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# THE VISCOELASTIC LUBRICATION PROBLEM OF MICRO-BEARING

## PROBLEM LEPKOSPRĘŻYSTEGO SMAROWANIA MIKROŁOŻYSK

**Key words:**

cylindrical micro-bearing, viscoelastic oil, pressure, friction forces

**Słowa kluczowe:**

lepko sprężysty olej, walcowe mikrołożysko, prędkość oleju, ciśnienie, siły tarcia

**Summary**

In this paper we show the method of solving the lubrication problem occurring in cylindrical micro-bearing gap. Oil velocity components, hydrodynamic pressure and friction forces are derived. We assume complete visco-elastic non-Newtonian lubricant directly near the cooperating surfaces. Viscoelastic oil properties are described by means of Rivlin-Ericksen constitutive relations. Up to now the influences of complete

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visco-elastic non-Newtonian oil on the pressure and friction forces in cylindrical micro-bearing gap were not considered in analytical way. Paper elaborates analytical formulae for velocity components, pressure distributions, friction forces in cylindrical microbearings.

## VISCOELASTIC STRESS STRAIN-RELATIONS AND EQUATIONS OF MOTION

Viscoelastic properties of the lubricant are described by means of Rivlin-Ericksen constitutive relations. Hence their stress-strain dependencies have the following form **[L. 4]**:

$$\mathbf{S} = -p\mathbf{I} + \eta_o \mathbf{A_1} + \alpha(\mathbf{A_1})^2 + \beta \mathbf{A_2} \tag{1}$$

where $\mathbf{A_1}$, $\mathbf{A_2}$ velocity deformation tensors have the form **[L. 4]**:

$$\mathbf{A_1} \equiv \mathbf{L} + \mathbf{L^T}, \mathbf{A_2} \equiv \text{grad}\, \mathbf{a} + (\text{grad}\, \mathbf{a})^\mathbf{T} + \mathbf{2L^TL}, \mathbf{a} \equiv \mathbf{L}\, \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \tag{2}$$

whereas: $\mathbf{A_1}$ – tensor of deformation of the first kind [$s^{-1}$], $\mathbf{A_2}$ – tensor of deformation of the second kind [$s^{-2}$], $\mathbf{I}$ – dimensionless unit tensor, $\mathbf{L}$ – tensor of gradient of fluid velocity vector [$s^{-1}$], $\mathbf{L^T}$ – transpose tensor of gradient of fluid velocity vector [$s^{-1}$], $\mathbf{S}$ – stress tensor [Pa], $\mathbf{a}$ – accele-ration vector [$m/s^2$], $\mathbf{p}$ – pressure during the flow [Pa], $\mathbf{t}$ – time [s], $\mathbf{v}$ – velocity vector [m/s], $\alpha$ – first pseudo-viscosity experimental coefficient of the fluid [$Pas^2$], $\beta$ – second pseudo-viscosity coefficient of the fluid [$Pas^2$], $\eta_o$– dynamic viscosity of motionless fluid [Pas]. The basic equations describing the fluid flow in joint gap are as follows: equations of conservation of momentum and continuity equation. Mentioned equations for the non-stationary, compressible flow with fluid density $\rho$ between two surfaces in the curvilinear orthogonal co-ordinates, have the following form **[L. 4]**:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho\left(\text{grad} \frac{1}{2} \mathbf{vv} - \mathbf{v} \times \text{rot}\mathbf{v}\right) = \text{Div}\mathbf{S}, \tag{3}$$

$$\partial\rho/\partial t + \text{div}\, \rho\mathbf{v} = 0 \tag{4}$$

In the case of Newtonian liquid flow we have $\alpha = 0$, $\beta = 0$ and the pressure function p is constant in the height direction. In the case of non-Newtonian viscoelastic liquid flow the pressure function is not constant in the gap height direction.

