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A new form of Boussinesq equations for long waves in water of non-uniform depth

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Abstract. The paper describes the non-linear transformation of long waves in shallow water of variable depth. Governing equations of the problem are derived under the assumption that the non-viscous fluid is incompressible and the fluid flow is a rotation free. A new form of Boussinesq-type equations is derived employing a power series expansion of the fluid velocity components with respect to the water depth. These non-linear partial differential equations correspond to the conservation of mass and momentum. In order to find the dispersion characteristic of the description, a linear approximation of these equations is derived. A second order approximation of the governing equations is applied to study a time dependent transformation of waves in a rectangular basin of water of variable depth. Such a case corresponds to a spatially periodic problem of sea waves approaching a near-shore zone. In order to overcome difficulties in integrating these equations, the finite difference method is applied to transform them into a set of non-linear ordinary differential equations with respect to the time variable. This final set of these equations is integrated numerically by employing the fourth order Runge – Kutta method.

Key words: long wave, Boussinesq equation, wave transformation, variable water depth.

1. Introduction

Water surface waves propagating from a deep water to a shallow water undergo changes resulting from variation of the water depth. Usually, lengths of such waves are large compared to the water depth, and the wave heights are of order of the shallow water depth. With respect to such conditions, in description of the phenomenon, we resort to Boussinesq equations which are basically a shallow water approximation to fully non-linear dispersive water waves. These equations enable us to eliminate the vertical dimension in the description of the phenomenon, i.e. a three dimensional problem is reduced to a two-dimensional one. The standard Boussinesq equations include the lowest order effects of non-linearity and frequency dispersion, and therefore, these equations are restricted to long waves propagating in a shallow water (Wei et al. [1]). In literature on the subject there is a number of ways of derivation of Boussinesq-type equations. Such equations for water of variable depth were derived by Peregrine [2], who introduced a depth averaged velocity as a dependent variable. He derived a system of equations for water of variable depth using an expansion procedure with respect to a small parameter. Since that work of Peregine, the depth averaging procedure, applied to the continuity and momentum equations, has become a standard in the derivation of Boussinesq-like equations (Nwogu [3]). Another, classical approach to the derivation of these equations is to follow the Laplace equation for the velocity potential, combined with boundary conditions at the bottom and the free surface of a fluid domain (Volcinger et al. [4], Madsen et al. [5]). In such a formulation the potential function is expressed in the form of a power series expansion with respect to the water depth (Wei et al. [1]). A detailed discussion on the description of the long waves propagating in fluid of variable depth may be found in Dingemans' monograph [6], which also contains a vast bibliography on the subject. Boussinesq-type equations are also discussed in Madsen and Schaffer [7], where a number of formulation of the problem, known from the literature on the subject, is reviewed. Like in the book of Volcinger et al. [4], the authors discussed the velocity potential formulation in terms of an infinite power series expansion.

As it has been mentioned above, the standard Boussinesq equations are restricted to a shallow water and long waves for which the assumption of weak dispersion and weak non-linearity is justified. In order to extend the range of applicability of the equations to deeper water, attempts have been made to improve the dispersion characteristic of these equations.

Witting [8] used the depth-averaged momentum equation for one horizontal direction, expressed in terms of the velocity at the free surface. He improved the dispersion characteristic of Boussinesq-type equations by retaining terms up to the fourth order in the Taylor's series expansion of the velocity, with coefficients of this expansion determined to give the best linear dispersion characteristics. That method however, cannot be easily extended to a more general two-dimensional case of the variable water depth. Madsen et al. [5] improved the dispersion characteristic of Boussinesq-type equations by adding a third order term to the momentum equation written for a fluid with a horizontal bottom. This term, derived from the long wave equations, was chosen to give the best possible linear dispersion relation. More recently, Nwogu [3] derived a new set of Boussinesq equations for water of variable depth, in which the velocity at a certain distance from the still water level has been used as the velocity variable instead

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of the commonly used depth-averaged velocity. The vertical velocity component was assumed to vary linearly over the water depth. In this way, the linear dispersion characteristic of Boussinesq-type equations has been improved. A modification of Boussinesq equations given by Nwogu is presented in Chen and Liu [9], where the derivation is based on a velocity potential at an arbitrary elevation and on a displacement of the free surface. For regular waves, consisting of a finite number of harmonics, a parabolic approximation of governing equations of the problem is derived.

In this paper an alternative derivation of Boussinesq-type equations for long waves propagating in water of variable depth is considered. In the presented approach, the three components of the velocity field are consistently expressed in the form of a power series expansion with respect to the water depth. As compared to the above mentioned papers, the formulation presented in this paper is unified and thus its accuracy in description of the phenomenon, together with linear dispersion characteristics, depends solely on the order of approximation, i.e. on a number of terms taken into account in the expansion procedure applied. In order to illustrate applicability of the equations derived, some examples of solutions of these equations for a transformation of waves in the two dimensional fluid domain are given.

2. Governing equations of the problem – preliminary remarks

In order to make the further discussion clear, let us consider the case shown schematically in Fig. 1.

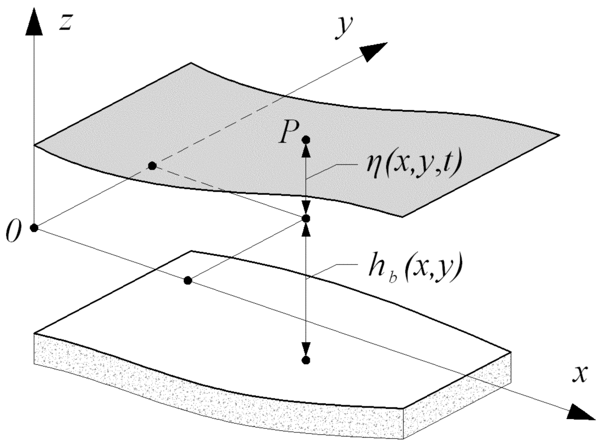

Fig. 1. A long wave propagating in fluid of variable depth

A time dependent wave of finite amplitude propagates in fluid over uneven bottom. The three components of the velocity field correspond to the Cartesian system of coordinate axes. It is assumed the potential motion of an inviscid, incompressible fluid. The governing equations of the fluid motion are the momentum equations and the equation of continuity. The latter equation reads

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

where $\vec{U} = (u, v, w)$ is the fluid velocity.

For the assumed potential motion, the following conditions hold:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0. \tag{2}$$

With regard to these conditions, the momentum (Euler's) equations are written in the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right) + \frac{\partial P}{\partial x} = 0,\\ &\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right) + \frac{\partial P}{\partial y} = 0,\\ &\frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right) + \frac{\partial P}{\partial z} + g^* = 0, \end{aligned} \tag{3}$$

where hereinafter $P = p/\rho$ is the pressure function, $\rho = const.$ is the fluid density and g^* is the gravitational acceleration.

For convenience, the continuity equation can also be expressed as

$$\nabla \cdot \vec{u} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

where $\vec{u} = (u, v)$ denotes the horizontal velocity and $\nabla = (\partial/\partial x + \partial/\partial y)$ means the nabla operator.

The solution of the momentum equations should satisfy dynamic and kinematical boundary conditions at the free surface, and a kinematical boundary condition at the bottom of the fluid domain. The boundary condition at the fluid bottom at $z = -h_b(x, y)$ is

$$w + \nabla h_b \cdot \vec{u}\big|_{z=-h_b} = w + ma \cdot u + mb \cdot v\big|_{z=-h_b} = 0, \tag{5}$$

where $h_b(x, y)$ is the still water depth, and

$$ma = \frac{\partial h_b}{\partial x}, \qquad mb = \frac{\partial h_b}{\partial y} \tag{6}$$

are slopes of the bottom.

At the free surface of the fluid, we have the kinematical boundary condition

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}u + \frac{\partial \eta}{\partial y}v - w\bigg|_{z=\eta(x,y,t)} = 0, \tag{7}$$

where

$$\eta(x, y, t) = h_w(x, y, t) - h_b(x, y) \tag{8}$$

is the free surface elevation, and $h_w = h(x, y, t)$ denotes the current water depth.

From substitution of (8) into equation (7) the following relation is obtained:

$$\frac{\partial h}{\partial t} + \nabla(h - h_b) \cdot \vec{u} - w\bigg|_{z=\eta} = 0 \tag{9a}$$

or

$$\frac{\partial h}{\partial t} + (h_{,x} - ma)u + (h_{,y} - mb)v - w\bigg|_{z=\eta} = 0. \tag{9b}$$

The subscripts x and y in this equation denote the partial derivatives with respect to x- and y coordinates. These last conditions are supplemented by the dynamic boundary condition at the free surface

$$P = p/\rho|_{z=\eta} = const. \tag{10}$$

In this equation p means the atmospheric pressure, taken here as a constant.

