received the "Laurea" degree in physics from the University of Turin, Turin, Italy in 1997. She is currently pursuing the Ph.D. degree at the Faculty of Electrical Engineering and Information Technology at the Chemnitz University of Technology, Chemnitz, Germany. She is supported by the European Research Network Training and Mobility of Researcher (T.M.R.) "Radar polarimetry – Theory and Applications". Her main interests center on radar polarimetry with special emphasis on polarimetric SAR image processing.
e-mail: carrea@infotech.tu-chemnitz.de
Faculty of Electrical Engineering
and Information Technology
University of Technology
Reichenhainer st 70
09126 Chemnitz, Germany

Gerd Wanielik received his diploma in 1979 from the Technical University of Darmstadt (Germany) and his Ph.D. in 1988 from the Technical University of Karlsruhe (Germany), all in electrical engineering. From 1979 he has worked as a scientist at the AEG-Telefunken Research Institute and later at the Daimler Chrysler Research Center. Since 1999 he is Professor of Communication Engineering at the Chemnitz University of Technology (Germany). His research interests include multisensor processing and systems, polarimetry, communication and navigation systems. He is a senior member of the IEEE, member of the ITG and DGON. He received the ITG Award in 1991 and the Daimler Chrysler Research Award in 1997.
e-mail: wanielik@infotech.tu-chemnitz.de
Faculty of Electrical Engineering
and Information Technology
University of Technology
Reichenhainer st 70
09126 Chemnitz, Germany