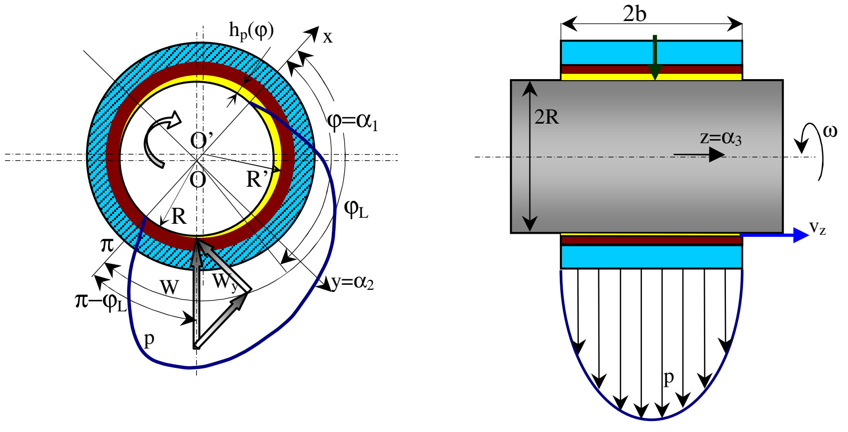


**Fig. 1. Geometry of slide micro-bearing surface**
Rys. 1. Geometria powierzchni mikrołożyska ślizgowego

We will be considered the steady incompressible liquid flow without adhesion forces for constant pressure in gap height direction with full viscoelastic properties for enough large Deborah Numbers $A_\alpha$, $A_\beta$ inside thin boundary layer. If viscous fluid forces are much greater than density forces, hence Strouhal Number and Deborah Numbers are as follows **[L. 4]**: $A_\alpha \equiv \alpha U/\eta_o L = O(1)$, $A_\beta \equiv \beta U/\eta_o L = O(1)$, for L – dimensional bearing length, U – peripheral journal velocity, are of the order of unity or larger than that. In this case liquid inertia forces and acceleration terms are neglected because: $Re\psi_1 << 1$. After boundary layer simplifications, i.e. when the terms of the order $\psi = \varepsilon/L \approx 1/100$ for radial clearance $\varepsilon$ are neglected. We take into account radial coordinates $(r,\varphi,z)$, with fluid velocity components $(v_\varphi, v_r, v_z)$, where $0 \le \varphi \le \varphi_k$, $0 \le r \le \varepsilon$, $0 \le z \le L$. The symbols R, $\varphi_k$ denote radius of the journal and the film end coordinate, respectively. The symbol L=2b denotes the bearing length (see **Fig. 1**).

## BOUNDARY CONDITIONS FOR MOVABLE JOURNAL

The flow of the liquid in cylindrical micro-bearing gap is generated by the rotation of the journal with the angular velocity ω and rotational speed 20000 rpm with oil viscosity 18 cp (see **Fig. 2a, b**).

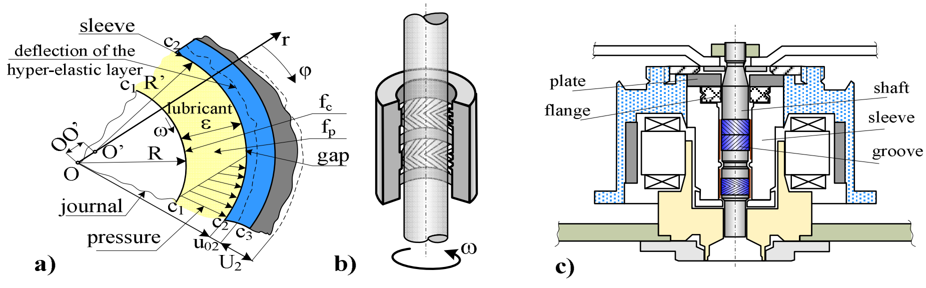


**Fig. 2. Micro-bearing: a) geometry of the gap 5–15 µm, b) micro-bearing shaft 3 mm, c) high speed HDD (20 000rpm)**

Rys. 2. Mikrołożysko: a) geometria szczeliny 5–15 µm, b) czop mikrołożyska 3 mm, c) szybko obrotowy HDD (20 000 obr/min) z zaprojektowanym wałkiem

**Fig. 2c** shows a cross sectional view of a tied HDD spindle **[L. 2]**. Hence the boundary conditions in cylindrical coordinates (r, φ, z) for the viscous lubricant flow on the journal surface in micro-bearing have the following form **[L. 1, 3–4]**:

$$v_\varphi(\varphi, r = 0, z) = \omega R,\ v_r(\varphi, r = 0, z) = 0,\ v_z(\varphi, r = 0, z) = 0 \qquad (5)$$

