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Solution of Extended Kelvin-Voigt Model

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Abstract: The great usefulness of uniaxial visco-elastic models, especially in highway engineering pavement theory, composites and other civil engineering disciplines were the reason for undertaking the trial to find a complete solution for the generalization of Kelvin-Voigt body. Here the elements of higher rank than velocities of strain and stress are considered. Carson's transformation simultaneously with residuum theorem are used for solutions derivation. The introduced procedure can be also used for more complicated differential or integral forms of constitutive equations, as well as for non homogenous initial conditions. The Burgers' body is examined. Finally, as an example the vibration of simple beam is shown.

Keywords: rheology, visco-elasticity, models.

1. Introduction

The origins of this treatise come from [1-6]. The list of scientific and application works concerning elementary visco-elastic models probably include more than thousand titles, due to that we limit the bibliography only to Reiner [7] and Nowacki [8] monographs in which the authors made a survey of rheological problems and models. It is important to note that Reiner made a full survey of rheological models in relation to physical rules, while Nowacki showed solving methods of rheological problems by means of Laplace transform and generalized functions.

The recalled monographs are rather old, but their contents has being repeated in many contemporary papers and can be treated as a rheology foundation. The advancement of formulated earlier problems are to find at [9].

Recently many of such models were refined by enhancing some aspect related to current problems [10], [11]. Especially the road and bridge engineering are the field of such models applications [12], [13].

In this approach we focus on benefits, which come from mathematical formalism, i.e. from admissible solutions' forms for assumed constitutive equation. Although the model was used in many works, the below results have been not noticed earlier and only due to that it seems to be interesting to present them.

On the basis of Hohenemser and Prager [7] postulate is assumed a general linear body, which is linear in Boltzmann sense. We confine our analyses to linear visco-elasticity, excluding inner constrains of Saint-Venant type used in Schwedoff model, for example. Only the differential form of constitutive relations are considered. For elementary models we have:

$$\dot{\sigma} a_1 + \sigma a_0 = \dot{\varepsilon} b_1 \quad \text{– Maxwell,} \tag{1.1}$$

$$\sigma a_0 = \varepsilon b_0 + \dot{\varepsilon} b_1 \text{ – Kelvina-Voigt,} \tag{1.2}$$

$$\ddot{\sigma} a_2 + \dot{\sigma} a_1 + \sigma a_0 = \dot{\varepsilon} b_1 + \ddot{\varepsilon} b_2 \text{ – Burgers,} \tag{1.3}$$

$$\dot{\sigma} a_1 + \sigma a_0 = \varepsilon b_0 + \dot{\varepsilon} b_1 \text{ – Zener;} \tag{1.4}$$

where a_0, a_1, a_2, b_0, b_1, b_2 – are visco-elastic material constants, σ, ε – are tensors of stress and strain, reduced here for analyzed one dimensional problem, number of dots over character means the rank of it's time derivative.

Additionally we assume that initial conditions are homogenous, i.e. at time moment $t_0 = 0$ we adopt

$$\sigma(0) = 0 \quad \dot{\sigma}(0) = 0 \quad \varepsilon(0) = 0 \quad \dot{\varepsilon}(0) = 0 \tag{2}$$

In mathematical sense (1.1-4) are a cutting of the following formula

$$\sum_{m=1,2,\dots}^{\infty} a_m \frac{\partial^{(m)} \sigma}{\partial t^{(m)}} = \sum_{n=1,2,\dots}^{\infty} b_n \frac{\partial^{(n)} \varepsilon}{\partial t^{(n)}} \tag{3}$$

Generalizing, (1.1-4) models could be derived from (3) when $m = 0, 1, 2$ and $n = 0, 1, 2$ i.e. –

$$\sum_{m=0,1,\dots}^{2} a_m \frac{\partial^{(m)} \sigma}{\partial t^{(m)}} = \sum_{n=0,1,\dots}^{2} b_n \frac{\partial^{(n)} \varepsilon}{\partial t^{(n)}}, \text{ furthermore } a_0 = 1. \tag{4}$$

For recognizing the properties of created constructive relation (4), the Carson transform is applied in the following form

$$C[f(t)] = p \int_{t=0}^{t \to \infty} f(t) \mathrm{e}^{-pt} \, dt = \tilde{f}(p) \tag{5}$$

