BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 60, No. 3, 2012
DOI: 10.2478/v10175-012-0073-y

Joint reactions in rigid or flexible body mechanisms with redundant constraints

M. WOJTYRA* and J. FRĄCZEK

Institute of Aeronautics and Applied Mechanics, Warsaw University of Technology, 24 Nowowiejska St., 00-665 Warsaw, Poland

Abstract. The problem of joint reactions indeterminacy, in engineering simulations of rigid body mechanisms is most often caused by redundant constraints which are defined as constraints that can be removed without changing the kinematics of the system. In order to find a unique set of all joint reactions in an overconstrained system, it is necessary to reject the assumption that all bodies are rigid. Flexible bodies introduce additional degrees of freedom to the mechanism, which usually makes the constraint equations independent. Quite often only selected bodies are modelled as the flexible ones, whereas the other remain rigid. In this contribution it is shown that taking into account flexibility of selected mechanism bodies does not guarantee that unique joint reactions can be found. Problems typical for redundant constraints existence are encountered in partially flexible models, which are not overconstrained. A case study of a redundantly constrained spatial mechanism is presented. Different approaches to the mechanism modelling, ranging from a purely rigid body model to a fully flexible one, are investigated and the obtained results are compared and discussed.

Key words: multibody system, redundant constraints, joint reactions, flexible bodies.

1. Introduction

The direct dynamic analysis of multibody systems consists in calculating motion resulting from external or internal loads [1, 2]. Positions, velocities and accelerations of bodies are determined during the analysis, and – in many cases – the joint reaction forces are calculated alongside these kinematic quantities. If motion is the only object of interest and all kinematic pairs are frictionless, it is possible to avoid calculation of joint reactions by eliminating them from equations of motion [3]. Nevertheless, joint reaction forces are quite frequently calculated because they are essential for various structural analyses of mechanisms bodies, like fatigue or static analysis, when stresses or strains are calculated. When Coulomb friction in joints is taken into account, calculation of reaction forces cannot be avoided, since friction forces depend directly on normal joint reactions [2, 4].

In mathematical models of multibody systems kinematic pairs are represented by constraint equations; the constraint equations are formulated for all joints, when absolute (Cartesian) or natural coordinates are used to describe the system, or only for selected loop-closing joints, in the case of relative joint coordinates [2, 5, 6]. The problem of finding physical joint reactions in a multibody system is equivalent to the problem of finding generalized constraint reactions. Mathematically, it is clear that if all of the constraint equations are independent, the constraint reaction forces can be uniquely determined. By contrast, when the constraint equations are dependent, the constraint reaction forces, in general, are not uniquely determined.

The problem of joint reactions indeterminacy, in engineering simulations of rigid multibody systems, is most often caused by redundant constraints which are defined as constraints that can be removed without changing the kinematics of the system. If redundant constraints are present in the multibody system, the constraint equations are dependent [2, 5, 7] By their nature, the redundant constraints cannot be uniquely selected. In multibody modelling one of the most common method of dealing with overconstrained systems consists in redundant constraints elimination. Selection of the removed constraints is usually based on the results of the constraint Jacobian matrix factorization [5, 7, 8]. It should be noted that the loads carried by the eliminated constraints are transferred to the constraints remaining in the model. Sometimes redundant constraint equations are preserved in the mathematical model [2]. In that case it is possible to use pseudo-inverse methods [9] or a penalty approach, often based on the augmented Lagrangian formulation [10, 11], to solve the indeterminate system of equations, however, this also results in choosing one of infinitely many possible solutions, and hence a nonunique solution is found.

It can be proved that in the case of an overconstrained rigid body mechanism, despite the fact that all constraint reactions cannot be uniquely determined, selected single constraint reactions or selected groups of reactions can be specified uniquely [12, 13]. In our previously published works [12, 14] an algebraic criterion, allowing detection of these joints for which reaction forces can be uniquely determined, was formulated. This criterion was followed by three numerical methods for finding such joints. It can be shown that these methods can be also used to check whether the simulated motion is unique (in terms of positions, velocities and accelerations of bodies) in the case of a redundantly constrained rigid body mechanism with Coulomb friction in joints [14] In

*e-mail: mwojtyra@meil.pw.edu.pl

general, the simulated motion depends on method of handling redundant constraints, however, if friction forces appear only in these joints for which reaction solution is unique, then the simulated motion of mechanism is also unique, and thus – in this special case – the direct dynamic problem is uniquely solvable.

The methods presented in [12] and [14] are focused on rigid body mechanisms, and are helpful when analyzing selected joint reactions. In order to find a unique set of all joint reactions in an overconstrained system it is necessary to reject the assumption that all bodies are rigid. Flexible bodies introduce additional degrees of freedom to the mechanism, which usually makes the constraint equations independent (the previously redundant constraints are no longer applied to the same degree of freedom). Theoretically, it is possible to account for flexibility of all mechanism parts, however, in the case of large number of moving bodies, this approach is ineffective due to a huge numerical cost. Therefore, quite often only selected bodies are replaced by flexible ones, whereas the other remain rigid. In many cases, a decision on which parts should be treated as flexible is difficult or, at least, is not straightforward. The choice of flexible parts is particularly problematical when in the investigated mechanism flexibility effects are negligible, and the only reason for introducing them into the multibody model is to avoid problems with joint reactions solvability.

The original contribution of this paper consists in showing that taking into account flexibility of selected mechanism bodies does not guarantee that unique joint reactions can be found. Problems typical for redundant constraints existence are encountered in partially flexible models, which – at least formally – are not overconstrained. In some cases introduction of flexibility is equivalent to (nonunique!) redundant constraints elimination, and thus nonunique reactions are found. In other cases redundant constraints can be still present in the system despite introducing flexibility of some bodies which obviously affects joint reaction solvability In some situations, in a partially flexible multibody system, formally there are no redundant constraints, however, the obtained reaction solution depends on the choice of parts modelled as flexible bodies. All these problems are discussed in the paper. For the sake of simplicity a case study of an overconstrained spatial mechanism is presented.