The micro-bearing sleeve surface is motionless both in the circumferential, φ, and longitudinal, z, direction. Hence velocity components in these directions equal zero:

$$v_\varphi(\varphi, r = \varepsilon, z) = 0,\ v_z(\varphi, r = \varepsilon, z) = 0,\ v_r(\varphi, r = \varepsilon, z) = 0 \qquad (6)$$

where ε – height of the liquid boundary layer [m].We have the dimensionless atmospheric pressure $p_A \approx 0$ in the angular coordinate $\varphi_p$ for the film origin, and on the unknown angular co-ordinate $\varphi_k$ for the film end, and on two face surfaces $z = -b_d$, $z = +b_d$. Thus we have the following boundary Reynolds conditions for the pressure function p:

$$p = p_A \approx 0 \text{ for}: \varphi = \varphi_p,\ \varphi = \varphi_k,\ \frac{\partial p}{\partial \varphi}(\varphi = \varphi_k) = 0, \text{ and for } z = \pm b_d \qquad (7)$$

## THE METHOD OF SOLVING LUBRICATION PROBLEM FOR MICRO-BEARING

In this intersection we show the method of solving the lubrication problem occurring in cylindrical micro-bearing gap. The equation of motion are considered in cylindrical coordinates by taking into account the equations (3), (4).We replace all the terms describing the visco-elastic properties of liquid, by the average terms obtained by the definite integration, in boundary layer direction, within limits from r = 0 to r = ε. Now we integrate twice the equation (3), (4) with respect to the variable r, and assume that the pressure functions p is unknown. On the obtained solutions we impose the boundary conditions (5), (6). As a result of that we obtain the following liquid velocity components in boundary layer between the journal movable surface for r=0 and the sleeve motionless surface for r=ε **[L. 4]**:

$$v_\varphi(r,\varphi,z) = \omega R\left(1-\frac{r}{\varepsilon}\right)+\frac{1}{2\eta}\frac{1}{R}\frac{\partial p}{\partial \varphi}\left(r^2-r\varepsilon\right)-\frac{1}{2\eta}\left(r^2-r\varepsilon\right)\left(\alpha O_{\alpha\varphi}+\beta O_{\beta\varphi}\right), \quad (8)$$

$$v_z(r,\varphi,z) = \frac{1}{2\eta}\frac{\partial p}{\partial z}\left(r^2-r\varepsilon\right)-\frac{1}{2\eta}\left(r^2-r\varepsilon\right)\left(\alpha O_{\alpha z}+\beta O_{\beta z}\right) \quad (9)$$

The auxiliary functions appearing in the solutions (8), (9) have the following form:

$$O_{\alpha\varphi}(v_\varphi, v_r, v_z) \equiv \frac{1}{\varepsilon}\int_0^\varepsilon \left\{ \frac{1}{R}\frac{\partial}{\partial\varphi}\left(\frac{\partial v_\varphi}{\partial r}\right)^2 + \frac{\partial}{\partial r}\left[2\frac{1}{R}\frac{\partial v_\varphi}{\partial r}\frac{\partial v_\varphi}{\partial \varphi}+2\frac{\partial v_r}{\partial r}\frac{\partial v_\varphi}{\partial r}+\frac{1}{R}\frac{\partial v_z}{\partial \varphi}\frac{\partial v_z}{\partial r}+\frac{\partial v_\varphi}{\partial z}\frac{\partial v_z}{\partial r}\right]+\frac{\partial}{\partial z}\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial v_z}{\partial r}\right)\right\}dr \quad (10)$$

$$O_{\beta\varphi}(v_\varphi, v_r, v_z) \equiv \frac{1}{\varepsilon}\int_0^\varepsilon \left\{ \frac{\partial}{\partial r}\left[\frac{1}{R}\frac{\partial^2 v_\varphi}{\partial\varphi\partial r}v_\varphi + v_r\frac{\partial^2 v_\varphi}{\partial r^2}+v_z\frac{\partial^2 v_\varphi}{\partial z\partial r}+2\frac{1}{R}\frac{\partial v_\varphi}{\partial\varphi}\frac{\partial v_\varphi}{\partial r}+\left(\frac{1}{R}\frac{\partial v_\varphi}{\partial\varphi}+\frac{\partial v_r}{\partial r}\right)\frac{\partial v_\varphi}{\partial r}+\frac{\partial v_z}{\partial r}\left(\frac{1}{R}\frac{\partial v_z}{\partial\varphi}+\frac{\partial v_\varphi}{\partial z}\right)+\frac{1}{R}\frac{\partial v_z}{\partial\varphi}\frac{\partial v_z}{\partial r}\right]+\frac{\partial^3 v_\varphi}{\partial r^2\partial t}\right\}dr$$

