BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 61, No. 1, 2013
DOI: 10.2478/bpasts-2013-0021

Stability criteria for a class of stochastic distributed delay systems

B. FICAK[1] and J. KLAMKA[2*]

[1] Institute of Mathematics, Jagiellonian University, 6 Lojasiewicza St., 30-348 Krakow, Poland
[2] Institute of Theoretical and Applied Informatics, 5 Bałtycka St., 41-100 Gliwice, Poland

Abstract. In the paper linear distributed delay stochastic systems are considered. Using theory of stochastic differential equations sufficient conditions for different kinds of stability are formulated and proved. The article attempts to generalise results presented in the paper [1] and thus theorems proved in [1] become a special case of a generalised approach. The considered class is wider – the function that influence dynamics of a problem can be a real solution of N-degree linear deterministic differential equation. Therefore the generalised reduction technique of distributed delay to lumped delay has to be applied. Criteria for numerous properties of the aforementioned class followed Mao theory designed for point delay systems [2, 3].

Key words: stochastic delay differential equations, stability, boundedness, distributed delay.

1. Introduction

The following paper develops methods classically applied to the investigation of stability and other properties of point delay equations to explore these features for a special class of distributed delay stochastic differential equations. Simultaneously, the paper attempts to extend results presented in [1] where only a narrow case is considered. The pondered case has numerous applications in various financial models [4]. Therefore the aforementioned class has been extended on rational functions that allow to fit a used model to special needs of modellers. The analysis of stability of such systems can be significantly facilitated by finding a proper reduction technique that reduce distributed delay to crisp delay. If the technique can be found, then well-known criteria that work for crisp delay. It is worth stressing that the distributed delay equations could be applied allow to describe mathematically dynamics of various complex processes but the theory of distributed delay systems is not as "well-equipped" as the crisp delay theory. There exist numerous theorems that allow to examine stability, asymptotic stability, asymptotic boundedness and many other features of point delay systems whilst the number of effective tools for analysis of these properties for distributed delay equations is very limited. The following paper extends the basic distributed delay class mentioned in [1] using rational functions, then finds a reduction technique and after successful reduction process uses Mao's theory that works under specific assumptions and conditions. Moreover, the article proves that criteria presented in paper [1] are special cases of conditions touched in this article. It is worth pointing out that the robust stability of differential linear control systems has been considered in the following papers [5, 6] and [7].

2. Preliminary

Throughout this article we use the following notation. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with embedded increasing and right continuous family $\{\mathcal{F}_t\}_{t\geq 0}$ of complete sub-σ-algebras of $\mathcal{F}$. Let $\mathcal{C}([-\tau;0],\mathbb{R}^n)$ be the space of continuous functions from $[-\tau;0]$ into $\mathbb{R}^n$ with the sup-norm defined as follows

$$\|\psi\|_s = \sup_{-\tau\leq u\leq 0} \|\psi(u)\|_{\mathbb{R}^n},\ \psi\in\mathcal{C}([-\tau;0],\mathbb{R}^n). \tag{1}$$

Let $\widetilde{B}$ be an element of $\mathcal{C}([-\tau;0],\mathbb{R}^n)$ whilst z be a continuous stochastic process $z:\Omega\times[-\tau;+\infty]\to\mathbb{R}^n$. For z we define a segment of a trajectory, i.e. continuous $\mathcal{F}_t$-adapted stochastic process $z_t:\Omega\to\mathcal{C}([-\tau;0],\mathbb{R}^n)$, where the following condition holds for $\forall t\geq 0$:

$$z_t(s,\omega) = z(t+s,\omega), \qquad s\in[-\tau;0]. \tag{2}$$

In further notation stochastic variable ω is omitted to simplify a notation e.g.

$$z(t) := z(t,\omega).$$

Let us consider the following class of stochastic differential delayed equations (SDDEs) with distributed delay:

$$\begin{cases} dz(t) = \Big(\widetilde{A}z(t) + \int\limits_{t-\tau}^{t} \widetilde{B}(t-u)z(u)du\Big)dt + \widetilde{C}z(t)dw(t),\\ z\big|_{[-\tau;0]} \equiv z_0,\ z_0\in\mathcal{C}([-\tau;0];\mathbb{R}^n), \end{cases} \tag{3}$$

where $\widetilde{A}$ and $\widetilde{C}$ are real constant matrices and $w(t)$ is n-dimensional Brownian motion. We assume that local Lipschitz and linear growth conditions are satisfied. These conditions

*e-mail: jerzy.klamka@iitis.pl

are indispensable for the existence and uniqueness of a solution of Problem (3). In this paper we assume that kernel B is rational.