This paper is divided into six sections. The next, i.e. the second section begins with presentation of the investigated mechanism, then solvability of its joint reactions is discussed and the forward dynamics problem is formulated. The third section is devoted to a rigid body version of the mechanism model; methods of redundant constraints handling are discussed and examples of both unique and nonunique reactions are presented. Partially flexible versions of the mechanism model are presented in the fourth section. Firstly, a simplified method of flexible rod modelling is presented. Then various possibilities of flexibility introduction are investigated and the obtained results are presented and discussed. The fifth section presents a fully flexible version of the mechanism model. Method of deformable bodies modelling is briefly summarised and then results of the investigated mechanism simulations are discussed. Concluding remarks are formulated in the last section.

2. The investigated mechanism

2.1. Structure, inertial properties and constraint equations. Various multibody models – ranging from fully rigid, through partially flexible, to fully flexible ones – are used in engineering practice (see, e.g., [15–17], respectively). In this paper, one and the same mechanism will be analyzed; firstly, as a system consisting of rigid bodies; then, as a mixed rigid-flexible system; and finally, as a fully flexible system. The investigation will be focused on joint reactions.

We study the problem of constraint reactions solvability in a spatial parallelogram mechanism (Fig. 1), consisting of the basis 0 and seven moving bodies 1–7. The bodies are connected by 12 spherical joints. The mechanism has seven degrees of freedom. The platform (part 7) can perform a translational motion (with constant orientation) with respect to the basis, moreover, links 1–6 can rotate along their longitudinal axes, not affecting the other bodies motion (the mechanism has 6 passive degrees of freedom).

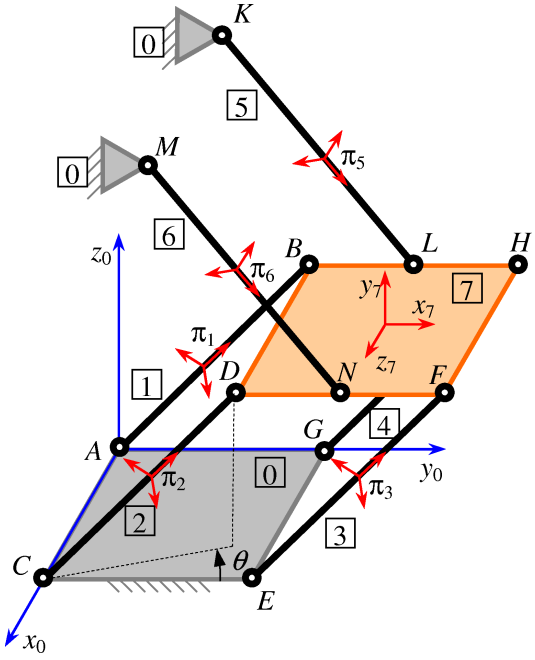

Fig. 1. A spatial parallelogram mechanism

For our considerations it is important that the theoretical mobility calculated using the Grübler-Kutzbach equation is 6, not 7. This indicates that dependent (redundant) constraints are present in the mechanism [18].

The global reference frame $\pi_0 = x_0y_0z_0$ is established on the basis 0 and the local reference frames $\pi_i = x_iy_iz_i$ are established on the other bodies (Fig. 1) at the respective centres of mass. The positions of joints K and M in the global frame are $\mathbf{r}_K = \begin{bmatrix} 0 & 0.25 & 1 \end{bmatrix}^T$ (m) and $\mathbf{r}_M = \begin{bmatrix} 0.5 & 0.25 & 1 \end{bmatrix}^T$ (m), respectively. Each link is a

cylindrical tube with inner diameter $d_1 = 0.028$ (m), outer diameter $d_2 = 0.03$ (m) and length $l = \sqrt{2}/2$ (m). The lower and the upper platforms are square plates 0.5×0.5 (m). The upper platform thickness equals 0.02 (m). The links 1–6 are made of aluminium (density 2700 (kg/m^3)), and the moving platform 7 is made of steel (density 7800 (kg/m^3)), thus they are characterized by the following masses and inertia matrices (moments of inertia are calculated with respect to local reference frames):

$$\begin{gathered} m_i \approx 0.34 \text{ (kg)}, \\ \mathbf{J}_i^{(i)} \approx diag\,(56,\ 56,\ 0.08) \cdot 10^{-3} (\text{kg m}^2), \\ i = 1,\ \ldots\ ,\ 6\,, \\ m_7 = 39 \text{ (kg)}, \\ \mathbf{J}_7^{(7)} \approx diag\,(3.25,\ 6.5,\ 3.25)\ (\text{kg m}^2). \end{gathered} \tag{1}$$

The mechanism is described by a vector of absolute (Cartesian) coordinates $\mathbf{q}$:

$$\begin{gathered} \mathbf{q} = \left[\begin{array}{ccccccc} \mathbf{q}_1^T & \mathbf{q}_2^T & \mathbf{q}_3^T & \mathbf{q}_4^T & \mathbf{q}_5^T & \mathbf{q}_6^T & \mathbf{q}_7^T \end{array}\right]^T, \\ \mathbf{q}_i^{=} \left[\begin{array}{cc} \mathbf{r}_i^T & \boldsymbol{\varphi}_i^T \end{array}\right]^T, \end{gathered} \tag{2}$$

where $\mathbf{r}_i = \left[\begin{array}{ccc} x_i & y_i & z_i \end{array}\right]^T$ is the position of the local frame π_i origin with respect to the global frame π_0, and $\boldsymbol{\varphi}_i = \left[\begin{array}{ccc} \alpha_i & \beta_i & \gamma_i \end{array}\right]^T$ are angles of $z-x-z$ Euler rotations describing the orientation of π_i with respect to π_0. The vector of absolute coordinates consists of $n = 42$ elements.