2. The solution of the problem

Operating with (5) onto (4) we arrive at

$$\tilde{\varepsilon} = \frac{\tilde{\sigma}}{b_2} \frac{a_2 p^2 + p a_1 + 1}{p_2 + p\beta_1 + \beta_0} = \frac{\tilde{\sigma}}{b_2} \frac{a_2 p^2 + p a_1 + 1}{(p - p_1)(p - p_2)} \tag{6}$$

where

$$\beta_1 = \frac{b_1}{b_2} \quad \beta_2 = \frac{b_0}{b_2} \tag{6.1}$$

p_1, p_2 – are the roots of the denominator at (6), where

$$p_{1,2} = \frac{1}{2}\left(-\beta_0 \pm \sqrt{\Delta}\right) \quad \Delta = (\beta_1)^2 - 4\beta_0 \tag{7}$$

We have to analyze the set of following cases –

(I) $\Delta > 0 \quad \rightarrow \qquad p_1 \neq p_2 \neq 0 \in R$

(II) $\Delta < 0 \quad \rightarrow \qquad p_1 = \overline{p}_2 \in C, \qquad p_{1,2} = \alpha_0 \pm i\alpha_1$

(III) $\Delta = 0 \quad \rightarrow \qquad p_{1,2} = -\frac{\beta_0}{2} \neq 0$

(IV) $\Delta = 0 \quad \rightarrow \qquad p_{1,2} = 0$

Looking for original $\varepsilon(t) = C^{-1}[\tilde{\varepsilon}]$ we modify the relation (6) to the appropriate form, adequate to apply convolution theorem –

$$\tilde{\varepsilon} b_2 = \frac{\tilde{\sigma}}{p}\left(p\frac{L(p)}{(p-p_1)(p-p_2)}\right) \tag{8}$$

where

$$L(p) = p^2 a_2 + p a_1 + 1 \tag{8.1}$$

By virtue of Carson transformation we have

$$C[\dot{f}] = -p\,f(0) + p\,\tilde{f} \rightarrow p\,\tilde{f} \tag{9}$$

which is valid in the case of homogenous initial conditions. Denoting

$$C[\dot{f}(t)] = p\frac{L(p)}{(p-p_1)(p-p_2)} \quad \text{and} \tag{10}$$

$$\tilde{f} = \frac{L(p)}{(p-p_1)(p-p_2)} \tag{11}$$

we can directly use the convolution form

$$\varepsilon\, b_2 = \int_{\tau=0}^{\tau=t} \sigma(\tau)\dot{f}(t-\tau)d\tau = \int_{\tau=0}^{\tau=t} \sigma(t-\tau)\dot{f}(\tau)d\tau \tag{12}$$

when the load function σ(t) is a known one.

To find the original $f(t)C^{-1}[\tilde{f}]$ the method based on residuum theorem connected with Jordan's lemma is adopted as follows

$$C^{-1}[\tilde{f}] = \sum \mathrm{Res}\left(\frac{\mathrm{e}^{pt}}{p}\frac{N(p)}{D(p)}\right) \tag{13}$$

where $N(p)$ and $D(p)$ mean respectively – numerator and denominator of rational expression.

The results obtained below are illustrated by using the load function which has a constant stress value $\sigma_0 \neq 0$ in the time interval $t \in \langle t_0, t_1 \rangle$ and is zero outside of the interval

$$\sigma = \sigma_0\left[H(t-t_0) - H(t-t_1)\right] \tag{14.1}$$

where $H(t)$ is step function. Excluding infinitesimal time interval surrounding t_0 and t_1- time moments the stress σ has constant value and this implies

$$a_2 = a_1 = 0 \text{ in the relation (4).} \tag{14.2}$$

3. Solutions in particular variants

3.1. Ad. (I)

Two roots of denominator in (6) are real and non zero. The values of these roots are singular points for relation (6), as well as for (8). Using (11) and (13) we arrive at

$$f_{(I)} = \sum \operatorname{Res}\left(\frac{e^{pt}}{p} \frac{L(p)}{(p-p_1)(p-p_2)} \right) \tag{15}$$

Additionally we have to consider the singular point $p = 0$.