The directional derivative of the pressure function with respect to arc length on the free surface gives the condition

$$\nabla P + \frac{\partial P}{\partial z}\nabla(h - h_b)\bigg|_{z=\eta} = 0, \tag{11}$$

which, for convenience, is rewritten in the form of two equations

$$\begin{aligned} \frac{\partial P}{\partial x} + (h_{,x} - ma)\frac{\partial P}{\partial z}\bigg|_{z=\eta} &= 0,\\ \frac{\partial P}{\partial y} + (h_{,y} - mb)\frac{\partial P}{\partial z}\bigg|_{z=\eta} &= 0. \end{aligned} \tag{12}$$

From the third momentum equation the following relation results

$$\frac{\partial P}{\partial z}\bigg|_{z=\eta} = -\left[\frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}(\vec{U})^2 + g^*\right]\bigg|_{z=\eta}. \tag{13}$$

Equations (12) and (13) enable to eliminate the pressure function from equations describing momentum equations at the free surface. On account of relations (12) and (13) the momentum equations lead to the following set of equations

$$\begin{aligned} &\frac{\partial u}{\partial t} + (h_x - ma)\left(g^* + \frac{\partial w}{\partial t}\right) + \\ &+ \frac{1}{2}\left[\frac{\partial}{\partial x}(\vec{U})^2 + (h_x - ma)\frac{\partial}{\partial z}(\vec{U})^2\right]\bigg|_{z=\eta} = 0,\\ &\frac{\partial v}{\partial t} + (h_y - mb)\left(g^* + \frac{\partial w}{\partial t}\right) + \\ &+ \frac{1}{2}\left[\frac{\partial}{\partial y}(\vec{U})^2 + (h_y - mb)\frac{\partial}{\partial z}(\vec{U})^2\right]\bigg|_{z=\eta} = 0. \end{aligned} \tag{14}$$

Equations (14) together with equations (5) and ((9b)) form the basic system of the partial differential equations of the problem considered. In order to find solutions of these equations we resort to power series representations of the velocity components.

3. Power series expansions of the velocity components

In what follows we consider the power series expansion of the velocity components

$$\begin{aligned} u = \sum_{n=0}^{\infty}(z + h_b)^n f^n,\ v = \sum_{n=0}^{\infty}(z + h_b)^n g^n,\\ w = \sum_{n=0}^{\infty}(z + h_b)^n \varphi^n, \end{aligned} \tag{15}$$

where $f^n = f^0, f^1, f^2, \cdots$, $g^n = g^0, g^1, g^2, \cdots$ and $\varphi^n = \varphi^0, \varphi^1, \varphi^2, \cdots$ are unknown functions dependent on (x, y, t). Hereinafter the superscripts of the functions, i.e. f^n, g^n and φ^n, denote the subsequent functions (do not denote powers of these functions).

From substitution of these equations into the boundary condition at the bottom of the fluid (at $z = -h_b$), the following relations result

$$z = -h_b,\ \to\ u = f^0,\ v = g^0,\ w = \varphi^0$$
$$\text{and } \varphi^0 = -\left(\frac{\partial h_b}{\partial x}f^0 + \frac{\partial h_b}{\partial y}g^0\right) = -(ma \cdot f^0 + mb \cdot g^0). \tag{16}$$

Substitution of the first and third relation (15) into the second of (2) gives

$$\sum_{n=1}^{\infty}(z + h_b)^{n-1}\left[nf^n - \left(\varphi^{n-1}_{,x} + nma\varphi^n\right)\right] = 0. \tag{17}$$

From this equation, the following formula is derived:

$$\varphi^{n-1}_{,x} + nma\varphi^n - nf^n = 0,\ n = 1, 2, \cdots \tag{18}$$

In a similar way, from the third of Eq. (2) one obtains

$$\varphi^{n-1}_{,y} + nmb\varphi^n - ng^n = 0,\ n = 1, 2, \cdots \tag{19}$$

Substitution of Eq. (15) into the continuity gives

$$\sum_{1}^{\infty}(z+h_b)^{n-1}\left(f^{n-1}_{,x} + nmaf^n + g^{n-1}_{,y} + nmbg^n + n\varphi^n\right) = 0. \tag{20}$$

From this relation the following formula is obtained

$$\begin{aligned} \varphi^n = -\frac{1}{n}\left(f^{n-1}_{,x} + g^{n-1}_{,y}\right) - (maf^n + mbg^n),\\ n = 1, 2, \cdots \end{aligned} \tag{21}$$

With the above formulae, the set of the functions $\varphi^0, \varphi^1, \varphi^2, \cdots$ may be expressed in terms of the functions $f^0, f^1, f^2, \cdots$ and $g^0, g^1, g^2, \cdots$ in the way as follows

$$\begin{aligned} \varphi^0 &= -\nabla h_b \vec{f}^0,\\ \varphi^n &= -\left(\frac{1}{n}\nabla \cdot \vec{f}^{n-1} + \nabla h_b \cdot \vec{f}^n\right),\ n = 1, 2, 3, \cdots \end{aligned} \tag{22}$$

where $\vec{f}^n = (f^n, g^n)$.

On the other hand, Eqs. (18) and (19) give

$$\vec{f}^n = \frac{1}{n}\nabla\varphi^{n-1} + \nabla h_b\varphi^n, \qquad n = 1, 2, \cdots \tag{23}$$

Equations (22) and (23) allow to write the recurrence formulae

$$\begin{aligned} \varphi^0 &= -\nabla h_b \vec{f}^0,\\ \varphi^n &= -\frac{1}{n}\frac{1}{1 + (\nabla h_b)^2}\left(\nabla \cdot \vec{f}^{n-1} + \nabla h_b \cdot \nabla\varphi^{n-1}\right),\\ \vec{f}^n &= \frac{1}{n}\nabla\varphi^{n-1} + \nabla h_b\varphi^n. \end{aligned} \tag{24}$$

Substitution of Eqs. (22) into the third formula (15) gives the new form of the vertical component of the velocity field

$$w = -\sum_{n=1}^{\infty}\left[(z+h_b)^{n-1}\nabla h_b \cdot \vec{f}^{n-1} + \frac{1}{n}(z+h_b)^n\,\nabla \cdot \vec{f}^{n-1}\right]. \tag{25}$$

For $z = \eta$, the last relation reads

$$w = -\sum_{n=1}^{\infty}\left(h^{n-1}\nabla h_b \cdot \vec{f}^{n-1} + \frac{1}{n}h^n \nabla \cdot \vec{f}^{n-1}\right). \tag{26}$$

All of the above formulae have been derived under the assumption that the series describing the velocity components are convergent. Moreover it is expected that the subsequent terms of the series rapidly decrease with increasing number of terms taken into account. The last feature is especially important in application of the procedure to a description of a specific problem, since it enables us to take only a few lowest order term into account and obtain a solution of acceptable accuracy. An estimation of convergences of these series and behaviour of their subsequent terms is given in the appendix.

With the formulae obtained, all the functions considered may be expressed in terms of the single vector function $\vec{f}^0(x, y, t)$ together with its spatial partial derivatives ranging from one to infinity. In practical calculations however, in order to simplify the description, we usually confine our attention to a few lowest order term derivatives of the component functions. In a description of long waves, propagating in a fluid with small variation of its depth, it is justified to ignore higher order terms and products of space derivatives. In order to make the further discussion clear, we attach here only the first three terms of the velocity series. For instance, following the recurrence formulae, we have

$$\begin{aligned}
&\varphi^0 = -\left(maf^0 + mbg^0\right),\\
&\varphi^1 = -\alpha\left(f^0_{,x} + g^0_{,y} + ma\varphi^0_{,x} + mb\varphi^0_{,y}\right)\\
&= -\alpha\left[f^0_{,x} + g^0_{,y} - ma\left(maf^0_{,x} + mbg^0_{,x}\right) + \right.\\
&\left. -mb\left(maf^0_{,y} + mbg^0_{,y}\right)\right)\right] \cong -\alpha\left(f^0_{,x} + g^0_{,y}\right),\\
&f^1 = \varphi^0_{,x} + ma\varphi^1 = -\frac{\partial}{\partial x}\left(maf^0 + mbg^0\right) +\\
&-\alpha ma\left(f^0_{,x} + g^0_{,y}\right) \cong -\left(maf^0_{,x} + mbg^0_{,x}\right) +\\
&-\alpha ma\left(f^0_{,x} + g^0_{,y}\right) = -\left[ma(1+\alpha)f^0_{,x} + mbg^0_{,x} + \right.\\
&\left. + \alpha mag^0_{,y}\right],\\
&g^1 = \varphi^0_{,y} + mb\varphi^1 = -\frac{\partial}{\partial y}\left(maf^0 + mbg^0\right) +\\
&-\alpha mb\left(f^0_{,x} + g^0_{,y}\right) =\\
&\cong -\left[maf^0_{,y} + \alpha mbf^0_{,x} + mb(1+\alpha)g^0_{,y}\right],
\end{aligned} \tag{27}$$

where

$$\alpha = \frac{1}{1 + (\nabla h_b)^2} = \frac{1}{1 + ma^2 + mb^2}. \tag{28}$$

For a specific problem considered, these formulae may be further simplified by assuming $\alpha = 1$.