In the mathematical model of the mechanism joints are represented by constraint equations. The constraint equations describing a spherical joint formed by bodies i and j at point P can be derived by requiring that the point P_i on body i coincides with point P_j on body j:

$$\boldsymbol{\Phi}^P (\mathbf{q}_i, \mathbf{q}_j) = \mathbf{r}_i + \mathbf{R}_i \mathbf{s}_P^{(i)} - \mathbf{r}_j - \mathbf{R}_j \mathbf{s}_P^{(j)} = \mathbf{0}_{3\times 1}, \tag{3}$$

where $\mathbf{s}_P^{(k)}$ is the position vector of point P in the local reference frame π_k, and:

$$\begin{gathered} \mathbf{R}_i = \mathbf{R}_i (\alpha_i, \beta_i, \gamma_i) = \mathbf{R}_z (\alpha_i)\ \mathbf{R}_x (\beta_i)\ \mathbf{R}_z (\gamma_i)\,, \\ \mathbf{R}_z (\psi) = \left[\begin{array}{ccc} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{array}\right], \\ \mathbf{R}_x (\psi) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{array}\right]. \end{gathered} \tag{4}$$

Equations (3) for all 12 kinematic pairs can be written jointly (vector equation (3) consists of 3 scalar equations, hence all joints of the mechanism are represented by $m = 36$ scalar constraint equations):

$$\begin{aligned} \boldsymbol{\Phi} (\mathbf{q}) = \Big[& \boldsymbol{\Phi}^{A^T}\ \boldsymbol{\Phi}^{B^T}\ \boldsymbol{\Phi}^{C^T}\ \boldsymbol{\Phi}^{D^T}\ \boldsymbol{\Phi}^{E^T}\ \boldsymbol{\Phi}^{F^T}\ \boldsymbol{\Phi}^{G^T} \cdots \\ & \cdots\ \boldsymbol{\Phi}^{H^T}\ \boldsymbol{\Phi}^{K^T}\ \boldsymbol{\Phi}^{L^T}\ \boldsymbol{\Phi}^{M^T}\ \boldsymbol{\Phi}^{N^T} \Big]^T \\ = & \left[\begin{array}{ccc} \Phi_1 & \cdots & \Phi_m \end{array}\right]^T = \mathbf{0}_{m\times 1}. \end{aligned} \tag{5}$$

The Jacobian matrix is obtained by differentiation of the constraint Eqs. (5). The partial derivative of Eq. (3) with respect to the vector of absolute coordinates $\mathbf{q}$ provides three rows to the Jacobian matrix. Since $\boldsymbol{\Phi}^P$ is a function of only $\mathbf{q}_i$ and $\mathbf{q}_j$, $\boldsymbol{\Phi}_{\mathbf{q}}^P$ may have nonzero elements only in the columns associated with $\mathbf{q}_i$ and $\mathbf{q}_j$. The possible nonzero entries are:

$$\begin{aligned} & \boldsymbol{\Phi}_{\mathbf{r}_i}^P = \mathbf{I}_{3\times 3}\,, \qquad \boldsymbol{\Phi}_{\mathbf{r}_j}^P = -\mathbf{I}_{3\times 3}\,, \\ & \boldsymbol{\Phi}_{\boldsymbol{\varphi}_i}^P = \left[\begin{array}{ccc} \boldsymbol{\Phi}_{\alpha_i}^P & \boldsymbol{\Phi}_{\beta_i}^P & \boldsymbol{\Phi}_{\gamma_i}^P \end{array}\right] \\ = & \left[\begin{array}{c} \boldsymbol{\Omega}_z \mathbf{R}_z (\alpha_i)\, \mathbf{R}_x (\beta_i)\, \mathbf{R}_z (\gamma_i)\, \mathbf{s}_P^{(i)} \quad \cdots \\ \cdots\ \mathbf{R}_z (\alpha_i)\, \boldsymbol{\Omega}_x \mathbf{R}_x (\beta_i)\, \mathbf{R}_z (\gamma_i)\, \mathbf{s}_P^{(i)} \quad \cdots \\ \cdots\ \mathbf{R}_z (\alpha_i)\, \mathbf{R}_x (\beta_i)\, \boldsymbol{\Omega}_z \mathbf{R}_z (\gamma_i)\, \mathbf{s}_P^{(i)} \end{array}\right]_{3\times 3}, \\ & \boldsymbol{\Phi}_{\boldsymbol{\varphi}_j}^P = \left[\begin{array}{ccc} \boldsymbol{\Phi}_{\alpha_j}^P & \boldsymbol{\Phi}_{\beta_j}^P & \boldsymbol{\Phi}_{\gamma_j}^P \end{array}\right] \\ = & -\left[\begin{array}{c} \boldsymbol{\Omega}_z \mathbf{R}_z (\alpha_j)\, \mathbf{R}_x (\beta_j)\, \mathbf{R}_z (\gamma_j)\, \mathbf{s}_P^{(j)} \quad \cdots \\ \cdots\ \mathbf{R}_z (\alpha_j)\, \boldsymbol{\Omega}_x \mathbf{R}_x (\beta_j)\, \mathbf{R}_z (\gamma_j)\, \mathbf{s}_P^{(j)} \quad \cdots \\ \cdots\ \mathbf{R}_z (\alpha_j)\, \mathbf{R}_x (\beta_j)\, \boldsymbol{\Omega}_z \mathbf{R}_z (\gamma_j)\, \mathbf{s}_P^{(j)} \end{array}\right]_{3\times 3}, \\ & \boldsymbol{\Omega}_x = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right], \qquad \boldsymbol{\Omega}_z = \left[\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]. \end{aligned} \tag{6}$$

The above formulae are used to calculate the constraint Jacobian matrix. There are $m = 36$ constraint equations and $n = 42$ coordinates, thus the Jacobian matrix has m rows and n columns:

$$\begin{aligned} \boldsymbol{\Phi}_{\mathbf{q}} (\mathbf{q}) &= \left[\frac{\partial \boldsymbol{\Phi} (\mathbf{q})}{\partial \mathbf{q}}\right]_{m\times n} \\ &= \left[\begin{array}{ccc} \dfrac{\partial \Phi_1}{\partial q_1} & \cdots & \dfrac{\partial \Phi_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \Phi_m}{\partial q_1} & \cdots & \dfrac{\partial \Phi_m}{\partial q_n} \end{array}\right]. \end{aligned} \tag{7}$$