Definition 1. A kernel $\widetilde{B}$ is called rational if it can be extended to B that is $n \times n$ matrix - valued function defined on $[0; +\infty)$ for which the Laplace transform

$$\mathcal{L}_t[B(t)](s) := \int_0^{+\infty} B(t)e^{-st}dt \tag{4}$$

is a rational function of s.

The above definition indicates that elements of matrix B may belong to the following set of functions

$$\begin{aligned} \{e^{\alpha t}, t, \cos(\alpha t), \sin(\alpha t), \cosh(\alpha t), \sinh(\alpha t), \\ e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}, \qquad \alpha, \beta \in \mathbb{R} \end{aligned} \tag{5}$$

and also can be a linear, finite combination of the enumerated elements.

The condition that kernel B is rational can be equivalently replaced by a condition that B is a real solution of the linear deterministic differential equation:

$$M_N B^{(N)}(u) + M_{N-1}B^{(N-1)}(u) + \ldots + M_0 B(u) = 0. \tag{6}$$

where $\{M_i\}_{i=0\ldots N} \subset \mathbb{R}$. Problem (3) is worth of further considering due to its numerous applications in finance. The above equation allows to describe dynamics of a financial item (such as price, value, rate of return etc.) taking into account past behaviour (continuously) of the item. An integral smoothes jumps and combined with embedded function B is a representative of a trend for process z. Subsequently, function B allows to make more important some values of z. For example values of B can be weights that make the nearest historical observations more important than older within trend indicating process. Therefore, Problem (3) can successfully embed signals that follow technical analysis (moving averages, exponentially weighted moving averages, trend and many others) into classic stochastic problem that describes dynamics. More details in [4]. The similar problem was considered in [4], but there was a condition that was sufficient for various practical applications of a problem but significantly limited:

$$\dot{B}(u) = MB(u). \tag{7}$$

Nevertheless, utility of the above equation with assumption (6) is also significant and can be used in simple financial or economic models.

3. Extended reduction technique

The main target of this Sec. is to present an effective procedure that reduces a distributed delay to lumped delay. It is worth reminding that if $B(\cdot)$ is rational then there exists a homogenous differential equation (Eq. (6)) that is solved by B. We denote N as an order of this equation. The main idea of the reduction will be based on transformation of Eq. (6) into the set of differential equations. Details of the whole reduction process were described in [8] and [9].

Now we recall two immediate lemmas. Please, find the proofs in [8] and [9].

Lemma 1. Let us consider problem (3). For the aforementioned distributed stochastic delay system and for $i = 1\ldots N+1$ we define the following set of functions

$$\xi_i(t) = A_i z(t) + \int_0^{\tau} B_i(u)z(t-u)du, \tag{8}$$

$$\xi_{N+1}(t) = z(t). \tag{9}$$

Then for $i = N+1\ldots 1$ we get

$$d\xi_i(t) = \xi_{i-1}(t)dt - B_i(\tau)z(t-\tau)dt + C_i z(t)dw(t), \tag{10}$$

where the A_i and $B_i(\cdot)$ satisfy the following conditions (**D** is a derivation operator).

$$A_{i-1} = A_i A + B_i(0), \tag{11}$$

$$B_{i-1}(u) = (\mathbf{D}B_i + A_i B)(u), \tag{12}$$

$$C_i = A_i C, \tag{13}$$

and

$$A_{N+1} = I, \tag{14}$$

$$B_{N+1}(\cdot) = 0, \tag{15}$$

As an immediate consequence we get the second lemma.

Lemma 2. If $B(\cdot)$ is rational, then there exists a positive integer N and matrices X_i, for $i = 1\ldots N+1$, such that

$$\xi_0(t) = -\sum_{i=1}^{N+1} X_i \xi_i(t). \tag{16}$$

Please, find details of proof in [8] and [9]. The main idea of the proof relies on existence of a sequence of matrices $X_1, \ldots, X_N$ such that

$$B_0(u) + \sum_{i=1}^{N} X_i B_i(u) = 0. \tag{17}$$

Now, we define $X_{N+1} = -\sum_{i=0}^{N} X_i A_i$. The below theorem is the main reduction theorem and allows to reduce a distributed delay to point delay.

Theorem 1 (Reduction procedure). If $B(\cdot)$ is a rational kernel, then the stochastic distributed delay problem (3) can be transformed into a stochastic delay equation with point delay in an $(N+1)n$-dimensional space. N is a rank of a homogenous differential equation and $B(\cdot)$ is its solution.