We obtain

$$p_0 \rightarrow \left[e^{pt} \frac{L(p)}{(p-p_1)(p-p_2)} \right]_{p=0} \tag{16.1}$$

$$p_1 \rightarrow \left[e^{\frac{pt}{p}} \frac{L(p)}{(p-p_2)} \right]_{p=p_1} \tag{16.2}$$

$$p_2 \rightarrow \left[e^{\frac{pt}{p}} \frac{L(p)}{(p-p_1)} \right]_{p=p_2} \tag{16.3}$$

The sought for function and its time derivative are read

$$f_{(I)} = \frac{1}{\beta_0} + \frac{1}{p_1 - p_2} \left[\frac{e^{p_1 t} L(p_1)}{p_1} - \frac{e^{p_2 t} L(p_2)}{p_2} \right] \tag{17}$$

$$\dot{f}_{(I)} = \frac{1}{p_1 - p_2} \left[e^{p_1 t} L(p_1) - e^{p_2 t} L(p_2) \right] \tag{18}$$

By virtue of (12) and on the basis of assumption (14.1) the strain process has form of functions –

$$\begin{aligned} b_2 \varepsilon_{(I)} &= \frac{\sigma_0}{p_2 - p_1} \left\{ \frac{L(p_1)}{p_1} \left[1 - e^{p_1 (t - t_0)} \right] - \frac{L(p_2)}{p_2} \left[1 - e^{p_2 (t - t_0)} \right] \right\} \rightarrow \\ &\underset{(14.2)}{\rightarrow} \quad \frac{\sigma_0}{p_2 - p_1} \left[\frac{1 - e^{p_1 (t - t_0)}}{p_1} - \frac{1 - e^{p_2 (t - t_0)}}{p_2} \right] \end{aligned} \tag{19.1}$$

when $t_0 \leq t \leq t_1$ and

$$\begin{aligned} b_2 \varepsilon_{(I)} &= \frac{\sigma_0}{p_2 - p_1} \left[\frac{L(p_1)}{p_1} e^{p_1 t} \left(e^{-p_1 t_1} - e^{-p_1 t_0} \right) - \frac{L(p_2)}{p_2} e^{p_2 t} \left(e^{-p_1 t_1} - e^{-p_1 t_0} \right) \right] \rightarrow \\ &\underset{(14.2)}{\rightarrow} \quad \frac{\sigma_0}{p_2 - p_1} \left[\frac{e^{p_1 t}}{p_1} \left(e^{-p_1 t_1} - e^{-p_1 t_0} \right) - \frac{e^{p_2 t}}{p_2} \left(e^{-p_2 t_1} - e^{-p_2 t_0} \right) \right] \end{aligned} \tag{19.2}$$

for $t > t_1$.

3.2. Ad. (II)

When the denominator (6) roots are conjugated complex, they could be presented in an alternative algebraic or exponential form

$$p_1 = \overline{p}_2 = \alpha_0 \pm i\,\alpha_1 = e^{(Y_0 \pm i\,Y_1)} = \rho_0\,\mathrm{e}^{(\pm i\,Y_1)} \qquad i = \sqrt{-1} \tag{20.1}$$

Together with $p_0 = 0$ the roots form the set of singular points necessary to get the original $f_{(II)}$. Appropriately for: p_0, p_1 and p_2 the residua are -

$$p_0 = 0 \quad \rightarrow \qquad \frac{1}{\beta_0} \tag{20.2}$$

$$p_1 = \alpha_0 + i\,\alpha_1 \qquad \rightarrow \qquad \mathrm{e}^{(\alpha_0 t)}\,\frac{\mathrm{e}^{(i\,\alpha_1 t)}}{2\,i\,\alpha_1}\,\frac{L(\alpha_0 + i\,\alpha_1)}{\alpha_0 + i\,\alpha_1} \tag{20.3}$$

$$p_2 = \alpha_0 - i\,\alpha_1 \qquad \rightarrow \qquad \mathrm{e}^{(\alpha_0 t)}\,\frac{\mathrm{e}^{(-i\,\alpha_1 t)}}{-2\,i\,\alpha_1}\,\frac{L(\alpha_0 - i\,\alpha_1)}{\alpha_0 - i\,\alpha_1} \tag{20.4}$$

Applying exponential form we can write

$$\frac{L(\alpha_0 \pm i\,\alpha_1)}{\alpha_0 \pm i\,\alpha_1} \underset{(14.2)}{\rightarrow} \frac{1}{\alpha_0 \pm i\,\alpha_1} = \mathrm{e}^{(A_0 \pm i\,A_1)} \tag{20.5}$$