Having the first order components, one may derive the second order approximations

$$\begin{aligned}
&\varphi^2 = -\frac{1}{2}\alpha\left(f^1_{,x} + g^1_{,y} + ma\varphi^1_{,x} + mb\varphi^1_{,y}\right)\\
&\cong \frac{1}{2}\alpha\left[ma(1+2\alpha)f^0_{,xx} + mbg^0_{,xx} + 2\alpha mbf^0_{,xy} + \right.\\
&\left. + 2\alpha mag^0_{,xy} + maf^0_{,yy} + mb(1+2\alpha)g^0_{,yy}\right],\\
&f^2 = \frac{1}{2}\varphi^1_{,x} + ma\varphi^2 \cong -\frac{1}{2}\alpha\left(f^0_{,xx} + g^0_{,xy}\right),\\
&g^2 = \frac{1}{2}\varphi^1_{,y} + mb\varphi^2 \cong -\frac{1}{2}\alpha\left(f^0_{,xy} + g^0_{,yy}\right).
\end{aligned} \tag{29}$$

In a similar way, the third order functions are derived

$$\begin{aligned}
&\varphi^3 = -\frac{1}{3}\alpha\left(f^2_{,x} + g^2_{,y} + ma\varphi^2_{,x} + mb\varphi^2_{,y}\right)\\
&\cong \frac{1}{6}\alpha^2\left(f^0_{,xxx} + g^0_{,xyx} + f^0_{,xyy} + g^0_{,yyy}\right)\\
&= \frac{1}{6}\alpha^2\left(\frac{\partial}{\partial x}\nabla^2 f^0 + \frac{\partial}{\partial y}\nabla^2 g^0\right),\\
&f^3 = \frac{1}{3}\varphi^2_{,x} + ma\varphi^3 \cong \frac{1}{6}\alpha\left[ma(1+3\alpha)f^0_{,xxx} + mbg^0_{,xxx}\right.\\
&+2\alpha mbf^0_{,xxy} + +3\alpha mag^0_{,xxy} + ma(1+\alpha)f^0_{,xyy}\\
&\left. +mb(1+2\alpha)g^0_{,xyy} + \alpha mag^0_{,yyy}\right],\\
&g^3 = \frac{1}{3}\varphi^2_{,y} + mb\varphi^3 \cong \frac{1}{6}\alpha\left[ma(1+2\alpha)f^0_{,xxy}\right.\\
&+mb(1+\alpha)g^0_{,xxy} + 3\alpha mbf^0_{,xyy} + 2\alpha mag^0_{,xyy}\\
&\left. +maf^0_{,yyy} + mb(1+3\alpha)g^0_{,yyy} + \alpha mbf^0_{,xxx}\right].
\end{aligned} \tag{30}$$

It may be seen that higher order components of the solution are described by higher order space derivatives of the fundamental functions $f^0(x, y, t)$ and $g^0(x, y, t)$. With regard to these functions, one may calculate the approximated vertical component of the velocity field

$$\begin{aligned}
&w \cong -\left(maf^0 + mbg^0\right) - \alpha(z + h_b)\left(f^0_{,x} + g^0_{,y}\right)\\
&+\frac{1}{2}\alpha(z + h_b)^2\left[ma(1+2\alpha)f^0_{,xx} + mbg^0_{,xx}\right.\\
&+2\alpha mbf^0_{,xy} + 2\alpha mag^0_{,xy} + maf^0_{,yy}\\
&\left. +mb(1+2\alpha)g^0_{,yy}\right]\\
&+\frac{1}{6}\alpha^2(z + h_b)^3\left(\frac{\partial}{\partial x}\nabla^2 f^0 + \frac{\partial}{\partial y}\nabla^2 g^0\right).
\end{aligned} \tag{31}$$

This equation may be further approximated to the following form:

$$\begin{aligned}
&w \cong -\left(maf^0 + mbg^0\right) - \alpha(z + h_b)\left(f^0_{,x} + g^0_{,y}\right)\\
&+\frac{1}{6}\alpha^2(z + h_b)^3\left(\frac{\partial}{\partial x}\nabla^2 f^0 + \frac{\partial}{\partial y}\nabla^2 g^0\right).
\end{aligned} \tag{32}$$

4. Boussinesq – type equations for long waves

In the preceding section, components of the power series solution have been derived. These components are described by complicated formulae dependent on the fundamental functions of the problem considered. With the help of these functions, the problem description has been reduced to the three independent functions: $f^0(x, y, t)$, $g^0(x, y, t)$ and $h(x, y, t)$. All theses functions should satisfy Eqs. (9) and (14). From substitution of the descriptions (25), (27) and (28) into these equations, final equations of the problem can be derived. For practical reasons, because of the complicated structure of these equations, in derivation of the final equations certain approximations into the description are introduced. In principle, we neglect products of derivatives of the basic functions and the bottom slope. Such a simplification is justified because these products are small numbers compared with remaining terms of the description. In this way, from the first Eq. (14) the following equation is obtained

$$\begin{aligned}&\frac{\partial f^0}{\partial t}-\frac{1}{2}\alpha h^2\frac{\partial}{\partial t}\left(f^0_{,xx}+g^0_{,xy}\right)+g\left(h_{,x}-ma\right)\\&\quad+\frac{1}{2}\frac{\partial}{\partial x}\left[(f^0)^2+(g^0)^2\right]\\&-h\left\{f^0\left[(ma+\alpha h_{,x})f^0_{,xx}+mbg^0_{,xx}+\alpha h_{,x}g^0_{,xy}\right]\right.\\&\left.+g^0\left[(ma+\alpha h_{,x})f^0_{,xy}+mbg^0_{,xy}+\alpha h_{,x}g^0_{,yy}\right]\right\}\\&\quad-\frac{1}{2}\alpha h^2\left\{\frac{\partial}{\partial x}\left[f^0\left(f^0_{,xx}+g^0_{,xy}\right)\right]\right.\\&\left.+\frac{\partial}{\partial x}\left[g^0\left(f^0_{,xy}+g^0_{,yy}\right)\right]-\alpha\frac{\partial}{\partial x}\left(f^0_{,x}+g^0_{,y}\right)^2\right\}=0.\end{aligned}\tag{33}$$

In a similar way, the second of relations (14) gives

$$\begin{aligned}&\frac{\partial g^0}{\partial t}-\frac{1}{2}\alpha h^2\frac{\partial}{\partial t}\left(f^0_{,xy}+g^0_{,yy}\right)+g^*\left(h_{,y}-mb\right)\\&\quad+\frac{1}{2}\frac{\partial}{\partial y}\left[(f^0)^2+(g^0)^2\right]\\&-h\left\{f^0\left[maf^0_{,xy}+(mb+\alpha h_{,y})g^0_{,xy}+\alpha h_{,y}f^0_{,xx}\right]\right.\\&\left.+g^0\left[maf^0_{,yy}+(mb+\alpha h_{,y})g^0_{,yy}+\alpha h_{,y}f^0_{,xy}\right]\right\}\\&\quad-\frac{1}{2}\alpha h^2\left\{\frac{\partial}{\partial y}\left[f^0\left(f^0_{,xx}+g^0_{,xy}\right)\right]\right.\\&\left.+\frac{\partial}{\partial y}\left[g^0\left(f^0_{,xy}+g^0_{,yy}\right)\right]-\alpha\frac{\partial}{\partial y}\left(f^0_{,x}+g^0_{,y}\right)^2\right\}=0.\end{aligned}\tag{34}$$