Proof. Below we present a sketch of the proof. Lemma 1 clearly indicates that

$$d\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \vdots\\ \xi_{N+1}(t)\end{bmatrix}=\begin{bmatrix}\xi_0(t)\\ \xi_1(t)\\ \vdots\\ \xi_N(t)\end{bmatrix}dt$$
$$-\begin{bmatrix}B_1(\tau)\\ \vdots\\ B_N(\tau)\\ 0\end{bmatrix}\xi_{N+1}(t-\tau)dt+\begin{bmatrix}C_1\\ \vdots\\ C_N\\ C_{N+1}\end{bmatrix}\xi_{N+1}(t)dw(t). \tag{18}$$

Invoking Lemma 2 we obtain the following equivalent and the unique point-delay form of Problem (3):

$$\underbrace{\begin{bmatrix}d\xi_1(t)\\ d\xi_2(t)\\ \vdots\\ d\xi_{N+1}(t)\end{bmatrix}}_{dx(t)}=$$
$$\underbrace{\begin{bmatrix}-X_1 & -X_2 & \dots & -X_{N+1}\\ I & 0 & \dots & 0\\ & \ddots & & \vdots\\ & & I & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \vdots\\ \xi_{N+1}(t)\end{bmatrix}}_{x(t)}dt \tag{19}$$
$$+\underbrace{\begin{bmatrix}0 & \cdots & 0 & -B_1(\tau)\\ \vdots & & & \vdots\\ 0 & \cdots & 0 & -B_N(\tau)\\ 0 & \cdots & 0 & 0\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}\xi_1(t-\tau)\\ \vdots\\ \xi_N(t-\tau)\\ \xi_{N+1}(t-\tau)\end{bmatrix}}_{y(t)}dt$$
$$+\underbrace{\begin{bmatrix}0 & \cdots & 0 & C_1\\ \vdots & & & \vdots\\ 0 & \cdots & 0 & C_N\\ 0 & \cdots & 0 & C_{N+1}\end{bmatrix}}_{C}\underbrace{\begin{bmatrix}\xi_1(t)\\ \vdots\\ \xi_N(t)\\ \xi_{N+1}(t)\end{bmatrix}}_{x(t)}dw(t).$$

Recapitulating, Problem (3) was reduced to the linear Problem (19) of the form

$$dx(t)=(Ax(t)+By(t))dt+Cx(t)dw(t). \tag{20}$$

It is worth pointing out that the form (19) is an equivalent form of problem (3) and thanks to the reduction process, theorems that work for point-delay theory may be applied to this problem. The next task is finding conditions that have to be satisfied to be able to use point-delay theorems for examination of asymptotic properties of the considered class of delay systems.

4. Stochastic boundedness and stability

The main target of this chapter is to show the conditions that provide asymptotic boundedness, stability, asymptotic stability and existence of the stochastic attractor for the extended Problem (3) with condition that kernel B is rational.

Let us recall the basic definitions. We consider a generalised SDDE with delay τ

$$dx(t)=\nu(x(t),y(t),t)dt+\theta(x(t),y(t),t)dw(t),\quad t\geq 0, \tag{21}$$

where $y(t)=x(t-\tau)$ with the initial condition $x|_{[-\tau;0]}\equiv\zeta_0$ where for ν and θ the local Lipschitz and linear growth conditions hold for x and y that provide existence and uniqueness of a solution $x(t)$.

4.1. Asymptotic boundedness.

Let us recall a difinition of asymptotic boundedness.

Definition 2. Let $p>0$. The SDDE (21) is said to be asymptotically bounded in p-th moment if there is a positive constant H such that

$$\limsup_{t\to\infty}\mathbb{E}|x(t)|^p\leq H,\ \forall\xi_0\in\mathcal{C}([-\tau;0];\mathbb{R}^n), \tag{22}$$

if $p=2$ we say that the SDDE is asymptotically bounded in mean square.

The below theorem presents a criterion that allows to check the asymptotic boundedness of the delay stochastic problem.

Theorem 2. Let $\sigma_0:=\sigma(A)$ denotes a spectrum of matrix A defined in Eq. (19). In case of $\sigma_0\subseteq\mathbb{R}$, a solution of Problem (3) with a condition of rational B is asymptotically bounded in mean square if the following condition holds

$$\lambda_{\max}(A)\leq-\left(\sqrt{\sum_{i=1}^{N}|B_i(\tau)|^2}+\frac{1}{2}\sum_{i=1}^{N+1}|C_i|^2\right), \tag{23}$$

In other cases the following condition

$$\lambda_{\max}(A+A^T)\leq-\left(2\sqrt{\sum_{i=1}^{N}|B_i(\tau)|^2}+\sum_{i=1}^{N+1}|C_i|^2\right), \tag{24}$$

provides an asymptotic boundedness, where $\lambda_{\max}$ is a maximum real part of eigenvalue.

Proof. The main idea of the proof is to show this property for an equivalent problem (19). The below Lemma 3 (proof in [3]) is very useful.