Using Euler's formulae, (20.4) and summing (20.1-20.3) we find the sought for function

$$f_{(II)} = \frac{1}{\beta_0} + \frac{\mathrm{e}^{(\alpha_0 t + \mathrm{A}_0)}}{i\,\alpha_1}\,sh\left[i(\alpha_1 t + A_1)\right] = \frac{1}{\beta_0} + \frac{\mathrm{e}^{(\alpha_0 t + \mathrm{A}_0)}}{\alpha_1}\sin(\alpha_1 t + A_1) \tag{21}$$

and its time derivative

$$\dot{f}_{(II)} = \mathrm{e}^{(\alpha_0 t + \mathrm{A}_0)}\left[\frac{\alpha_0}{\alpha_1}\sin(\alpha_1 t + A_1) + \cos(\alpha_1 t + A_1)\right] \tag{22}$$

The load function (14.1) yield the strain process as follows

$$b_2\varepsilon_{(II)} = \sigma_0\,\mathrm{e}^{(B_0)}\left\{\mathrm{e}^{\alpha_0(t-t_0)}\left[\frac{\alpha_0}{\alpha_1}\sin\big(\alpha_1(t-t_0)+B_1\big)+\cos\big(\alpha_1(t-t_0)+B_1\big)\right]-\left[\frac{\alpha_0}{\alpha_1}\sin(B_1)+\cos(B_1)\right]\right\} \tag{23.1}$$

for $t_0 \le t \le t_1$ and when $t > t_1$

$$b_2\varepsilon_{(II)} = \sigma_0\,\mathrm{e}^{[\alpha_0 t + B_0]}\left\{\mathrm{e}^{-\alpha_0 t_1}\left[\frac{\alpha_0}{\alpha_1}\sin\big(\alpha_1(t-t_1)+B_1\big)+\cos\big(\alpha_1(t-t_1)+B_1\big)\right]+\right.$$
$$\left. -\,\mathrm{e}^{-\alpha_0 t_0}\left[\frac{\alpha_0}{\alpha_1}\sin\big(\alpha_1(t-t_0)+B_1\big)+\cos\big(\alpha_1(t-t_0)+B_1\big)\right]\right\} \tag{23.2}$$

where the simplifying symbols B_0, B_1 mean

$$\frac{\mathrm{e}^{(A_0 \pm i\,A_1)}}{\alpha_0 \pm i\,\alpha_1} \underset{(14.2)}{\rightarrow} \frac{1}{(\alpha_0 \pm i\,\alpha_1)^2} = \mathrm{e}^{(B_0 \pm i\,B_1)} \tag{23.3}$$

3.3. Ad. (III)

In this variant we have double real non null root $p_1 = p_2 = -\beta_0/2$ which, together with $p_0 = 0$, are also singularity points for (13). The residuum for p_0 we obtain, as previously,

$$p_0 = 0 \to \frac{1}{\beta_0} \tag{24.1}$$

To find residuum for a double root we use the following rule

$$p_1 = p_2 = -\frac{\beta_0}{2} \to \left\{ \frac{d}{d\,p}\left[\frac{\mathrm{e}^{pt}}{p} L(p) \right] \right\}_{p=p_1} = \frac{\mathrm{e}^{p_1 t}}{(p_1)^2}\left[L(p_1)(t-1) + p_1 \dot{L}(p_1) \right] \tag{24.2}$$

The sought for $f_{(III)}$ - function is read

$$f_{(III)} = \frac{1}{\beta_0} + 4\frac{\mathrm{e}^{-\frac{\beta_0 t}{2}}}{(\beta_0)^2}\left[L\left(-\frac{\beta_0}{2}\right)(t-1) - \frac{\beta_0}{2}\dot{L}\left(-\frac{\beta_0}{2}\right) \right] \tag{25}$$

and its time derivative

$$\dot{f}_{(III)} = 4\frac{\mathrm{e}^{-\frac{\beta_0 t}{2}}}{(\beta_0)^2}(\vartheta_1 t + \vartheta_0) \tag{26}$$

where we denoted

$$\vartheta_1 = -\frac{\beta_0}{2} L\left(-\frac{\beta_0}{2}\right) \underset{(14.2)}{\to} -\frac{\beta_0}{2} \tag{26.1}$$

$$\vartheta_0 = \left(\frac{\beta_0}{2}\right)^2 \dot{L}\left(-\frac{\beta_0}{2}\right) + L\left(-\frac{\beta_0}{2}\right)\left(\frac{\beta_0}{2} + 1\right) \underset{(14.2)}{\to} \frac{\beta_0}{2} + 1 \tag{26.2}$$