Let vector of absolute coordinates $\mathbf{q}^0$ describe the mechanism configuration characterized by angle $\theta = 0$ (see Fig. 1). It can be calculated that:

$$r = rank \left(\boldsymbol{\Phi}_{\mathbf{q}} \left(\mathbf{q}^0\right)\right) = 35 < m = 36. \tag{8}$$

The same result can be obtained for any other non-singular configuration of the mechanism. The obtained inequality indicates that Jacobian matrix is rank-deficient, i.e. that constraint equations are dependent. The number of independent constraint equations equals 35, thus the investigated mechanism has $42 - 35 = 7$ degrees of freedom. This result confirms the

outcome of an intuitive analysis of the mechanism mobility. Since the Jacobian matrix analysis proves that the mechanism is overconstrained, one can state that some or all constraint reaction forces cannot be uniquely determined (holding assumption of rigid bodies).

2.2. Joint reactions solvability analysis for the rigid body model. In the case of rigid body model, we consider the joint reaction to be solvable when an unique reaction solution exist. The problem of joint reactions solvability in rigid body mechanisms was addressed in our previously published works [12] and [14]. Methods allowing for detection of these joints for which reaction forces can be uniquely determined were developed. For convenience of the reader, these methods have been briefly summarized here, and then applied to the investigated mechanism.

The joint reactions solvability analysis is based on the concept of a direct sum, known in linear algebra [19]. We say that a linear vector space Z is a *direct sum* of subspaces X and Y, which is denoted as $Z = X \oplus Y$, when two conditions are fulfilled:

(1°) Z is a *sum* of subspaces X and Y ($Z = X + Y$), which means that any vector $\mathbf{z} \in Z$ can be represented as $\mathbf{z} = \mathbf{x} + \mathbf{y}$, where $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

(2°) If $\mathbf{x}_1$+$\mathbf{y}_1$ = $\mathbf{x}_2$+$\mathbf{y}_2$, provided that $\mathbf{x}_1 \in X$, $\mathbf{x}_2 \in X$, $\mathbf{y}_1 \in Y$ and $\mathbf{y}_2 \in Y$, then $\mathbf{x}_1$ = $\mathbf{x}_2$ and $\mathbf{y}_1$ = $\mathbf{y}_2$.

To check whether reactions in a selected joint (labelled P) can be uniquely determined, three linear vector spaces must be defined: $Z = span\left(\mathbf{\Phi_q}\right)$ – a linear space spanned by all rows of the constraint Jacobian matrix $\mathbf{\Phi_q}$, $X = span\left(\mathbf{\Phi_q}^P\right)$ – a linear space spanned by these rows of $\mathbf{\Phi_q}$ that correspond to the selected joint P, Y – a linear space spanned by the remaining rows of $\mathbf{\Phi_q}$.

It was proven [12] that the following statement is true: *if the linear space Z is a direct sum of subspaces X and Y ($Z = X \oplus Y$), then reaction forces in the selected joint P are uniquely solvable*.

The joint reaction solvability criterion based on direct sum condition is purely mathematical, however, it forms a basis for numerical methods that enable detection of kinematic pairs with uniquely solvable reactions. Three numerical methods of reaction solvability analysis were proposed in [12]. All of them are based on constraint Jacobian matrix investigation, which makes it possible to easily implement them in general-purpose multibody software. The first method consists in dividing the Jacobian matrix into submatrices and calculating their ranks, the second is based on *singular value decomposition* of the Jacobian matrix. To analyse the mechanism investigated in this paper, the third method, based on *QR decomposition* [8] of the Jacobian matrix, was used.

Firstly, the *QR* decomposition of the constraint Jacobian matrix $\mathbf{\Phi_q}$ (see Eq. (7)) was performed to obtain:

$$\mathbf{\Phi_q}(\mathbf{q})\mathbf{E} = \mathbf{QR}, \tag{9}$$

where $\mathbf{Q}$ is an orthogonal $(m \times m)$ matrix, $\mathbf{E}$ is an orthogonal $(n \times n)$ permutation matrix (the permutation is chosen so that absolute values of diagonal elements in $\mathbf{R}$ are decreasing) and $\mathbf{R}$ is a rectangular $(m \times n)$ upper triangular matrix.

Then, for each joint P, submatrices $\mathbf{\Phi}_\mathbf{q}^P$ and $\mathbf{Q}^P$, corresponding to the joint, were extracted from matrices $\mathbf{\Phi_q}$ and $\mathbf{Q}$, respectively. For example, in the case of joint B, submatrices $\mathbf{\Phi}_\mathbf{q}^B$ and $\mathbf{Q}^B$ consist of 4^{th}, 5^{th} and 6^{th} rows of respective matrices (see Eq. (5)).

Finally, for each joint P, matrix $\mathbf{B}^P$was calculated:

$$\left[\mathbf{B}^P\right]_{n\times(m-r)} = \left(\mathbf{\Phi}_\mathbf{q}^P\right)^T \mathbf{Q}_{col}^P, \tag{10}$$

where matrix $\mathbf{Q}_{col}^P$ consists of last $(m - r)$ columns of $\mathbf{Q}^P$.

In paper [12] it was proven that reactions in joint P can be uniquely determined when matrix $\mathbf{B}^P$ is a zero matrix. Otherwise, the joint reactions cannot be determined uniquely.

The joint reactions solvability analysis can be performed in any non-singular configuration of the mechanism . In our case, the configuration was described by vector $\mathbf{q}^0$. Equation (9) was employed to calculate matrix $\mathbf{B}$ for each joint of the mechanism. It was found that $\mathbf{B}$ is a zero matrix only for joints K, L, M and N, and it is a nonzero matrix for the other joints. According to the criterion mentioned above, the reactions in joints K, L, M and N can be determined uniquely, and the reactions in the remaining joints are not solvable. This qualitative information will be supported by quantitative results in the subsequent sections.