Lemma 3. Let us consider the point delay problem (21) with the initial data $\xi_0\in\mathcal{C}([-\tau,0],\mathbb{R}^n)$. Assume that there exist a function $V\in\mathcal{C}^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ and positive constants $p,\alpha,c_1,c_2,\lambda_1,\lambda_2$ with $\lambda_1>\lambda_2$ such that

$$c_1|x|^p\leq V(x,t)\leq c_2|x|^p,\qquad\forall(x,t)\in\mathbb{R}^n\times\mathbb{R}_+ \tag{25}$$

and

$$LV(x,t)\leq-\lambda_1|x|^p+\lambda_2|y|^2+\alpha,\quad\forall(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+. \tag{26}$$

Then

$$\limsup_{t\to\infty} \mathbb{E}|x(t;\xi)|^p \leq \frac{\alpha}{c_1\lambda}, \qquad \forall\xi \in \mathcal{C}([-\tau,0],\mathbb{R}^n), \quad (27)$$

where

$$\lambda \in (0, \lambda_1 - \lambda_2),$$

is the unique root of the equation

$$\lambda c_2 + \lambda_2 e^{\lambda\tau} = \lambda_1,$$

where

$$\begin{aligned} LV(x,y,t) &= V_t(x,t) + V_x(x,t)f(x,y,t) \\ &+ \frac{1}{2}\text{trace}[g^T(x,y,t)V_{xx}(x,t)g(x,y,t)]. \end{aligned} \quad (28)$$

Proof (Theorem 2). Now we consider a case where $\sigma_0 \subseteq \mathbb{R}$. Let us define the following Lyapunov-type function $V(x,t) = |x|^2$ (then $c_1 = c_2 = 1$).
For further proceeding we need to calculate LV:

$$\begin{aligned} LV(x,t) &= 2x^T(Ax+By) + |Cx|^2 = \\ & 2x^TAx + 2x^TBy + |C|^2|x|^2 \\ &\leq 2x^T(A)x + 2|x||B||y| + |C|^2|x|^2 \\ &\leq 2x^T(A)x + |B|(|x|^2+|y|^2) + |C|^2|x|^2 \\ &\leq (2\lambda_{\max}(A) + |B| + |C|^2)|x|^2 + |B||y|^2. \end{aligned}$$

If

$$-(2\lambda_{\max}(A) + |B| + |C|^2) \geq |B|$$

namely

$$\lambda_{\max}(A) \leq -\left(|B| + \frac{1}{2}|C|^2\right), \quad (29)$$

then the linear SDDE (19), according to Lemma 3, is asymptotically bounded in mean square. It is obvious that λ_{max} is negative.

We need to calculate $|B|$ and $|C|$:

$$|B|^2 = \text{tr}(B^TB) = \sum_{i=1}^{N} |B_i(\tau)|^2 \quad (30)$$

that is equivalent to

$$|B| = \sqrt{\sum_{i=1}^{N} |B_i(\tau)|^2}. \quad (31)$$

Similarly we calculate $|C|$:

$$|C|^2 = \text{tr}(C^TC) = \sum_{i=1}^{N+1} |C_i|^2 \quad (32)$$

that is equivalent to

$$|C| = \sqrt{\sum_{i=1}^{N+1} |C_i|^2}. \quad (33)$$

Recapitulating, for the extended reduction technique condition (29) has the following form

$$\begin{aligned} &\lambda_{\max}\left(\begin{bmatrix} -X_1 & -X_2 & \cdots & -X_{N+1} \\ I & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}\right) \leq \\ &-\left(\sqrt{\sum_{i=1}^{N} |B_i(\tau)|^2} + \frac{1}{2}\sum_{i=1}^{N+1}|C_i|^2\right), \end{aligned} \quad (34)$$

In other cases ($\sigma_0 \subseteq \mathbb{R}$ does not hold) the above inequality has a form

$$\begin{aligned} LV(x,t) &= 2x^T(Ax+By) + |Cx|^2 = \\ & 2x^TAx + 2x^TBy + |C|^2|x|^2 \\ &\leq 2x^T(A)x + 2|x||B||y| + |C|^2|x|^2 \\ &\leq x^T(A+A^T)x + |B|(|x|^2+|y|^2) + |C|^2|x|^2 \\ &\leq (\lambda_{\max}(A+A^T) + |B| + |C|^2)|x|^2 + |B||y|^2. \end{aligned}$$

Recalling Lemma 3 the following inequality

$$-(\lambda_{\max}(A+A^T) + |B| + |C|^2) \geq |B|$$

provides an asymptotic boundedness in mean square. But the inequality is equivalent to

$$\begin{aligned} &\lambda_{\max}\left(\begin{bmatrix} -X_1 & -X_2 & \cdots & -X_{N+1} \\ I & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}\right. \\ &\left. + \begin{bmatrix} -X_1 & -X_2 & \cdots & -X_{N+1} \\ I & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}^T\right) \leq \\ &-\left(2\sqrt{\sum_{i=1}^{N} |B_i(\tau)|^2} + \sum_{i=1}^{N+1}|C_i|^2\right), \end{aligned} \quad (35)$$

what completes the proof.