The strain process for the load function (14.1-2) and (26) implies the following result

$$b_2 \varepsilon_{(III)} = \frac{4\sigma_0}{(\beta_0)^2}\left\{ -\frac{2}{\beta_0} + \mathrm{e}^{-\frac{\beta_0(t-t_0)}{2}}\left[\frac{2}{\beta_0} + (t-t_0) \right] \right\} \tag{27.1}$$

when $t_0 \le t \le t_1$ and

$$b_2 \varepsilon_{(III)} = \frac{4\sigma_0}{(\beta_0)^2}\mathrm{e}^{-\frac{\beta_0 t}{2}}\left\{ \mathrm{e}^{\frac{\beta_0 t_0}{2}}\left[\frac{2}{\beta_0} + (t-t_0) \right] - \mathrm{e}^{\frac{\beta_0 t_1}{2}}\left[\frac{2}{\beta_0} + (t-t_1) \right] \right\} \tag{27.2}$$

when $t > t_1$.

3.4. Ad. (IV)

Similar as above we have double real, but in this case its value is zero $p_1 = p_2 = 0$. Taking into account $p_0 = 0$ we have a triple singularity point, this implies the residuum value as

$$p_0 = p_1 = p_2 = 0 \rightarrow \left\{ \frac{1}{2!} \frac{d^2}{d\,p^2} \left[\mathrm{e}^{pt} \, L(p) \right] \right\}_{p=0} = \left\{ \frac{\mathrm{e}^{pt}}{2} \left[t^2 L(p) + 2t\dot{L}(p) \right] + \ddot{L}(p) \right\}_{p=0} \quad (28)$$

The functions $f_{(IV)}$ and its time derivative $\dot{f}_{(IV)}$ have the forms

$$f_{(IV)} = \frac{1}{2}\left(t^2 + 2ta_1 + 2a_2 \right) \quad (29)$$

$$\dot{f}_{(IV)} = t + a_1 \quad (30)$$

Assuming (14.1-2) and (30) we arrive at

$$b_2 \varepsilon_{(IV)} = \sigma_0 \left[\frac{t^2 - {t_0}^2}{2} + (t - t_0) a_1 \right] \rightarrow \sigma_0 \frac{t^2 - {t_0}^2}{2} \quad (31.1)$$

for $t_0 \le t \le t_1$ and

$$b_2 \varepsilon_{(IV)} = \sigma_0 \frac{t - t_0}{2} \left[2t - (t_1 - t_0) + 2a_1 \right] \rightarrow \frac{\sigma_0}{2} \left[2t^2 - t(t_0 + t_1) + t_0 (t_1 - t_0) \right] \quad (31.2)$$

since $t > t_1$.

4. Burgers model

Treating the above results as a generalization we can obtain a particular models, here the Burgers model. In the case of (I), setting up

$$\beta_0 = 0 \quad (32)$$

we obtain

$$p_1 = 0 \quad \text{and} \quad p_2 = -\beta_1 = -\frac{b_1}{b_2} \quad (33)$$

Again, we have a double singularity point, now for

$$p_0 = p_1 = 0 \quad (34)$$

the second one, non zero, is p_2. The values of residua are

$$p_0 = p_1 = 0 \rightarrow \left\{ \frac{d}{d\,p} \frac{\left[\mathrm{e}^{pt} \, L(p) \right]}{p + \beta_1} \right\}_{p=0} = \left\{ \frac{\mathrm{e}^{pt} \left\{ \left[t\,L(p) + \dot{L}(p) \right](p + \beta_1) - L(p) \right\}}{(p + \beta_1)^2} \right\}_{p=0} \quad (35.1)$$

$$p_2 = -\beta_1 \rightarrow \left\{ \frac{\mathrm{e}^{pt} \, L(p)}{p^2} \right\}_{p=-\beta_1} \quad (35.2)$$

The above results in

$$f_{(B.)} = \frac{1}{(\beta_1)^2} \left[(t + a_1)\beta_1 - 1 + \mathrm{e}^{-\beta_1 t} \, L(-\beta_1) \right] \quad \text{and} \quad (36)$$

$$\dot{f}_{(B.)} = \frac{1}{\beta_1}\left[1 - e^{-\beta_1 t} L(-\beta_1)\right] \tag{37}$$