In the last equations, only the second order terms (the second order power of the water depth) have been retained. From the substitution of Eqs. (25)–(27) into relation (9), the following equation is derived

$$\begin{aligned}&\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hf^0)+\frac{\partial}{\partial y}(hg^0)\\&-\frac{1}{6}\alpha h^3\left[\frac{\partial}{\partial x}\left(\nabla^2 f^0\right)+\frac{\partial}{\partial y}\left(\nabla^2 g^0\right)\right]=0.\end{aligned}\tag{35}$$

It may be seen, that the equations derived have still complicated structure. In specific cases however, it may be justified to introduce further approximations into these equations. For instance, in the momentum equations, one may disregard terms, corresponding to the first power of the water depth, as small quantities compared with other terms in the description. Moreover, it is justified to neglect also the last terms in these equations. With such an approach, instead of these equations we consider the following ones

$$\begin{aligned}&\frac{\partial f^0}{\partial t}-\frac{1}{2}\alpha h^2\frac{\partial}{\partial t}\left(f^0_{,xx}+g^0_{,xy}\right)+g^*\left(h_{,x}-ma\right)\\&\quad+\frac{1}{2}\frac{\partial}{\partial x}\left[(f^0)^2+(g^0)^2\right]\\&\quad-\frac{1}{2}\alpha h^2\left\{\frac{\partial}{\partial x}\left[f^0\left(f^0_{,xx}+g^0_{,xy}\right)\right]\right.\\&\left.\quad+\frac{\partial}{\partial x}\left[g^0\left(f^0_{,xy}+g^0_{,yy}\right)\right]\right\}=0\end{aligned}\tag{36}$$

and

$$\begin{aligned}&\frac{\partial g^0}{\partial t}-\frac{1}{2}\alpha h^2\frac{\partial}{\partial t}\left(f^0_{,xy}+g^0_{,yy}\right)\\&+g^*\left(h_{,y}-mb\right)+\frac{1}{2}\frac{\partial}{\partial y}\left[(f^0)^2+(g^0)^2\right]\\&\quad-\frac{1}{2}\alpha h^2\left\{\frac{\partial}{\partial y}\left[f^0\left(f^0_{,xx}+g^0_{,xy}\right)\right]\right.\\&\left.\quad+\frac{\partial}{\partial y}\left[g^0\left(f^0_{,xy}+g^0_{,yy}\right)\right]\right\}=0.\end{aligned}\tag{37}$$

In our further discussion, we confine our attention to Eqs. (35), (36) and (37) which are assumed to be sufficiently accurate in the description of the propagation of long waves in water of a small, non-uniform depth.

5. Linear dispersion characteristics

In order to assess the dispersion characteristics of the equations derived in the preceding section, a linearized version of these equations for the case of the constant water depth is considered. The linear momentum and continuity equations are

$$\begin{aligned}&\frac{\partial f^0}{\partial t}-\frac{1}{2}h_0^2\frac{\partial}{\partial t}\left(f^0_{,xx}+g^0_{,xy}\right)+g^*\eta_{,x}=0,\\&\frac{\partial g^0}{\partial t}-\frac{1}{2}h_0^2\frac{\partial}{\partial t}\left(f^0_{,xy}+g^0_{,yy}\right)+g^*\eta_{,y}=0,\\&\frac{\partial\eta}{\partial t}+h_0\left(\frac{\partial f^0}{\partial x}+\frac{\partial g^0}{\partial y}\right)+\\&-\frac{1}{6}h_0^3\left[\frac{\partial}{\partial x}\left(\nabla^2 f^0\right)+\frac{\partial}{\partial y}\left(\nabla^2 g^0\right)\right]=0,\end{aligned}\tag{38}$$

where $h_0=const.$ is still water depth and $\eta(x,t)$ is the free surface elevation.

For our purposes it is sufficient to take into account a monochromatic wave propagating in one direction, say along x coordinate and, instead of Eq. (38), consider the following ones

$$\begin{aligned}&\frac{\partial f^0}{\partial t}-\frac{1}{2}h_0^2\frac{\partial^3 f^0}{\partial t\partial x^2}+g^*\eta_{,x}=0,\\&\frac{\partial\eta}{\partial t}+h_0\frac{\partial f^0}{\partial x}-\frac{1}{6}h_0^3\frac{\partial^3 f^0}{\partial x^3}=0.\end{aligned}\tag{39}$$

Consider now a small amplitude periodic wave with the frequency ω and the wave number k:

$$\begin{aligned}f^0&=u_0\exp\left[i(kx-\omega t)\right],\\\eta&=a_0\exp\left[i(kx-\omega t)\right],\end{aligned}\tag{40}$$

where u_0 and a_0 denote amplitudes of the respective variables.

From substitution of (40) into Eq. (39) a homogeneous system of algebraic equations in terms of u_0 and a_0 is obtained. A nontrivial solution of these equations is obtained by letting their determinant vanish, which gives the dispersion relation as

$$\begin{aligned}\omega^2 &= g^* k(kh_0)\frac{1+(kh_0)^2/6}{1+(kh_0)^2/2} \\ &\cong g^* k(kh_0)\left[1-\frac{1}{3}(kh_0)^2+\frac{1}{6}(kh_0)^4\right].\end{aligned} \tag{41}$$

This formula is close to the standard linear dispersion formula for Stokes first order theory

$$\begin{aligned}\omega^2 &= g^* k\tanh(kh_0) \\ &= g^* k(kh_0)\left[1-\frac{1}{3}(kh_0)^2+\frac{2}{15}(kh_0)^4-\cdots\right].\end{aligned} \tag{42}$$

On account of Eqs. (41) and (42) one may calculate the associated phase speeds $c_f = \omega/k$ of a periodic wave. The phase speed of the Boussinesq model is given by

$$c_{fB} = \sqrt{gh_0}\sqrt{\frac{1+(kh_0)^2/6}{1+(kh_0)^2/2}}, \tag{43}$$

while the standard Stokes first-order theory leads to the formula

$$c_{fS} = \sqrt{gh_0}\sqrt{\tanh(kh_0)/(kh_0)}. \tag{44}$$

The ratio of these speeds reads

$$Rc = \frac{c_{fB}}{c_{fS}} = \sqrt{\frac{(kh_0)}{\tanh(kh_0)}\frac{1+(kh_0)^2/6}{1+(kh_0)^2/2}}. \tag{45}$$

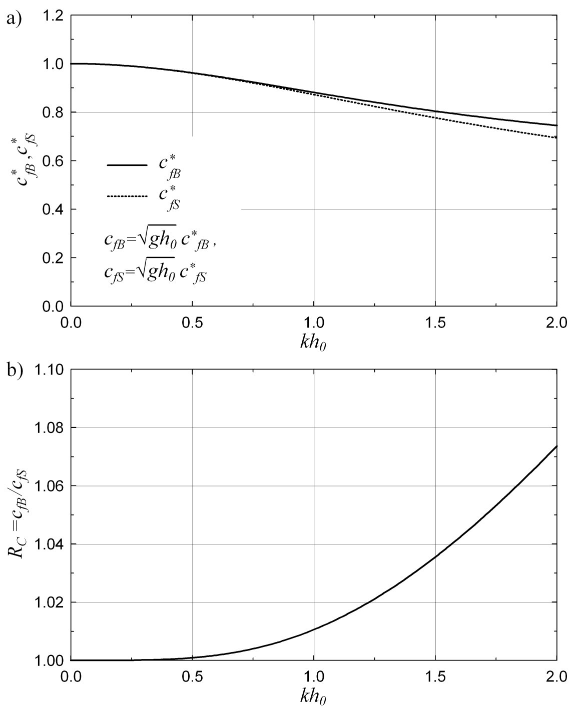

Fig. 2. Phase speeds of the Boussinesq model and the Stokes theory (a), and their ratio (b) versus the parameter (kh_0)

To illustrate a range of applicability of the description mentioned, in Fig. 2 the graphs of the phase speeds and their ratio in terms of the parameter (kh_0) are presented. From the plots in this figure it is possible to evaluate the difference between the Boussinesq approximation and the standard Stokes theory. Having the dispersion relation it is a simple task to calculate the group velocity $c_g = d\omega/dk$ associated with the Boussinesq approach. The phase velocity characterises the dispersion feature of the Boussinesq – type equations derived. Non-linearity effects, associated with these equations depend of course on the order of approximation in the description of this phenomenon.