Please, find examples in chapter 5.

4.2. Moment exponential stability.

Definition 3. For $p > 0$ the SDDE (21) is said to be exponentially stable in p-th moment if the p-th moment Lyapunov exponent is negative, namely

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\log\left(\mathbb{E}|x(t)|^p\right) < 0, \\ &\forall\xi_0 \in \mathcal{C}([-\tau,0],\mathbb{R}^n). \end{aligned} \quad (36)$$

If $p = 2$ we say that the SDDE is exponentially stable in mean square. The below theorem proves that the condition (23) and

(24) is sufficient for the stochastic delay problem (3) to be exponentially stable in mean square.

Theorem 4. Let us consider extended Problem (3) i.e. kernel B is rational. Problem (3) is exponentially stable in mean square if Condition (23) holds (in case of $\sigma_0 \subseteq \mathbb{R}$) or Condition (24) holds in other cases.

Proof. We examine this property for the equivalent crisp delay problem (19) similarly as in the proof of Theorem 2.

To show that the above theorem is true we will use the following criterion on the p-th moment exponential stability (proof in[3]).

Lemma 5. Let us consider the point delay problem (21) with initial data $\xi_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Assume that there exist a function $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ and positive constants $p, c_1, c_2, \lambda_1, \lambda_2$ with $\lambda_1 > \lambda_2$ such that

$$c_1|x|^p \leq V(x,t) \leq c_2|x|^p, \qquad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+$$

and

$$LV(x,t) \leq -\lambda_1|x|^p + \lambda_2|y|^2, \qquad \forall (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+.$$

Then

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}|x(t)|^p\right) \leq -\lambda, \qquad \forall \xi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n),$$

where

$$\lambda \in (0; \lambda_1 - \lambda_2)$$

is the unique root to the equation

$$\lambda c_2 + \lambda_2 e^{\lambda\tau} = \lambda_1,$$

where

$$LV(x,y,t) = V_t(x,t) + V_x(x,t)f(x,y,t) + \frac{1}{2}\text{trace}[g^T(x,y,t)V_{xx}(x,t)g(x,y,t)].$$

Similarly to the proof of Theorem 2 we assume that $V(x,t) := |x|^2$. Calculating $LV(x,t)$ for the equivalent Problem (19) and using matrix inequalities we obtain

$$LV(x,t) \leq (2\lambda_{\max}(A) + |B| + |C|^2)|x|^2 + |B||y|^2,$$

in case of $\sigma_0 \subseteq \mathbb{R}$ or in the other cases

$$LV(x,t) \leq (\lambda_{max}(A + A^T) + |B| + |C|^2)|x|^2 + |B||y|^2.$$

If the following conditions hold:

$$-(2\lambda_{\max}(A) + |B| + |C|^2) \geq |B| \Leftrightarrow$$

$$\lambda_{\max}(A) \leq -(|B| + \frac{1}{2}|C|^2) \Leftrightarrow$$

$$\Leftrightarrow \lambda_{\max}\left(\begin{bmatrix} -X_1 & -X_2 & \cdots & -X_{N+1} \\ I & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}\right) \leq - \left(\sqrt{\sum_{i=1}^{N} |B_i(\tau)|^2} + \frac{1}{2}\sum_{i=1}^{N+1} |C_i|^2\right).$$

and

$$-(\lambda_{\max}(A + A^T) + |B| + |C|^2) \geq |B| \Leftrightarrow$$

$$\lambda_{\max}(A + A^T) \leq -(2|B| + |C|^2) \Leftrightarrow$$

$$\Leftrightarrow \lambda_{\max}\left(\begin{bmatrix} -X_1 & -X_2 & \cdots & -X_{N+1} \\ I & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} + \begin{bmatrix} -X_1 & -X_2 & \cdots & -X_{N+1} \\ I & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}^T\right) \leq - \left(2\sqrt{\sum_{i=1}^{N} |B_i(\tau)|^2} + \sum_{i=1}^{N+1} |C_i|^2\right),$$

then the assumptions of Lemma 5 are satisfied and therefore the theorem is true.

Examples in Sec. 5.

4.3. Almost sure exponential stability.

Definition 4. The SDDE (21) is said to be almost surely exponentially stable if the sample Lyapunov exponent is almost surely (a.s.) negative, namely

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(|x(t)|\right) < 0 \tag{37}$$

for any $\xi_0 \in \mathcal{C}([-\tau; 0], \mathbb{R}^n)$.