Considering the load function (14.1) we arrive at

$$b_2\varepsilon = \frac{\sigma_0}{(\beta_1)^2}\left\{(t - t_0)\beta_1 - \left[1 - e^{-\beta_1(t - t_0)}\right]L(-\beta_1)\right\} \tag{38.1}$$

when $t_0 \le t \le t_1$ and

$$b_2\varepsilon_{(B.)} = \frac{\sigma_0}{(\beta_1)^2}\left[-\beta_1(t_1 - t_0) + e^{-\beta_1 t}\left(e^{\beta_1 t_1} - e^{\beta_1 t_0}\right)L(-\beta_1)\right] \tag{38.2}$$

if $t > t_1$; now, taking into account (14.2) we get the Burgers model

$$\varepsilon_{(B.)} = \sigma_0 \frac{b_2}{(b_1)^2}\left[\frac{b_1}{b_2}(t - t_0) - 1 + e^{-\frac{b_1}{b_2}(t - t_0)}\right] \quad t_0 \le t \le t_1 \tag{39.1}$$

$$\varepsilon_{(B.)} = -\sigma_0 \frac{b_2}{(b_1)^2}\left\{\frac{b_1}{b_2}(t_1 - t_0) + e^{-\frac{b_1}{b_2}t}\left[e^{\frac{b_1}{b_2}t_1} - e^{\frac{b_1}{b_2}t_0}\right]\right\} \tag{39.2}$$

for $t > t_1$.

5. Example - Maxwell model - simple beam vibration

That subject was investigated by many authors, for example Nowacki [8], looking for common effects comparing elasticity and viscoelasticity in Maxwell and Kelvin-Voigt models. Now the topic of consideration is to examine the vibration process at time limit at infinity.

The dynamical equilibrium equation for elastic infinitesimal beam element is as follows –

$$\left[EJ\frac{\partial^4}{\partial x^4} + \frac{\gamma A}{g}\frac{\partial^2}{\partial t^2}\right]w(x,t) = q_1(x,t) \tag{40}$$

where: w – beam deflection, A – constant beam cross-section, EJ – bending rigidity, q_1 – load linear density.

Denoting beam span by L, we introduce dimensionless coordinates –

$$x = \xi L \quad w = \omega L \quad t = t_o \tau \tag{41}$$

where t_o - positive constant with time units, that involves –

$$\frac{EJ}{(L)^3}\omega^{IV}(\xi,\tau) + \frac{\gamma AL}{g(t_o)^2}\ddot{\omega}(\xi,\tau) = q_1(\xi,\tau) \tag{25}$$

where: γ - material weight density, g – gravity acceleration, ω^{IV} - four order derivative according to ξ, $\ddot{\omega}$ - the second rank dimensionless time parameter derivative.

Assuming:

$$\left(t_o\right)^2=\frac{\gamma A\left(L\right)^4}{J\,g} \quad \text{and} \quad q=q_1\frac{\left(L\right)^3}{J} \tag{26. 1-2}$$

we obtain –

$$E\,\omega^{IV}+\ddot{\omega}=q \tag{27}$$

With the help of Carson transformation according to τ, and involving Alfrey's analogy we can turn into viscoelastic problem –

$$\tilde{\mathrm{E}}\,\tilde{\omega}^{IV}+\left(p\right)^2\tilde{\omega}=\tilde{q} \tag{28}$$

since the initial conditions are homogenous. In the case of Maxwell model we obtain –

$$\tilde{\mathrm{E}}=p\frac{1}{E}+\frac{1}{\eta} \tag{29}$$

where $\tilde{\mathrm{E}}$ is not Young modulus (E) transformation.

Let the load function be Dirac's impulse –

$$q=q_\xi q_\tau=\delta\left(\xi-\frac{1}{2}\right)\delta\left(\tau-\tau_o\right) \tag{30}$$

that yield

$$\tilde{q}=q_\xi\tilde{q}_\tau \tag{31}$$

We look for the solution expanding the unknowns and load function into Fourier sine series according to ξ –

$$\tilde{\omega}\left(\xi,p\right)=\sum_{j=1,2,\ldots}^{\infty}\tilde{\omega}_j\left(p\right)\sin j\pi\xi \;\; \tilde{q}\left(\xi,p\right)=\tilde{q}_\tau\left(p\right)\sum_{j=1,2,\ldots}^{\infty}\left(q_\xi\right)_j\sin j\pi\xi \tag{32.1-2}$$

$$\left(q_\xi\right)_j=2\int_0^1\delta\left(\xi-\frac{1}{2}\right)\sin j\pi\xi\,d\xi=2\sin\frac{j\pi}{2} \tag{33}$$