6. Reduction of the partial differential equations to a system of ordinary differential equations

Equations derived in Sec. (4) form the system of non-linear partial differential equations with respect to the independent variables (x, y, t). In order to find a solution to these equations we resort to a discrete formulation allowing us to replace the partial differential equations by a system of ordinary differential equations. In derivation of the latter equations, the continuous fluid domain is substituted by a set of nodal points and, the spatial derivatives in the original equations are substituted by finite difference quotients written at these points. Such an approximation may be directly applied to finite fluid domains. In the case of an infinite fluid domain however, such a formal approach to the problem leads to an infinite set of the ordinary differential equations. In order to overcome this difficulty we may confine our attention to a finite fluid domain, obtained from the infinite one by an artificial boundary, with appropriate boundary conditions assumed at the boundary between the finite and infinite parts of the domain. These boundary conditions should transmit or absorb waves approaching the boundary. In this way it possible to construct a solution in the finite domain, having properties of a solution in the infinite fluid domain.

In order to make the further discussion clear, we confine our attention to finite fluid domains. Thus, let us consider a rectangular fluid domain shown schematically in Fig. 3, which corresponds to a periodic bathymetry of a sea shore zone. The motion of the fluid is induced by a piston type generator, placed at the boundary at $x = 0$. Till the initial moment of time, the generator – fluid system is at rest. At this initial moment of time, say at $t = 0$, the generator starts to move. At the generator face $x = x_g(t)$ we have the boundary condition that the fluid velocity $f^0(x_g, t)$ equals the generator velocity $\dot{x}_g(t)$. At the boundaries at $y = 0$ and $y = L_2$, the normal components of the fluid velocity are equal to zero. The boundary at $x = L_1$ admits reflection of water waves from this boundary. The last boundary condition corresponds to a cliff boundary of a sea shore zone. It is also assumed that in a range of time considered, the water depth at all points of the fluid domain is greater than zero i.e.

$$h(x, y, t) = h_b(x, y) + \eta(x, y, t) > 0.$$

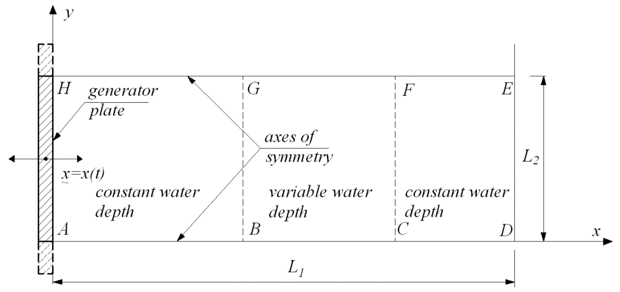

Fig. 3. A rectangular fluid domain corresponding to periodic boundary conditions of a sea shore zone

As it has been mentioned above, the rectangular fluid domain is substituted by a system of nodal points, and the continuous functions f^0, g^0 and h are substituted by their values at these points. In the procedure applied, the space derivatives of the dependent variables are substituted by their finite difference quotients. In order to solve boundary conditions at the fluid domain boundary, additional virtual nodal points are placed outside this boundary. Such points are also used in substitution of mixed space derivatives entering momentum equations by their finite difference analogues. In order to solve the boundary condition at the generator plate, we have to calculate finite difference quotients for a non-uniform spacing of nodal points in the vicinity of the generator plate. In calculation of higher order difference quotients at boundary of the fluid domain, we also make use of the Gregory-Newton extrapolation formula (Chan & Street, [10])

$$f_j \cong \frac{11}{3} f_{j+1} - 5f_{j+2} + 3f_{j+3} - \frac{2}{3} f_{j+4}, \tag{46}$$

where $f_j, \cdots, f_{j+4}$ denote values of a given function at the consecutive nodal points.

With respect to the finite difference approximation, instead of the continuous functions $f^0(x,y,t)$, $g^0(x,y,t)$ and $h(x,y,t)$ we operate with the time dependent vectors $\mathbf{f^0}(t)$, $\mathbf{g^0}(t)$ and $\mathbf{h^0}(t)$. It should be stressed that each of these vectors has its own number of components. In deriving the final system of ordinary differential equations with respect to the time variable, it is convenient to divide the momentum and continuity equations into linear and non-linear parts. Following the finite difference approach, the respective parts of Eqs. (35), (36) and (37) are transformed into the following set of matrices:

Equation (36)

$$\begin{aligned}
&\frac{\partial f^0}{\partial t} - \frac{1}{2}\alpha h^2 \frac{\partial}{\partial t} f^0_{,xx} \;\rightarrow\; \mathbf{AU}(\mathbf{f^0}, \mathbf{h}, t)\frac{d\mathbf{f^0}}{dt},\\
&-\frac{1}{2}\alpha h^2 \frac{\partial}{\partial t} g^0_{,xy} \;\rightarrow\; \mathbf{AV}(\mathbf{g^0}, \mathbf{h}, t)\frac{d\mathbf{g^0}}{dt},\\
&g^*(h_{,x} - ma) + \frac{1}{2}\frac{\partial}{\partial x}\left[(f^0)^2 + (g^0)^2\right]\\
&-\frac{1}{2}\alpha h^2 \frac{\partial}{\partial x}\left[f^0(f^0_{,xx} + g^0_{,xy})\right.\\
&\left.+g^0(f^0_{,xy} + g^0_{,yy})\right] \;\rightarrow\; \mathbf{NLA}(\mathbf{f^0}, \mathbf{g^0}, \mathbf{h}, t).
\end{aligned} \tag{47}$$

Equation (37)

$$\begin{aligned}
&\frac{\partial g^0}{\partial t} - \frac{1}{2}\alpha h^2 \frac{\partial}{\partial t} g^0_{,yy} \;\rightarrow\; \mathbf{BV}(\mathbf{g^0}, \mathbf{h}, t)\frac{d\mathbf{g^0}}{dt},\\
&-\frac{1}{2}\alpha h^2 \frac{\partial}{\partial t} f^0_{,xy} \;\rightarrow\; \mathbf{BU}(\mathbf{f^0}, \mathbf{h}, t)\frac{d\mathbf{f^0}}{dt},\\
&g^*(h_{,y} - mb) + \frac{1}{2}\frac{\partial}{\partial y}\left[(f^0)^2 + (g^0)^2\right]\\
&-\frac{1}{2}\alpha h^2 \frac{\partial}{\partial y}\left[f^0(f^0_{,xx} + g^0_{,xy})\right.\\
&\left.+g^0(f^0_{,xy} + g^0_{,yy})\right] \;\rightarrow\; \mathbf{NLB}(\mathbf{f^0}, \mathbf{g^0}, \mathbf{h}, t).
\end{aligned} \tag{48}$$

Equation (35)

$$\begin{aligned}
&\frac{\partial h}{\partial t} \;\rightarrow\; \frac{d\mathbf{h}}{dt},\\
&\frac{\partial}{\partial x}(hf^0) + \frac{\partial}{\partial y}(hg^0) +\\
&-\frac{1}{6}\alpha h^3\left[\frac{\partial}{\partial x}\left(\nabla^2 f^0\right) + \frac{\partial}{\partial y}\left(\nabla^2 g^0\right)\right] \;\rightarrow\; \mathbf{NB}(\mathbf{f^0}, \mathbf{g^0}, t).
\end{aligned} \tag{49}$$

With the above descriptions, the final system of ordinary differential equations reads

$$\begin{aligned}
&\mathbf{AU}(\mathbf{f^0}, \mathbf{h}, t)\frac{d\mathbf{f^0}}{dt} + \mathbf{AV}(\mathbf{g^0}, \mathbf{h}, t)\frac{d\mathbf{g^0}}{dt}\\
&\quad + \mathbf{NLA}(\mathbf{f^0}, \mathbf{g^0}, \mathbf{h}, t) = \mathbf{0},\\
&\mathbf{BU}(\mathbf{f^0}, \mathbf{h}, t)\frac{d\mathbf{f^0}}{dt} + \mathbf{BV}(\mathbf{g^0}, \mathbf{h}, t)\frac{d\mathbf{g^0}}{dt}\\
&\quad + \mathbf{NLB}(\mathbf{f^0}, \mathbf{g^0}, \mathbf{h}, t) = \mathbf{0},\\
&\frac{d\mathbf{h}}{dt} + \mathbf{NB}(\mathbf{f^0}, \mathbf{g^0}, \mathbf{h}, t) = \mathbf{0}.
\end{aligned} \tag{50}$$