The following lemmas are crucial to show that the class of differential delay systems (3) is almost sure asymptotically stable if the well-known conditions (23) and (24) hold.
In the previous paragraphs we proved that conditions (23) and (24) imply asymptotic boundedness and mean square exponential stability. The below two lemmas confirm that these conditions provide almost surely asymptotical stability but the additional inequalities must hold.

Lemma 6. Assume that there exists a constant K for the Eq. (6) such that

$$\begin{aligned} &|f(x,y,t)| + |g(x,y,t)| \leq K(|x| + |y|), \\ &\forall (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+. \end{aligned} \tag{38}$$

Let $p > 0$, $\lambda > 0$ and the initial data $\xi \in \mathcal{C}$. If

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}|x(t)|^p\right) < -\lambda, \qquad \forall \xi_0 \in \mathcal{C}, \tag{39}$$

then

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(|x(t)|\right) < -\frac{\lambda}{p}, a.s. \tag{40}$$

Lemma 7. Let (38) hold. Assume also that Conditions (23) or (24) of Theorem (4.3) are satisfied. Then Problem (3) is almost surely exponentially stable.

Lemma (4.7) shows that Condition (38) is always valid.

Lemma 8. For Eq. (19) Property (38) holds.

Proof (Lemma 8). We define K as follows

$$K := \max\{|A| + |C|, |B|\}.$$

For Eq. (3) the inequality (38) is in the form:

$$|Ax + By| + |Cx| \le |A||x| + |B||y| + |C||x| = (|A| + |C|)|x| + |B||y| \le K(|x| + |y|),$$

what completes the proof.

Recapitulating, Lemma 8 provides that inequality (38) holds. Simultaneously, Theorem 4 indicates that Conditions (23) and (24) are sufficient for holding (39). From Lemma 8 we have that under Conditions (23) and (24) the delay differential system is almost surely exponential stable.

5. Examples and applications

5.1. Example 1. We shall prove that the problem that has been considered in [1] is a special case of Problem (3). Therefore, let us consider the following class of equations

$$\begin{cases} dz(t) = \Big(\widetilde{A}z(t) + \int\limits_{t-\tau}^{t} \widetilde{B}(t-u)z(u)du\Big)dt + \widetilde{C}z(t)dw(t), \\ z\big|_{[-\tau;0]} \equiv z_0, \qquad z_0 \in \mathcal{C}([-\tau;0];\mathbb{R}^n), \end{cases} \tag{41}$$

and $\widetilde{B}(u)$ satisfies

$$\dot{\widetilde{B}}(u) = M\widetilde{B}(u), \tag{42}$$

where M, $\widetilde{A}$ and $\widetilde{C}$ are constant matrices and $w(t)$ is n-dimensional Brownian motion whilst the trace norm is defined as follows $|M| = \sqrt{\mathrm{trace}(M^TM)}$.

We define a function $v : [-\tau;\infty] \to \mathbb{R}^n$ as a superposition of B function and z.

$$v(t) := S^{-1}(\widetilde{B} * z)(t),$$

where S is an invertible matrix. It means that v has the following form

$$v(t) := S^{-1} \int\limits_{t-\tau}^{t} \widetilde{B}(u)z(t-u)du. \tag{43}$$

Let us substitute $v(t)$ to Problem (41) and use Duhamel differential to calculate $dv(t)$ The simplified reduction procedure presented in [1] leads to the following matrix equation with lumped delay

$$\begin{bmatrix} dz(t) \\ dv(t) \end{bmatrix} = \left(\begin{bmatrix} \widetilde{A} & S \\ S^{-1}\widetilde{B}(0) & S^{-1}MS \end{bmatrix}\begin{bmatrix} z(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -S^{-1}\widetilde{B}(\tau) & 0 \end{bmatrix}\begin{bmatrix} z(t-\tau) \\ v(t-\tau) \end{bmatrix}\right)dt + \begin{bmatrix} \widetilde{C} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z(t) \\ v(t) \end{bmatrix}dw(t), \tag{44}$$

where function $v : [-\tau;\infty] \to \mathbb{R}^n$ is defined as follows

$$v(t) := S^{-1} \int\limits_{t-\tau}^{t} \widetilde{B}(u)z(t-u)du. \tag{45}$$

Below we attempt to conduct the extended reduction process for Eq. (3) and our task is to check whether Eq. (26) is a special form of the Eq. (3) obtained via extended reduction.