For more information on joint reactions solvability analysis the reader is referred to our papers [12] and [14] where this problem is discussed in greater detail. Note that mechanisms similar to the one presented in Fig. 1 serve as examples in these papers. It should be emphasised that flexibility of bodies – crucial for this contribution – is not taken into account in [12] and [14].

2.3. Simulated mechanism motion. Different variants of mechanism model were investigated, and for each variant one forward dynamics simulation was performed to observe motion and joint reaction forces. It was assumed that gravitational forces ($\mathbf{g} = \begin{bmatrix} 0 & 0 & -9.81 \end{bmatrix}^T$ (m/s^2)) are present and external constant force $\mathbf{F}_7 = \begin{bmatrix} 2000 & 3000 & 3000 \end{bmatrix}^T$ (N) and torque $\mathbf{T}_7 = \begin{bmatrix} 300 & 0 & 500 \end{bmatrix}^T$ (Nm) are applied to the moving platform (part 7) centre. The equations of motion were integrated numerically with initial configuration described by q^0 and zero initial velocity. All calculations were conducted using self-written procedures, and then validated using commercially available multibody software.

Different joint reactions were calculated for different variants of simulated mechanism, however, in each case, the same mechanism motion (to within numerical precision) was obtained. This confirms well-known fact that, in the case of an overconstrained mechanism with frictionless joints, the individual reactions are not unique, but their resultant effect (when motion is concerned) is unique.

The time history of the platform 7 centre of mass position relative to the global frame obtained during simulations is presented in Fig. 2.

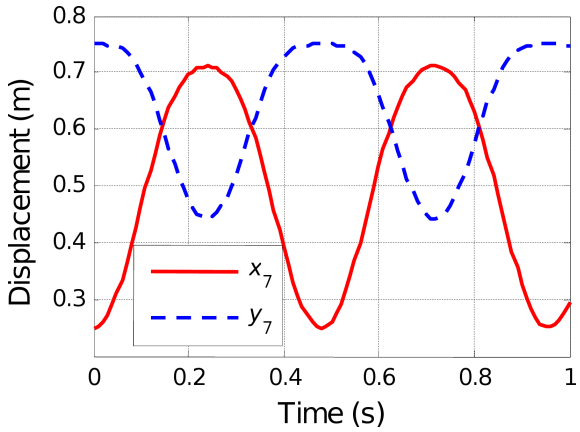

Fig. 2. Position (coordinates x and y) of the moving platform 7 centre of mass vs. time

3. Joint reactions in a rigid body model of the mechanism

3.1. Methods of redundant constraints handling. In multibody modelling one of the most popular methods of dealing with overconstrained systems consists in elimination of redundant constraint equations from the mathematical description of a multibody system. Thus, only the subset of independent equations is analyzed. In a given set of constraint equations, the redundant ones can be selected in many ways. Most often the selection is made automatically, according to the results of constraint Jacobian matrix factorization [1, 7, 8]. It is also possible to build a kinematically equivalent model without redundancies. In fact, building a model with modified kinematic structure is equivalent to "manual" elimination of redundant constraints.

It should be emphasized that, regardless of the used method of redundant constraints elimination, the reaction forces associated with eliminated constraints are arbitrarily set to zero. Obviously, in real mechanisms reactions associated with constraints neglected during analysis are not likely to be constantly equal zero. Moreover, setting the reactions of eliminated constraints to zero, transfers their loads to the constraints that remain in the mathematical model. Consequently, redundant constraints elimination affects not only the reactions of eliminated constraints but the reactions of remaining constraints as well. When solving for joint reactions is concerned, redundant constraints elimination can be interpreted as choosing one of infinitely many solutions.

Another possible method of handling redundant constraint equations is to preserve them in the mathematical model of a mechanism [2]. In that case, the system of equations used to calculate joint reaction forces is indeterminate. It is possible to use pseudo-inverse methods [9] or a penalty approach, often based on the augmented Lagrangian formulation, to solve the indeterminate system of equations [10, 11]. It can be shown, however, that also in the case of these methods, one of infinitely many possible solutions is arbitrarily chosen, and hence a non-unique solution is found [20].

It is worth noting that the redundant constraints elimination method is frequently implemented in general purpose multibody packages. The software users are usually advised to replace overconstrained models with kinematically equivalent models without redundant constraints. If the software user fails to follow the advice, the redundant constraints detected in the model are automatically eliminated. The problem that model after redundant constraints elimination does not reflect the real system and some of calculated reactions are non-unique usually is not signalized to the multibody software user.

In the next section the redundant constraints elimination method is exploited. Joint reactions calculated for different selections of eliminated constraints are investigated.

3.2. Unique and non-unique reactions. Qualitatively it is known that selected reactions are unique, whereas the other are not, since the constraint analysis was performed in Subsec. 2.2. The simulations presented in this section were made to show these properties quantitatively, and to establish reference results for the subsequent simulations.

To solve the direct dynamics problem, redundant constraints detected in the rigid body model of the mechanism were eliminated. Since redundant constraint equations cannot be selected uniquely, several possible variants of elimination were studied. Several simulations (one for each variant) were performed to observe the joint reaction forces. The simulations showed that reaction forces in joints K, L, M and N were not affected by the choice of eliminated redundant constraints. By contrast, it was observed that for the other joints different reactions were obtained for different choices of eliminated constraints. These findings corroborate correctness of joint reactions solvability analysis.