In these equations: $\mathbf{AU}$ and $\mathbf{BV}$ are square matrices, $\mathbf{AV}$ and $\mathbf{BU}$ are rectangular matrices and $\mathbf{h}$, $\mathbf{NLA}$, $\mathbf{NLB}$ and $\mathbf{NB}$ are vector matrices. The dimensions of these matrices correspond to number of components of the dependent variables. For numerical reasons, this system of the ordinary differential equations is transformed into another form. Left multiplication of the first equation of (50) by transpose of the matrix $\mathbf{AU}$ and the second – by transpose of $\mathbf{BV}$, leads to the final form of these equations

$$\begin{aligned}
&\mathbf{AM}\frac{d\mathbf{w}}{dt} + \mathbf{NA} = \mathbf{0},\\
&\frac{d\mathbf{h}}{dt} + \mathbf{NB} = \mathbf{0},
\end{aligned} \tag{51}$$

where

$$\mathbf{AM} = \begin{bmatrix} \mathbf{AU^T}\cdot\mathbf{AU} & \mathbf{AU^T}\cdot\mathbf{AV} \\ \mathbf{BV^T}\cdot\mathbf{BU} & \mathbf{BV^T}\cdot\mathbf{BV} \end{bmatrix}, \tag{52}$$

$$\mathbf{w} = \begin{Bmatrix} \mathbf{f^0} \\ \mathbf{g^0} \end{Bmatrix}, \tag{53}$$

$$\mathbf{NA} = \begin{Bmatrix} \mathbf{AU^T}\cdot\mathbf{NLA} \\ \mathbf{BV^T}\cdot\mathbf{NLB} \end{Bmatrix}. \tag{54}$$

The matrices of the non-linear coupled system of Eq. (51) depend on the unknown vectors $\mathbf{w}$, $\mathbf{h}$. Therefore, in order to find a solution of the equations we resort to a numerical integration of them in the time domain by means of the fourth order Runge-Kutta method (Björck, Dahlquist, [11]).

7. Examples of numerical solutions

In order to learn more about the applicability of the equations derived, in this section, Eq. (51) are integrated for chosen cases of generation of the waves in rectangular fluid domains of variable depth. The initial - value problem considered is shown schematically in Fig.3. The water wave, generated by a piston type generator, placed at the left boundary of the fluid domain, propagates through the area of variable water depth. Two cases of bottom bathymetry are considered. The first one is described by a continuous 'bell-shaped surface', and the second one forms a rigid underwater obstacle of a trapezoidal shape, placed at a certain distance from the generator plate. The generator starts to move at the initial moment of time and its motion is assumed in the following form (Wilde & Wilde, [12]):

$$x_g(t) = A_g \left[A(\tau)\cos(\omega t) + D(\tau)\sin(\omega t)\right], \qquad (55)$$

where A_g is the generator amplitude, ω is the angular frequency, $\tau = \eta t$ is the non-dimensional time factor, η is a parameter responsible for a growth in time of the generator displacement, and the terms in the square brackets are defined as

$$\begin{aligned} A(\tau) &= \frac{1}{3!}\tau^3 \exp(-\tau), \\ D(\tau) &= 1 - \left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3\right)\exp(-\tau). \end{aligned} \qquad (56)$$

Equations (55) and (56) allow to calculate the generator velocity $\dot{x}_g(t)$ and its acceleration $\ddot{x}_g(t)$. One can check, that for $\eta = 2$, assumed in our calculations, the generator motion approaches the case of a steady state harmonic generation within a few first periods of time. In addition, the displacement, velocity and acceleration of the generator face are equal to zero at the initial moment of time at $t = 0^+$. For an assumed length of the generated wave, the associated frequency may be obtained from equation (42).

Numerical calculations have been carried out for rectangular fluid domains of dimensions $L_1 \times L_2 = (66.9 \text{ m} \times 5.7 \text{ m})$ with 4237 nodal points and $(42.3 \text{ m} \times 5.7 \text{ m})$ with 2679 nodal points, and the maximum still water depth $h_0 = 0.60$ m. The smallest still water depth in the area of variable bottom bathymetry was equal to 0.20 m. In the discrete approach, a constant spacing of nodal points $\Delta x \times \Delta y = 0.30 \text{ m} \times 0.30 \text{ m}$ has been used. At the same time, the horizontal spacing at the generator face depended on time i.e. $\Delta x^* = \Delta x - x_g(t)$. In the Runge-Kutta numerical integration, the assumed time step was chosen to be $\Delta t = 0.05$ s. This time step satisfies the Courant condition, which requires that the ratio of the wave celerity to the 'net velocity' be less than one, i.e. $c_f \Delta t \Big/ \sqrt{(\Delta x)^2 + (\Delta y)^2} < 1$.

Integration of the problem equations gives the velocity field as well as the free surface elevation dependent on the time coordinate. Some of the results obtained in computations are shown in the subsequent Fig. 4 and Fig. 5, where the plots illustrate transformation of surface waves propagating through the area of constant and variable water depth. From the plots in may be seen that for a small range of time, measured from the starting point, one may assume the condition that at the right hand boundary (segment DE in Fig. 3) the fluid is at rest.

The equations of the problem, derived in the preceding sections, correspond to a mechanical system without any dissipation of energy. Therefore, for the harmonic generation of the fluid motion in the rectangular domain it may happen, that the generation frequency is equal, or close, to a frequency inherent for standing water waves in the domain. In such a case one may expect a radical growth of the dependent variables with passing time. In order to answer the question about such a possibility, it is reasonable to consider a classical linear problem of standing water waves in a rectangular fluid domain with reflecting boundary conditions assumed at its boundaries. Thus, let us consider the steady state harmonic motion of the fluid in the rectangular fluid domain $(L_1 \times L_2 \times h_0)$, where L_1 and L_2 are horizontal dimensions and $h_0 = const.$ is the still water depth. In order to calculate the eigenfrequencies of the fluid domain, associated with Eq. (38), let us consider the following steady state solution of these equations

$$\begin{aligned} f^0 &= F \exp\left[i(k_1 x + k_2 y - \omega t)\right], \\ g^0 &= G \exp\left[i(k_1 x + k_2 y - \omega t)\right], \\ \eta &= H \exp\left[i(k_1 x + k_2 y - \omega t)\right], \end{aligned} \qquad (57)$$

where F, G and H are complex amplitudes of the dependent variables and k_1, k_2 and ω are real numbers. From substitution of these equations into Eq. (38) the dispersion relation (42) can be derived.

The wave numbers k_1 and k_2 are not arbitrary. From the assumed conditions at boundaries of the fluid domain it follows that

$$k_1^2 + k_2^2 = k^2 = \left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2, \quad m, n = 0, 1, 2, \cdots \qquad (58)$$

If the ratio of the two sides of the domain is a rational number, that is, $L_1 = pL$ and $L_2 = qL$, where p and q are integers, the following is obtained:

$$(k)^2_{mn} = \left(\frac{\pi}{L}\right)^2 \left[\left(\frac{m}{p}\right)^2 + \left(\frac{n}{q}\right)^2\right], \quad m, n = 0, 1, 2, \cdots \qquad (59)$$

Equations (58) and (59) show that, in a general case, there is more than one set of (m, n) which correspond to the same wave number k (to the same angular frequency ω). The associated eigenmodes are described by Eq. (57). If $L_1 > L_2$, the lowest mode ($m = 1$, $n = 0$) corresponds to the lowest frequency. The other modes corresponding to this one-dimensional motion $(m, n = 0)$ follow the condition

$$L_1 = m\frac{\lambda}{2} = (1, 2, 3, \cdots)\frac{\lambda}{2}, \qquad (60)$$

where λ is the wave length.