(11)

$$O_{\alpha z}(v_\varphi, v_r, v_z) \equiv \frac{1}{\varepsilon}\int_0^\varepsilon \left\{ \frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial r}\right)^2 + \frac{\partial}{\partial r}\left[ 2\frac{\partial v_z}{\partial r}\frac{\partial v_z}{\partial z} + 2\frac{\partial v_r}{\partial r}\frac{\partial v_z}{\partial r} + \frac{\partial v_\varphi}{\partial r}\frac{\partial v_\varphi}{\partial z} + \right.\right.$$
$$\left.\left. + \frac{1}{R}\frac{\partial v_\varphi}{\partial r}\frac{\partial v_z}{\partial \varphi}\right] + \frac{1}{R}\frac{\partial}{\partial \varphi}\left(\frac{\partial v_\varphi}{\partial r}\frac{\partial v_z}{\partial r}\right)\right\} dr \qquad (12)$$

$$O_{\beta z}(v_\varphi, v_r, v_z) \equiv \frac{1}{\varepsilon}\int_0^\varepsilon \left\{ \frac{\partial}{\partial r}\left[ \frac{v_\varphi}{R}\frac{\partial^2 v_z}{\partial \varphi \partial r} + v_r\frac{\partial^2 v_z}{\partial r^2} + v_z\frac{\partial^2 v_z}{\partial z \partial r} + 2\frac{\partial v_z}{\partial r}\frac{\partial v_z}{\partial z} + \right.\right.$$
$$\left.\left. \left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z}\right)\frac{\partial v_z}{\partial r} + \left(\frac{\partial v_\varphi}{\partial z} + \frac{1}{R}\frac{\partial v_z}{\partial \varphi}\right)\frac{\partial v_\varphi}{\partial r} + \frac{\partial v_\varphi}{\partial r}\frac{\partial v_\varphi}{\partial z}\right] + \frac{\partial^3 v_z}{\partial r^2 \partial t}\right\} dr. \qquad (13)$$

We integrate once the continuity equation (4) with respect to the variable r, i.e. in the layer height direction. Imposing the boundary conditions (5), i.e. $v_r=0$ for $r=0$ and imposing condition (6) for $r=\varepsilon$, and putting the liquid velocity components (8), (9), thus we obtain the following modified Reynolds equation:

$$\frac{1}{R^2}\frac{\partial}{\partial \varphi}\left(\frac{\varepsilon^3}{\eta}\frac{\partial p}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(\frac{\varepsilon^3}{\eta}\frac{\partial p}{\partial z}\right) = 6\omega\frac{\partial \varepsilon}{\partial \varphi} + \frac{1}{R}\frac{\partial}{\partial \varphi}\left[\frac{\varepsilon^3}{\eta}\left(\alpha O_{\alpha\varphi} + \beta O_{\beta\varphi}\right)\right] +$$
$$+ \frac{\partial}{\partial z}\left[\frac{\varepsilon^3}{\eta}\left(\alpha O_{\alpha z} + \beta O_{\beta z}\right)\right] \qquad (14)$$

The pressure function $p(\varphi,z)$ is determined from the modified Reynolds equation (14).

We put velocity components (8), (9) into the continuity equation (7). After integrating the terms on the left hand side of the modified Reynolds equation (14) and putting

this relation into (7), then the vertical component of the liquid velocity attains the final form:

$$v_r(r,\varphi,z) = \frac{1}{6}\left(\varepsilon r^2 - r^3\right)\left[\frac{1}{R^2}\frac{\partial}{\partial \varphi}\left(\frac{1}{\eta}\frac{\partial p}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\eta}\frac{\partial p}{\partial z}\right)\right] +$$
$$+ \frac{1}{6}\left(\varepsilon r^2 - r^3\right)\left\{\frac{1}{R}\frac{\partial}{\partial \varphi}\left[\frac{1}{\eta}\left(\alpha O_{\alpha\varphi} + \beta O_{\beta\varphi}\right)\right] + \frac{\partial}{\partial z}\left[\frac{1}{\eta}\left(\alpha O_{\alpha z} + \beta O_{\beta z}\right)\right]\right\} \qquad (15)$$

Now we show iterative steps of solutions of the liquid velocity profiles (8), (9), (15).