For $N = 1$ we define:

$$\xi_1(t) := A_1 z(t) + \int\limits_0^{\tau} B_1(u)z(t-u)du, \tag{46}$$

$$\xi_2(t) := z(t). \tag{47}$$

Applying the reduction process presented in Sec. 3 we calculate

$$B_2(u) = 0, \qquad A_2 = 1, \tag{48}$$

$$B_1(u) = \dot{B}_2(u) + A_2B(u) = B(u), \quad A_1 = A_2A + B_2(0) = A, \tag{49}$$

$$B_0(u) = \dot{B}_1(u) + A_1B(u) = MB(u) + AB(u) = (M + A)B(u), \tag{50}$$

$$A_0 = A_1A + B(0) = A^2 + B(0). \tag{51}$$

Following Verriest's theory ([9]) we can find matrices X_i. We put

$$X_0 = 1. \tag{52}$$

In order to calculate the matrix X_1 we consider the following equation

$$B_0(u) = -X_1B_1(u). \tag{53}$$

Applying Eqs. (32), (33), (34) and (35) we obtain

$$(M + A)B(u) = -X_1B(u). \tag{54}$$

Immediately we have

$$X_1 = -M - A. \tag{55}$$

In order to calculate X_2 we use Lemma 2:

$$X_2 = -X_0A_0 - X_1A_1 = -A_0 + (M + A)A_1 = MA - B(0). \tag{56}$$

Finally, we obtain the reduced form (via extended reduction) of Eq. (26).

$$\begin{bmatrix} d\xi_1(t) \\ d\xi_2(t) \end{bmatrix} = \left(\begin{bmatrix} M + A & B(0) - MA \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 0 & -B(\tau) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1(t-\tau) \\ \xi_2(t-\tau) \end{bmatrix}\right)dt + \begin{bmatrix} 0 & C_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}dw(t). \tag{57}$$

It is worth reminding that the Eq. (57) is an equivalent form of Eq. (44):

$$\begin{bmatrix} d\xi_2(t) \\ d\xi_1(t) \end{bmatrix} = \left(\begin{bmatrix} \widetilde{A} & S \\ S^{-1}\widetilde{B}(0) & S^{-1}MS \end{bmatrix}\begin{bmatrix} \xi_2(t) \\ \xi_1(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -S^{-1}\widetilde{B}(\tau) & 0 \end{bmatrix}\begin{bmatrix} \xi_2(t-\tau) \\ \xi_1(t-\tau) \end{bmatrix}\right)dt + \begin{bmatrix} \widetilde{C} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_2(t) \\ \xi_1(t) \end{bmatrix}dw(t). \tag{58}$$

It is easy to find non-singular transformations that allow to convert matrix

$$\begin{bmatrix} \widetilde{A} & S \\ S^{-1}\widetilde{B}(0) & S^{-1}MS \end{bmatrix} \text{ into } \begin{bmatrix} M+A & B(0)-MA \\ 1 & 0 \end{bmatrix}.$$

The aforementioned considerations clearly indicate that a sufficient condition for stability for the generalized problem i.e. B is rational

$$\lambda_{\max}\left(\begin{bmatrix} \widetilde{A} & S \\ S^{-1}\widetilde{B}(0) & S^{-1}MS \end{bmatrix}\right) \leq -\left(|S^{-1}\widetilde{B}(\tau)| + \frac{1}{2}|\widetilde{C}|^2\right), \tag{59}$$

is also a sufficient condition for the problem described in article [1] and for $A = 0$ there is no difference between them.

5.2. Example 2. Let us consider a numerical example with $\tau = 1$

$$dz(t) = -\frac{1}{4}\left(\int\limits_{t-1}^{t} e^{-(t-u)}z(u)du\right)dt + \widetilde{C}z(t)dw(t), \tag{60}$$

where $\widetilde{A} \equiv 0$ and $\widetilde{B}(u) = -\frac{1}{4}e^{-u}$, $M = -1$ that follows from Condition (6). The matrix from Eq. (19) has the following form

$$\begin{bmatrix} -X_1 & -X_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{4} \\ 1 & 0 \end{bmatrix} \tag{61}$$

There are two eigenvalues for the aforementioned matrix (49).

$$\lambda_1 = \lambda_2 = -\frac{1}{2}. \tag{62}$$

The negative values of eigenvalues clearly indicate that for sufficiently small $|\widetilde{C}|$ the inequality (48) is satisfied and therefore assumptions of theorems of boundedness and moment exponential stability are fulfilled.