$\lambda = 7.6325m,\ h = 0.60m,\ ho = 0.40m,\ A_g = 0.04m$

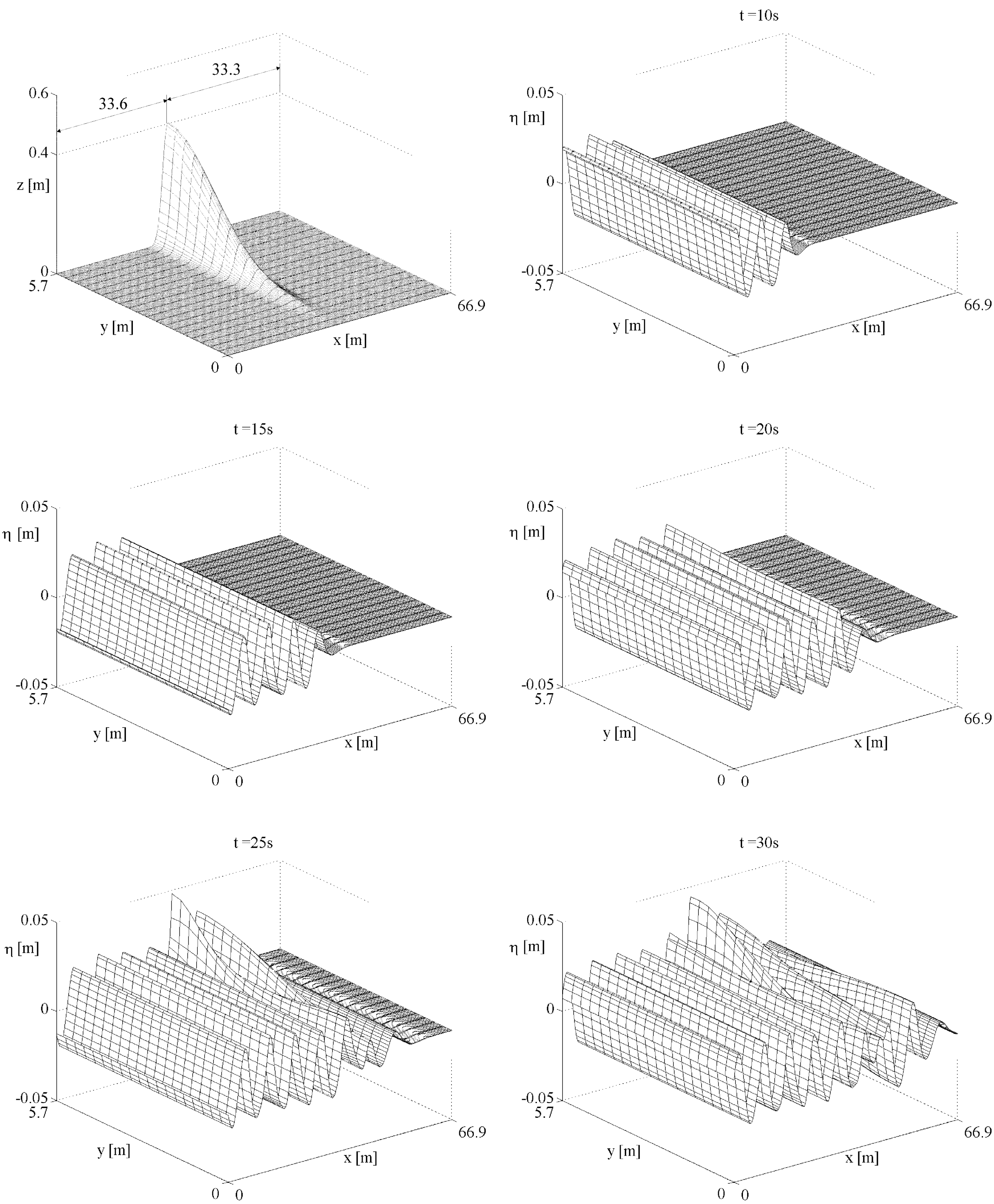

Fig. 4. Transformation of a surface wave propagating through the area of a continuous 'bell-shaped' variation of the bottom bathymetry

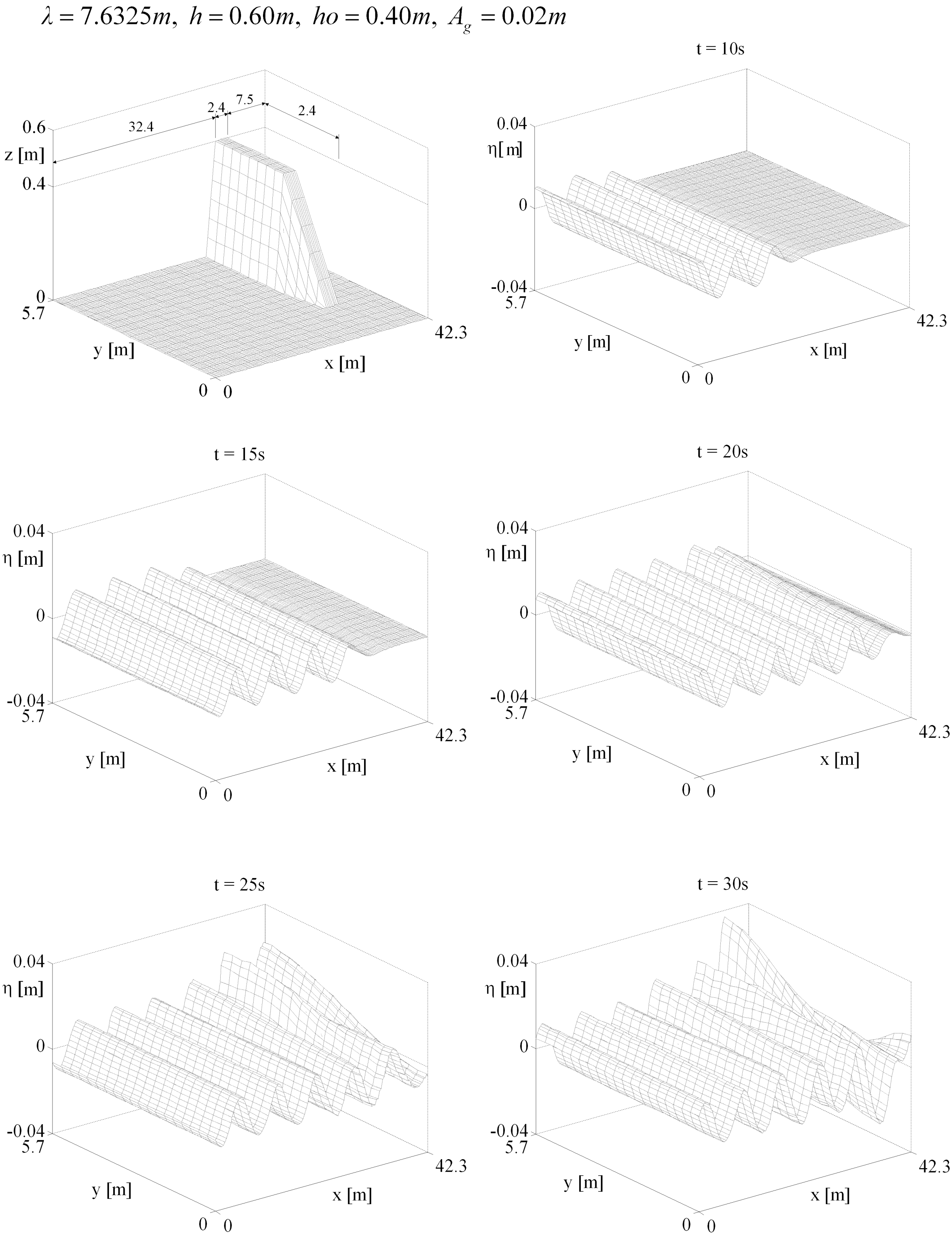

Fig. 5. Transformation of a surface wave propagating over an underwater obstacle of a trapezoidal shape