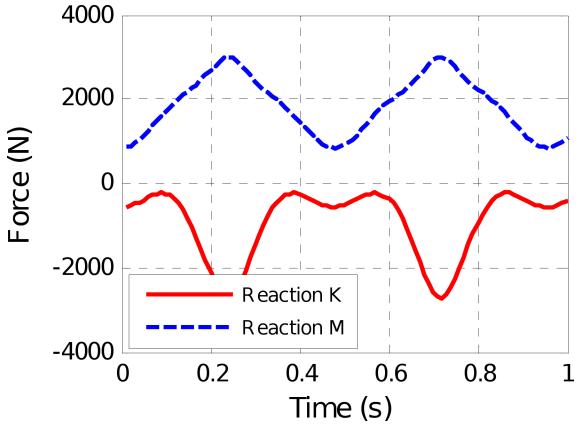

Fig. 3. Joint reaction forces calculated using rigid body model – examples of uniquely solvable reactions

Some exemplary results are shown in Fig. 3 and Fig. 4. The reactions in joints K and M are presented in Fig. 3 (the same results from all simulations), and reactions in joint G obtained for three different selections of eliminated constraints are presented in Fig. 4 (note that constraint numbering can be

identified in Eq. (5)). It is worth noting that the alternatively eliminated constraint equations represent joints B, C and F, nevertheless, elimination affected reaction force in joint G, as well as some other reactions (not presented here).

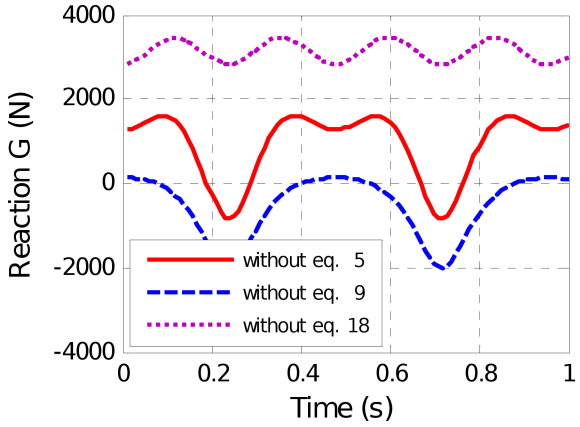

Fig. 4. Joint reaction forces calculated using rigid body model – examples of nonunique reactions

It should be clarified that spherical joint reaction is a three-component force vector, however, at least in our case, the component along longitudinal link axis is dominating (for all links and all joints). Therefore, in all figures presenting joint reactions (in the whole article), only the axial component is shown, which makes the presented results easier to analyse.

4. Joint reactions in partially flexible versions of the mechanism

4.1. Simplified modelling of a flexible rod. The mechanism links 1–6 are carrying mainly axial loads. Hence, it was possible to build a simplified model of a flexible rod in which only axial deflections are considered and other elastic effects are neglected. This simplification allowed for fast simulations using rigid-body approach. Note that more sophisticated, FEM-based models of mechanism parts flexibility were also built, and they are described in Sec. 5.

In the simplified model each link was split into two identical overlapping links with halved density (Fig. 5). These links were connected by a translational joint and a massless spring-damper element attached to both links was added.

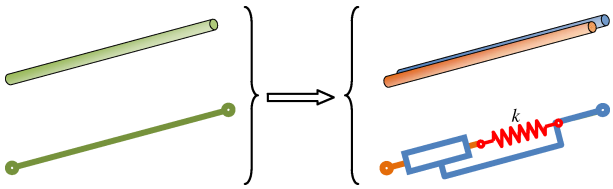

Fig. 5. A rod split into two overlapping rods connected by a translational joint and a spring – general view and kinematic scheme

The spring stiffness k was equal to the axial stiffness of the rod (cylindrical tube made of aluminium), and calculated as:

$$k = \frac{E \cdot \pi \left(d_2^2 - d_1^2\right)}{4\,l} \approx 17 \cdot 10^6 \text{ (N / m)}, \qquad (11)$$

where E is the aluminium Young modulus ($6.9{\cdot}10^{10}$ (Pa)), l is the undeformed rod length, d_1 and d_2 are inner and outer tube diameters, respectively (see Subsec. 2.1 for details). Small viscous damping was added to avoid high-frequency vibrations.

4.2. Effects of flexibility introduction. Different variants of flexible links placement in the mechanism structure were analysed. The moving platform (part 7) was regarded as much stiffer than the other mechanism parts, i.e., it was assumed that only flexibility of links determines joint reactions distribution. Hence, the moving platform was modelled as a rigid body during all simulations presented in this section (in the next section the platform flexibility was taken into account), and only selected or all links were modelled as flexible bodies. The simplified model of a flexible rod was utilised.

Partially flexible models with redundant constraints.

In the first series of simulations the upper links, i.e. *KL* and/or *MN*, were modelled as flexible bodies. It was found that, despite introducing some elasticity effects, redundant constraints were still present in the partially flexible mechanism. This finding should be emphasised, since it shows that partial flexibility introduction may be irrelevant to the redundant constraints occurrence.

As in the case of the fully rigid model, redundant constraints were eliminated, and several possible variants of elimination were examined. The simulations showed that taking into account flexibility of the upper links does not change practically the joint reactions. Negligibly small differences between the joint reactions calculated using the rigid and the partially flexible model are observed (less than 0.3% of force magnitude), since the models are similar but not equivalent to each other. As in the fully rigid model, reactions in joints K, L, M and N are not affected by the choice of eliminated constraints, whereas the other reactions depend on this choice.

The obtained results coincide with the results presented in Subsec. 3.2, thus quantitative examples are omitted here.

Flexibility introduction as an equivalent of redundant constraints elimination.

During the second series of simulations exactly one, arbitrarily selected, lower link was modelled as a flexible body; all four possibilities were tested. This time no redundant constraints were detected, thus – at least formally – the mechanism was not overconstrained anymore. Detecting no redundant constraints may be a bit misleading when joint reactions uniqueness is considered. Since three lower links are rigid, the fourth, flexible link cannot change its length, and thus is unable to carry axial load. Hence, in this case, introduction of flexibility may be regarded as being equivalent to – arbitrary, and thus nonunique! – redundant constraints elimination.

The simulations showed that reactions in joints K, L, M and N are not affected by the choice of the link treated as a flexible body (moreover, these reactions coincide with those obtained using the rigid body model), whereas the other joint reactions depend on this choice. The exemplary results are presented in Fig. 6.