We restrict the velocity components $v_\varphi$, $v_z$ to the velocity functions: $v_\varphi^{(o)}, v_z^{(o)}$ in Newtonian flow only, where the terms $O_{\alpha x}$, $O_{\alpha z}$, $O_{\beta x}$, $O_{\beta z}$ are omitted.

The pressure function $p^{(o)}$ for Newtonian lubricant is determined by the following equation:

$$\frac{1}{R^2}\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3}{\eta}\frac{\partial p^{(o)}}{\partial\varphi}\right)+\frac{\partial}{\partial z}\left(\frac{\varepsilon^3}{\eta}\frac{\partial p^{(o)}}{\partial z}\right)=6\omega\frac{\partial\varepsilon}{\partial\varphi} \tag{16}$$

If we replace the functions $v_x$, $v_y$, $v_z$ by the functions $v_x^{(o)}, v_y^{(o)}, v_z^{(o)}$ in the relations (10), (11), (12), (13), then we obtain approximate value of the functions $O_{\alpha x}$, $O_{\alpha z}$, $O_{\beta x}$, $O_{\beta z}$ presented in the following form:

$$\begin{aligned}
O_{\alpha\varphi}^{(o)} &\equiv O_{\alpha\varphi}(v_\varphi = v_\varphi^{(o)}, v_r = v_r^{(o)}, v_z = v_z^{(o)}),\\
O_{\beta\varphi}^{(o)} &\equiv O_{\beta\varphi}(v_\varphi = v_\varphi^{(o)}, v_r = v_r^{(o)}, v_z = v_z^{(o)})\\
O_{\alpha z}^{(o)} &\equiv O_{\alpha z}(v_\varphi = v_\varphi^{(o)}, v_r = v_r^{(o)}, v_z = v_z^{(o)}),\\
O_{\beta z}^{(o)} &\equiv O_{\beta z}(v_\varphi = v_\varphi^{(o)}, v_r = v_r^{(o)}, v_z = v_z^{(o)})
\end{aligned} \tag{17}$$

The functions (17) have average values in the directions of height of liquid boundary layer. If we replace the symbols $O_{\alpha x}$, $O_{\alpha z}$, $O_{\beta x}$, $O_{\beta z}$ by the functions (17), and if we replace the functions p by the functions $p^{(1)}$ in the solutions (8), (9), (15), then we obtain the following approximate values $v_\varphi^{(1)}, v_r^{(1)} v_z^{(1)}$ of the non-Newtonian liquid velocity components.

The pressure $p^{(1)}$ for non-Newtonian lubricant determines the following modified equation:

$$\frac{1}{R^2}\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3}{\eta}\frac{\partial p^{(1)}}{\partial\varphi}\right)+\frac{\partial}{\partial z}\left(\frac{\varepsilon^3}{\eta}\frac{\partial p^{(1)}}{\partial z}\right)=6\omega\frac{\partial\varepsilon}{\partial\varphi}+\frac{1}{R}\frac{\partial}{\partial\varphi}\left[\frac{\varepsilon^3}{\eta}\left(\alpha O_{\alpha\varphi}^{(o)}+\beta O_{\beta\varphi}^{(o)}\right)\right]+$$
$$+\frac{\partial}{\partial z}\left[\frac{\varepsilon^3}{\eta}\left(\alpha O_{\alpha z}^{(o)}+\beta O_{\beta z}^{(o)}\right)\right] \tag{18}$$

for $0 \le \varphi \le \pi$, $0 \le z \le L_d$, $0 \le r \le \varepsilon$. The distributions of the liquid velocity components (8), (9), (15), in thin boundary layers or micro-bearing gap are illustrated in **Fig. 3**.