By virtue of series properties, we have –

$$\underset{j=1,2,\ldots,\infty}{\forall}\quad \tilde{\omega}_j=\frac{\tilde{q}_\tau\left(q_\xi\right)_j}{\left(p\right)^2+\tilde{E}\left(j\pi\right)^4}=\frac{\tilde{q}_\tau\left(q_\xi\right)_j}{\left(p\right)^2+p\dfrac{\left(j\pi\right)^4}{E}+\dfrac{\left(j\pi\right)^4}{\eta}}=\frac{\tilde{q}_\tau\left(q_\xi\right)_j}{\tilde{f}\left(p\right)=\left(p-p_1\right)\left(p-p_2\right)} \tag{34}$$

The roots of $\tilde{f}\left(p\right)=0$ are as follows –

$$p_{1,2}=-\frac{\left(j\pi\right)^4}{2E}\left(1\mp\sqrt{1-\frac{1}{\eta}\left[\frac{2E}{\left(j\pi\right)^2}\right]^2}\right)=\alpha\left(1\mp\sqrt{\Delta}\right) \tag{35}$$

Applying convolution theorem we can find Fourier coefficients ω_j –

$$\omega_j = \left(q_\xi\right)_j \int_0^\tau \delta\left(\theta - \tau_o\right) C^{-1}\left[\frac{p}{\left(p-p_1\right)\left(p-p_2\right)}\right]\left(\tau - \theta\right) d\theta =$$

$$= \left(q_\xi\right)_j H\left(\tau - \tau_o\right) C^{-1}\left[\frac{p}{\left(p-p_1\right)\left(p-p_2\right)}\right]\left(\tau - \tau_o\right) \tag{36}$$

where $C^{-1}[\cdot]$ is the symbol of retransformation.

Both roots (35) are *j* index function. Having in mind that E and η are positive we arrive at –

$$\lim_{j\to\infty} \sqrt{\Delta} = \lim_{j\to\infty} \sqrt{1 - \frac{1}{\eta}\left[\frac{2E}{\left(j\pi\right)^2}\right]^2} = 1 \tag{37}$$

and in consequence for large enough *j* value the roots are real and equal to –

$$p_{1\infty} = 0 \quad p_{2\infty} = -\frac{\left(j\pi\right)^4}{E} \tag{38.1-2}$$

We have to analyze three potential variants –

I. $\Delta > 0 \rightarrow 1 > \frac{1}{\eta}\left[\frac{2E}{\left(j\pi\right)^2}\right]^2$ $\rightarrow$ the roots are real and negative, $p_1 < p_2$,

II. $\Delta < 0 \rightarrow 1 < \frac{1}{\eta}\left[\frac{2E}{\left(j\pi\right)^2}\right]^2$ $\rightarrow$ the roots are conjugative complex $p_1 = \bar{p}_2$ when $\mathrm{Re}\left(p_1\right) < 0$,

III. $\Delta = 0 \rightarrow 1 = \frac{1}{\eta}\left[\frac{2E}{\left(j\pi\right)^2}\right]^2$ $\rightarrow$ we have dual real root $p_1 = p_2 = -\frac{\left(j\pi\right)^4}{2E}$, it could be only for one j index value.

The complexity of the problem consists in simultaneous occurrence of all (I-III) variants. Simplifying, let us assume that we found j_* by solving III and $j_* \notin N$. j_* is dividing j domain into two parts where -

- Variant I is valid for $_{-}j < j_*$ and
- Variant II when $_{+}j > j_*$.

Additionally, we can state j_* is not large and we can neglect the condition (38.1-2) which obeys $j \rightarrow \infty$.

Variant I is associated with hard viscous damping. Variant II describes decaying beam vibration.

In our problem the Jordan's lemma is fulfilled and we can apply residual theorem. The original for the variant I has the form -

$$C^{-1}\left[\frac{p}{(p-p_1)(p-p_2)}\right]=\sum \mathrm{Res}\,\frac{e^{p\tau}}{\left[p-\alpha\left(1-\sqrt{\Delta}\right)\right]\left[p-\alpha\left(1+\sqrt{\Delta}\right)\right]}=\frac{e^{\alpha\tau}}{\alpha\sqrt{\Delta}}sh\left(\alpha\sqrt{\Delta}\,\tau\right) \quad (39)$$

For variant II, with the help of Euler's formulae, we arrive at –

$$C^{-1}\left[\frac{p}{(p-p_1)(p-p_2)}\right]=C^{-1}\left[\frac{p}{\left[p-\alpha\left(1-i\sqrt{|\Delta|}\right)\right]\left[p-\alpha\left(1+i\sqrt{|\Delta|}\right)\right]}\right]=$$

$$=\sum \mathrm{Res}\,\frac{e^{p\tau}}{\left[p-\alpha\left(1-i\sqrt{|\Delta|}\right)\right]\left[p-\alpha\left(1+i\sqrt{|\Delta|}\right)\right]}=e^{\alpha\tau}\frac{\sin\left(\alpha\sqrt{|\Delta|}\tau\right)}{\alpha\sqrt{|\Delta|}} \quad (40)$$

where $i=\sqrt{-1}$.