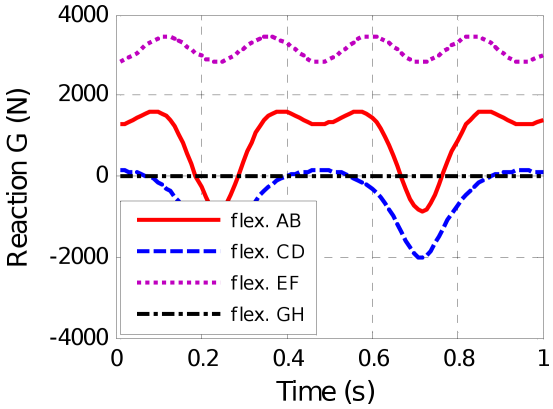

Fig. 6. Joint reaction forces in a partially flexible mechanism (with one lower link flexible)

The elasticity effects of upper links were introduced to the model and the series of simulations (with exactly one lower link flexible) was repeated. It was found that accounting for flexibility effects in upper links resulted in negligible changes in the obtained results. Thus, for the observed joint reaction forces it is irrelevant whether the upper links are modeled as rigid or flexible bodies.

Partially flexible models with non-unique joint reactions.

Two out of four lower links were modelled as flexible bodies during the next series of simulations; all six possible combinations were analysed. No redundant constraints were detected. The flexible bodies were able to carry axial loads, thus this time accounting for elasticity cannot be regarded as being equivalent to redundant constraints elimination.

The "elastic degrees of freedom" were active, nevertheless – as in the previous cases – it was found that arbitrary choice of flexible links placement in the mechanism structure affects the obtained joint reaction forces. Only the reactions in joints K, L, M and N were not affected by this choice and were the same as in the previous simulations (to be exact: only negligibly small differences were observed). Exemplary results are presented in Fig. 7.

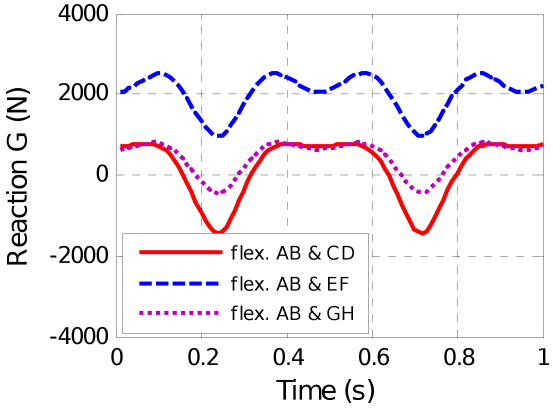

Fig. 7. Joint reaction forces in a partially flexible mechanism (with two lower links flexible)

The simulations were repeated for a model with two flexible lower links and one or two flexible upper links. It was found that for the obtained reactions it was practically unimportant whether the upper links were rigid or flexible, since only negligible differences were observed.

During the next series of simulations three out of four lower links were modelled as flexible ones; all four possibilities were tested. Essentially, the findings concerning reactions uniqueness were no different from the obtained earlier. Exemplary results are presented in Fig. 8. Again, flexibility of the upper links showed negligible influence on the obtained results.

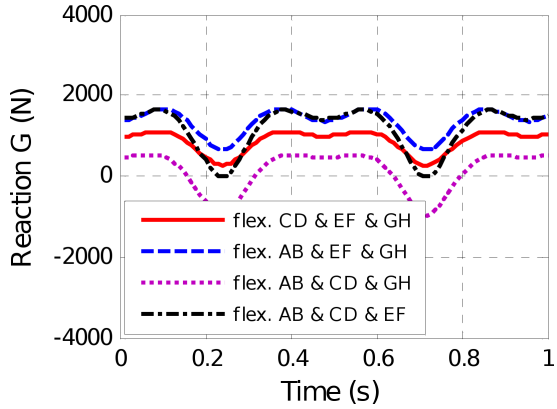

Fig. 8. Joint reaction forces in a partially flexible mechanism (with three lower links flexible)

Partially flexible models with unique joint reactions.

All lower links were modelled as flexible bodies during the final simulations of the partially flexible model. Since it was earlier assumed that the moving platform elasticity effects are negligible when compared with the links elasticity effects, the platform was modelled as a rigid body. Firstly, the upper links were modelled as rigid bodies, then flexibility of one or both of them was taken into account. Only negligibly small differences were found between results obtained for rigid upper links and for flexible upper links. As in all previous variants of the model, it was irrelevant whether the upper links were modelled as rigid or flexible bodies.

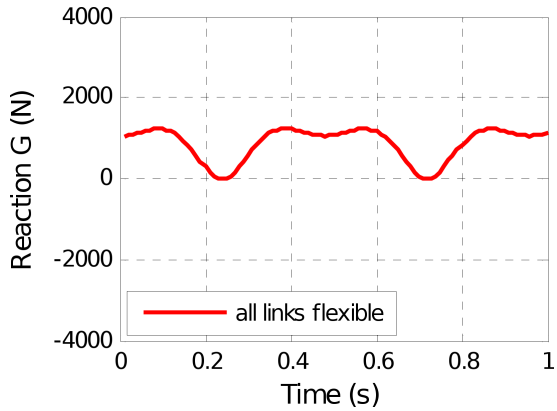

Fig. 9. Joint reaction forces in a partially flexible mechanism (with all lower links flexible)

If the assumptions that the joint friction is negligible and links flexibility effects are more significant than the platform

flexibility effects are valid, then joint reaction forces calculated in the final series of simulations are supposed to be the "true" ones, i.e. they closely correspond to the reactions expected to appear in the real system. Thus, the model version with four flexible lower links may be regarded as the one which predicts the unique reaction solution.

It was found that reactions in joints K, L, M and N were practically the same as calculated for other variants of the model (Fig. 3). Since in every simulation in this series the same results were obtained, only one set of reactions in lower links joints was found. The exemplary results for joint G are presented in Fig. 9. It is worth noting that the proper reactions in joint G (and some other joints, not presented in the figure) were not found using other, previously described, versions of the partially flexible model.