Friction forces in φ and z directions occurring on the sleeve surface can be obtained by using the following relations:

$$F_{R\varphi}^{(1)} = R\int_0^{\pi}\int_0^{L_d}\left(\eta\frac{\partial v_{\varphi}^{(1)}}{\partial r}\right)_{r=\varepsilon} dr d\varphi = \frac{\pi\omega R^2\eta}{\varepsilon}L_d + \frac{\varepsilon}{2}\int_0^{\pi}\int_0^{L_d}\frac{\partial p^{(1)}}{\partial\varphi}d\varphi dz - \frac{\varepsilon R}{2}\int_0^{\pi}\int_0^{L_d}\left(\alpha O_{\alpha\varphi} + \beta O_{\beta\varphi}\right)d\varphi dz \tag{19}$$

$$F_{Rz}^{(1)} = R\int_0^{\pi}\int_0^{L_d}\left(\eta\frac{\partial v_{z}^{(1)}}{\partial r}\right)_{r=\varepsilon} dr d\varphi = \frac{\varepsilon R}{2}\int_0^{\pi}\int_0^{L_d}\frac{\partial p^{(1)}}{\partial z}d\varphi dz - \frac{\varepsilon R}{2}\int_0^{\pi}\int_0^{L_d}\left(\alpha O_{\alpha z} + \beta O_{\beta z}\right)d\varphi dz \tag{20}$$

The last terms on the right hand side of the equations (19), (20) describe simultaneous influence of liquid visco-elastic properties on the vector components of friction forces in micro-bearing gap.

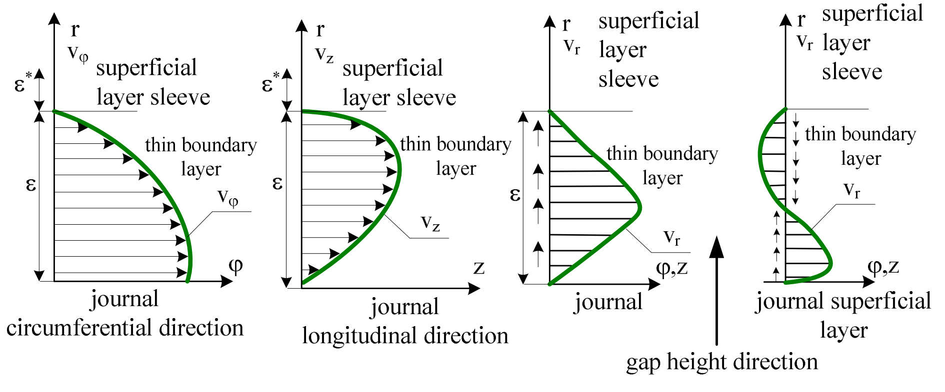


**Fig. 3. Distributions of the liquid velocity components in thin micro bearing gap**

Rys. 3. Rozkłady składowych wektora prędkości cieczy w cienkiej szczelinie mikrołożyska

## CONCLUSIONS

This paper presents the method of non-classic solution of velocity, pressure and friction forces determination in slide cylindrical micro-bearing taking into account the full terms of viscoelastic Rivlin Ericksen constitutive relations concerning the influences of bearing superficial layer mechanical properties on the oil dynamic features.

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**Recenzent:**
**Jarosław SĘP**

### Streszczenie

**Ze względu na bardzo małą wysokość szczeliny mikrołożyska, jak również bardzo małą grubość warstwy smarującej w mikrołożyskach ślizgowych mamy do czynienia z wpływem parametrów mechanicznych warstwy wierzchniej na wartość lepkości dynamicznej czynnika smarującego. Ten wpływ ujawnia się poprzez rozbudowane i zmody-**

**fikowane związki konstytutywne o modelu Rivlina-Ericksena. Związki takie wprowadzone do równań ruchu cieczy smarującej wzbogacają te równania o nowe człony, które nie ulegają pominięciu po dokonaniu typowych uproszczeń łożyskowej warstwy granicznej i są związane z liczbami Deboraha, które w rozpatrywanych przypadkach nie są małe. Uwzględnienie tych członów o charakterze lepkosprężystym związanych z wpływem parametrów mechanicznych warstwy wierzchniej na lepkość czynnika smarującego przy rozwiązaniu hydrodynamicznego smarowania mikrołożysk ślizgowych został przedstawiony w pracy [L. 4].**

**Specyfika i odrębność warunków brzegowych w mikrołożyskach ślizgowych w porównaniu z warunkami brzegowymi aplikowanymi w klasycznym hydrodynamicznym smarowaniu polega na uwzględnieniu wpływu właściwości mechanicznych warstwy wierzchniej na przepływ czynnika smarującego w bardzo cienkiej warstewce filmu o wartościach poniżej jednego mikrometra.**