In case of large j values we have –

$$\lim_{j\to\infty}\Delta\to 1 \quad \text{and} \quad \lim_{j\to\infty}\left(e^{\alpha\tau}\frac{\sin\left(\alpha\sqrt{\Delta}\tau\right)}{\alpha\sqrt{\Delta}}\right)\to 0 \quad (41)$$

On the basis of (36) we get –

$$\omega_{-j}=2\sin\frac{_{-}j\pi}{2}H(\tau-\tau_o)\frac{e^{\alpha(\tau-\tau_o)}}{\alpha\sqrt{\Delta}}sh\left[\alpha\sqrt{\Delta}(\tau-\tau_o)\right] \quad (42)$$

and

$$\omega_{+j}=2\sin\frac{_{+}j\pi}{2}H(\tau-\tau_o)\frac{e^{\alpha(\tau-\tau_o)}}{\alpha\sqrt{|\Delta|}}\sin\left[\alpha\sqrt{|\Delta|}(\tau-\tau_o)\right] \quad (43)$$

The solution of the problem has the following form –

$$\omega(\xi,\tau)=\sum_{_{-}j=1,2,\ldots}^{Int(j_{gr.})}\omega_{-j}\sin\left(_{-}j\pi\xi\right)+\sum_{_{+}j=Int(j_{gr.})+1,\ldots}^{\infty}\omega_{+j}\sin\left(_{+}j\pi\xi\right) \quad (44)$$

Now we can find the limit of (44) for $\xi = 1/2$ and $\tau\to\infty$ arriving at

$$\lim_{\text{Ä}\to\infty}\text{É}\left(\frac{1}{2},\text{Ä}\right)=2\lim_{\text{Ä}\to\infty}\left\{\sum_{_{-}\mathrm{j}=1,2,\ldots}^{\mathrm{Int(j_{gr.})}}\left(\sin\frac{_{-}\mathrm{j\text{À}}}{2}\right)^2\frac{\mathrm{e}^{\pm(\text{Ä}-\text{Ä}_o)}}{\pm\sqrt{\text{”}}}\mathrm{sh}\left[\pm\sqrt{\text{”}}\,(\text{Ä}-\text{Ä}_o)\right]+\right.$$

$$\left.+\sum_{_{+}\mathrm{j}=\mathrm{Int(j_{gr.})}+1,\ldots}^{\infty}\left(\sin\frac{_{+}\mathrm{j\text{À}}}{2}\right)^2\frac{\mathrm{e}^{\pm(\text{Ä}-\text{Ä}_o)}}{\pm\sqrt{|\text{”}|}}\sin\left[\pm\sqrt{|\text{”}|}(\text{Ä}-\text{Ä}_o)\right]\right\}=0. \quad (45)$$

The obtained result has proved that Maxwell model is reversible.

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Rozwiązanie uogólnionego modelu Kelvina-Voigta

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Streszczenie: Użyteczność jednowymiarowych modeli lepko-sprężystych, szczególnie w zagadnieniach nawierzchni drogowych, kompozytach i innych dziedzinach inżynierii lądowej stała się przyczyną podjęcia próby znalezienia kompletnego rozwiązania uogólnionego modelu Kelvina-Voigta, przy czym w modelu uwzględniono także przyspieszenia tak naprężeń jak i odkształceń. Do uzyskania rozwiązań wykorzystano transformację Carsona oraz twierdzenie o residuach. Zastosowana procedura może być także użyta w przypadkach bardziej złożonych związków konstytutywnych w formie różniczkowej lub całkowej, jak również przy niejednorodnych warunkach początkowych. Rozpatrzono szczególny przypadek analizowanego uogólnienia tj. model Burgersa. Jako aplikację zamieszczono przykład analizy drgań belki swobodnej.

Słowa kluczowe: reologia, modele lepko-sprężyste.