5.3. Example 3. The below example presents financial applications of class (3). The whole equation was described in details in [4]. It is worth reminding that one of the most necessary tools for options pricing is Black-Scholes equation of the following form

$$dS(t) = rS(t)dt + \sigma S(t)dw(t), \tag{63}$$

where S is an underlying asset, $S(t)$ denotes a price of underlying instrument in time t, σ is a volatility whilst r denotes risk-free rate. The main task of Eq. (63) is to describe dynamics of an underlying asset $S(t)$ of a derivative instrument. The equation is analytically solvable and allows to price a derivative in risk-neutral world that provides a unique price of a derivative security. Empirical research confirms that Eq. (63) does not describe all classes of commonly traded securities properly. Therefore, it should be better to fit instruments' properties. Thus, there exists separate models for various assets. The main motivation to use Eq. (3) instead of (63) has been taking into consideration signals of technical analysis. Technical analysis claims that the future prices can be forecasted with past behavior of an instrument. Therefore, the delay stochastic equation of the form (3) is a reasonable class of equations for stock prices that take into consideration past signals. Generality of function $\widetilde{B}$ and $\widetilde{A}$ allows to use various techniques of a technical analysis. The wider range of $\widetilde{B}$ and $\widetilde{A}$, the more acceptable tools can be used. Detailed discrete and continuous models were wider described and investigated in [4]. Moreover, options were priced in [4] in the delayed model (exponential moving averages were sources of technical analysis signals) and results were compared to classic theory. The general assumption was that the option prices calculated on a delayed model indicated much better long-term value of an option and became more useful for analysts who examine long-term portfolio value or valuate a portfolio holder. Stability of such models is also an important issue. The below equation presents one model

$$dS(t) = r\left(\frac{1}{\tau}\int\limits_{-\tau}^{0} S_t(u)du\right)dt + \sigma\left(\frac{1}{\tau}\int\limits_{-\tau}^{0} S_t(u)du\right)dw(t), \tag{64}$$

where $\frac{1}{\tau}\int\limits_{-\tau}^{0} S_t(u)du$ is a continuous moving average.

This subsection is only one of many applications of class (3). The similar class of stochastic distributed delay equations are also used in analysis of computer networks [14].

6. Recapitulation

The main motivation for this paper has been an extension of a range of tools that can be used to analyse asymptotic properties of the problem that can be effectively used in various models and allows to describe dynamics taking into account past behaviour of the stochastic process. In finance the aforementioned model allows to take into consideration technical analysis signals, for instance exponential moving averages etc. The article [1] presented tools that can be used to examine properties such as stability, asymptotic stability and boundedness of a special class of distributed delay stochastic equations that were strictly limited by a special condition. In the above paper there has also been showed a special inequality that provided ability of using point delay tools to analyse distributed delay stochastic equations. This article extends considered class on rational functions and presents conditions that should be satisfied to be able to use theorems inherited from crisp delay "world" to investigate stability, stochastic boundedness and other properties.

Acknowledgements. This work was supported by the National Science Centre under grants N N514 513340 and N N519 442339.

References

[1] B. Ficak, "Point delay methods applied to the investigation of stability for a class of distributed delay systems", *Systems and Control Letters* 56, 223–229 (2007).

[2] X. Mao, "Attraction, stability and boundedness for stochastic differential equations", *Nonlinear Anal.* 47 (7), 4795–4806 (2001).

[3] X. Mao, (1994), "Exponential stability of stochastic differential equations", in *Monographs and Textbooks in Pure and Applied Mathematics Series*, pp. 307, Marcel Dekker, New York, 1994.

[4] B. Ficak, "Application and mathematical properties of time delay diffusion models", *J. Mathematical Finance* (2013), (to be published).

[5] M. Buslowicz, "Robust stability of the new general 2D model of a class of continuous-discrete linear systems", *Bull. Pol. Ac.: Tech.* 58 (4), 561–565 (2010).

[6] M. Buslowicz, "Robust stability of positive discrete-time linear systems of fractional order", *Bull. Pol. Ac.: Tech.* 58 (4), 567–572 (2010).

[7] M. Buslowicz, "Robust stability of positive continuous-time linear systems with delays", *Int. J. Appl. Math. Comput. Sci.* 20 (4), 665–670 (2010).

[8] E. Verriest, "Linear systems with rational distributed delay: reduction and stability", *Proc. 1999 Eur. Control Conf.* DA-12, CD-ROM (1999).

[9] E. Verriest, "Stochastic stability of a class of distributed delay system", *44th IEEE Conf. on Decision and Control* 1, CD-ROM (2005).

[10] P. Florchinger, "Lyapunov-like techniques for stochastic stability", *SIAM J. Control and Optimization* 33, 1151–1169 (1995).

[11] V.B. Kolmanovskii, V.R. Nosov, *Stability of Functional Differential Equations*, Academic Press, New York, 1986.

[12] X. Mao, *Stochastic Differential Delayed Equations*, Springer, Berlin, 2003.

[13] X. Mao, *Stochastic Differential Equations and Applications*, pp. 366, Horwood Publishing, Chichester, 1997.

[14] J. Tancula and J. Klamka, "Examination of robust stability of computer networks", *Proc. 6-th Working Int. Conf. "Performance Modelling and Evaluation of Heterogenous Networks"* 1, CD-ROM (2010).