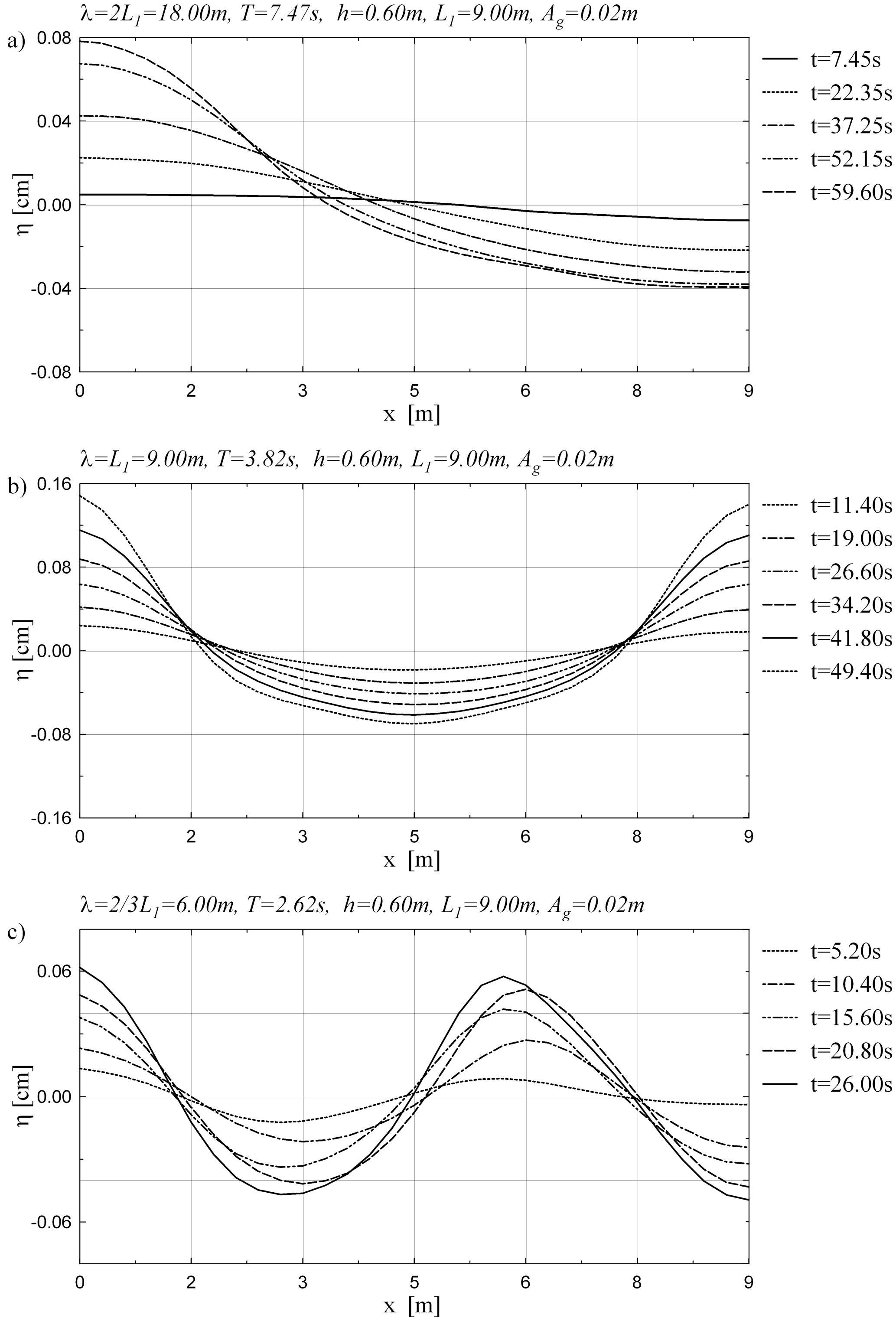

Fig. 6. Evolution in time of the free surface in a rectangular fluid domain. Generation frequency corresponds to the resonance range as described by equation (60)

It should be stressed that in the discussed non-linear cases of propagation of long waves in water of variable depth, the results obtained may only serve as a certain indicators of the resonance ranges of the time-dependent problem considered.

In order to illustrate the phenomenon, numerical solutions were carried out for cases corresponding to ranges as defined by the last formula. Some results of these computations are presented in Fig. 6, where the plots show evolution in time of the free surface elevation. From the plots it may be seen how the generation with frequency close to the resonance range influences the numerical solution.

8. Conclusions

The Boussinesq equations developed in this paper allows to describe main features of long waves propagating in water of small variable depth. Because of their complicated structure, the analysis has been confined to lowest order terms in the power series expansion of the variables with respect to the water depth. In a rather formal way it is possible to take into account higher order terms, but this comes at a cost. Moreover, with the higher order terms, one can face more difficulties in preparing approximate description of the derivatives entering the equations derived. As compared with formulations presented in literature on the subject, the theory presented above is, in a sense, unified, but its application to a more complicated geometry encountered in practice may face difficulties, resulting from a solution to boundary and initial conditions. The dispersion properties of the description may be considered as being sufficiently accurate for long waves, for which $k_0h < 1$. With this condition ensured, the relative error of the wave celerity of the theory presented above compared to the phase velocity of Stokes solution is less than 1%.

Appendix

The discussed problem of long waves, propagating in a fluid of non-uniform depth, may be characterised by two important parameters: $\mu = h_0/\lambda$, i.e. the ratio of a typical still water depth to a typical wave length, and $\varepsilon = a_0/h_0$, i.e. the ratio of the wave amplitude to the reference water depth. For the waves considered, the first of these parameters is assumed to be a small quantity (typically $\mu < 1/10$), whereas the second parameter is a quantity of order one (Dingemans, [6]). In addition, the bottom slope $|dh_b/dx| = |dh_b/dy| = O(\mu)$ is assumed to be also a small quantity.

In order to estimate convergence of the series in Eq. (15), let us consider the first of these series for the case of a constant bottom slope ($ma = const.$, $mb = const.$). For points of the free surface, the x- component of the velocity field reads

$$u = \sum_{n=0}^{\infty} h^n f^n(x, y, t) \tag{A1}$$

where $h = h_b + \eta$ is the water depth.

From the recurrence formulae (Eq. 24) one obtains

$$\begin{aligned}
\varphi^0 &= -maf^0 - mbg^0, \\
\varphi^n &= -\frac{1}{n}\left(\mu_1 f^{n-1}_{,x} + \mu_1 g^{n-1}_{,y} + \mu_2 \varphi^{n-1}_{,x} + \mu_3 \varphi^{n-1}_{,y}\right), \\
f^n &= \frac{1}{n}\left[\mu_4 \varphi^{n-1}_{,x} - \mu_5 \varphi^{n-1}_{,y} - \mu_2 f^{n-1}_{,x} - \mu_2 g^{n-1}_{,y}\right], \\
g^n &= \frac{1}{n}\left[-\mu_6 \varphi^{n-1}_{,x} + \mu_5 \varphi^{n-1}_{,y} - \mu_3 f^{n-1}_{,x} - \mu_3 g^{n-1}_{,y}\right], \\
& n = 1, 2, \cdots,
\end{aligned} \tag{A2}$$

where

$$\begin{aligned}
\mu_1 = \alpha &= \frac{1}{1 + ma^2 + mb^2}, \\
\mu_2 &= \alpha ma, \\
\mu_3 &= \alpha mb, \\
\mu_4 &= 1 - ma^2, \\
\mu_5 &= \alpha mamb, \\
\mu_6 &= 1 - mb^2.
\end{aligned} \tag{A3}$$

It is important to note that all these multipliers are less than one, i.e. $|\mu_k| < 1$, $k = 1, 2, \cdots, 6$. Moreover, from the recurrence formulae it follows that the higher order components f^n, g^n and φ^n of the velocity depend on higher order derivatives of the fundamental functions f^0, g^0 and φ^0. At the same time, an examination of these formulae shows that, when starting from the lowest components, the multiplier $1/n$ in Eq. (A2) leads to the multiplier $1/(1 \cdot 2 \cdot \cdots \cdot n) = 1/n!$ for the n-th term. Therefore, this term, entering equation (A1), may be expressed in the following form

$$\begin{aligned}
h^n f^n = \frac{h^n}{n!}\Bigg(\beta_1^n \frac{\partial^n}{\partial x^{r1} \partial y^{r2}} \varphi^0 \\
+ \beta_2^n \frac{\partial^n}{\partial x^{r3} \partial y^{r4}} f^0 + \beta_3^n \frac{\partial^n}{\partial x^{r5} \partial y^{r6}} g^0\Bigg), \\
n = 1, 2, \cdots,
\end{aligned} \tag{A4}$$

where $r1 + r2 = r3 + r4 = r5 + r6 = n$, $(n = 1, 2, \cdots)$ and β_1^n, β_2^n and β_3^n are products of the multipliers $\mu_1, \mu_2, \cdots, \mu_6$.

It may be seen that all the multipliers β_k^n, $(k = 1, 2, 3)$ rapidly decrease ($|\beta_k^n| < 1$) with increasing number $n = 1, 2, \cdots$. For the long waves considered, appreciate changes of the water depth as well as the fluid velocity, may occur only in a region of the wave length. Therefore, it is justified to assume that the subsequent partial derivatives of φ^0, f^0 and g^0, with respect to the space coordinates, rapidly decrease with the number $n = 1, 2, \cdots$ in such a way, that the following inequality holds

$$\begin{aligned}
\Bigg|\beta_1^n \frac{\partial^n}{\partial x^{r1} \partial y^{r2}} \varphi^0 + \beta_2^n \frac{\partial^n}{\partial x^{r3} \partial y^{r4}} f^0 \\
+ \beta_3^n \frac{\partial^n}{\partial x^{r5} \partial y^{r6}} g^0\Bigg| < D,
\end{aligned} \tag{A5}$$

where $D > 0$ is a bounded number.

Following this assumption one obtains

$$|u| = \sum_{n=0}^{\infty} h^n |f^n(x, y, t)| < D \sum_{n=0}^{n=\infty} \frac{h^n}{n!} = D \exp(h). \tag{A6}$$

Thus, under the assumption (A5), the series in this equation is absolutely convergent. In a similar way, one may show that the remainder series in Eq. (15) are also convergent.

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