5. Joint reactions in a fully flexible version of the mechanism

5.1. Modelling of flexible parts. To account for elasticity of mechanism parts FEM models of the links and the platform were built. Then, the FEM models of parts were incorporated to the multibody model of the mechanism. The floating frame of reference (FFR) approach was utilised [21]. In the FFR formulation, the motion of a point of a body is composed of the motion of its reference (rigid motion) and the motion of the point with respect to its reference (deformation). It is assumed that only small, linear body deformations relative to a local reference frame are considered, while that local reference frame is undergoing large, non-linear global motion.

To reduce the number of degrees of freedom responsible for deformations the Craig-Bampton [22] method was used, thus in the multibody model the deformations of flexible parts were represented in modal basis. In the Craig-Bampton method the modes are partitioned into the constraint modes, and the fixed-boundary normal modes. The software applied to simulations uses orthogonalized Craig-Bampton modes [23].

In our case, to calculate the constraint modes for the links and the platform, the points coinciding with spherical joint centers were selected as the attachment nodes. Moreover, the point of external load application (at the platform) was selected to be an attachment node. Thirty normal modes were utilized in the case of the platform, and twenty in the case of links. Examples of modal shapes are presented in Fig. 10.

5.2. Simulation results. All simulations described in Sec. 4 were repeated. This time, to account for links elasticity effects, FEM-based models were utilised instead of simplified rigid-body models. The obtained results were practically the same as in the previous simulations (in the case of calculated mechanism motion and joint reaction forces only negligibly small differences were observed). This shows that – in our particular case – simplified modelling of flexible rods was a good enough approximation.

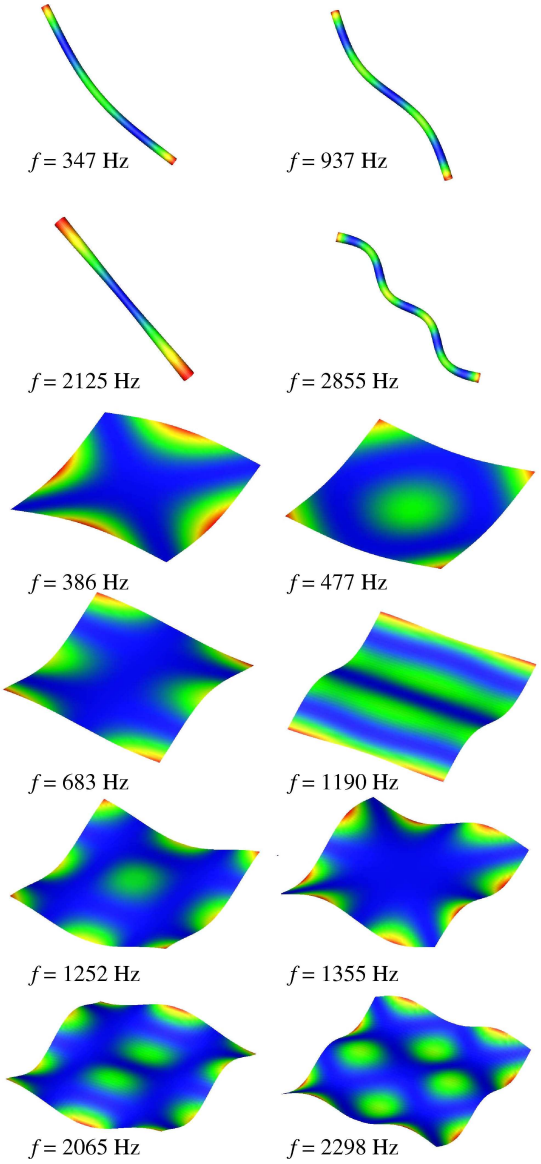

Fig. 10. Examples of platform and link modal shapes and corresponding frequencies

Finally, during the last simulation all moving bodies, including the platform, were modelled as flexible. The reactions calculated during this simulation can be regarded as the "true" ones, since the multibody system was modelled in its full complexity, and no arbitrary decisions on where to introduce flexibility were made. The joint reaction forces calculated during the final simulation were (almost) the same as observed during earlier simulations with flexibility effects of all four lower links taken into account. This confirms that the model version with four flexible lower links may be regarded as the one which predicts the unique reaction solution, and that it is irrelevant whether upper links are modelled as rigid or flexible bodies. An example of a lower link joint reaction is presented in Fig. 9.

It is worth noting that reactions in joints K, L, M and N predicted by the fully flexible model (some examples are presented in Fig. 3) agree with the ones calculated for other

variants of the model, including the fully rigid one. It is not surprising, since it was proved [12] that these reactions can be determined uniquely using rigid body model, thus accounting for flexibility should not change them radically. To find the unique reaction solution for other joints it was necessary to account for flexibility of all lower links.

It should be mentioned that simulations for a less symmetric mechanism (i.e., mechanism with links of different stiffnesses) were also performed. The obtained results were similar to those presented in the paper. In order to calculate realistic joint reactions, it was necessary to account for stiffness of all lower links. On the other hand, it was not important whether upper links were modelled as rigid or flexible bodies.

6. Conclusions and discussion

If redundant constraints exist in a multibody system, it is not possible to determine uniquely all constraint reactions using a rigid body model. Moreover, in the case of friction, if the joint reaction force cannot be uniquely determined, then the joint friction cannot be uniquely determined as well [14]. As a result, simulated motion may not be unique. The methods proposed in [12] enable detection of joints for which reactions and friction forces can be uniquely determined, despite the existence of redundant constraints. Thus, for some overconstrained mechanisms it is possible to gain information about loads on selected joints and bodies using rigid body approach. In general, however, to find a unique set of all joint reaction forces in an overconstrained multibody system it is necessary to abandon the assumption that all bodies are rigid.

The presented study shows that it is neither obvious nor straightforward which parts of the investigated mechanism should be modelled as flexible bodies to obtain the unique reaction solution. The most important point is that not always taking into account flexibility of selected bodies guarantees that calculated reactions are the unique, "true" ones. It is shown that, in many cases, problems typical for overconstrained systems can be observed when analysing models (formally) without redundant constraints. The results obtained for partially flexible models can be misleading or completely erroneous.

The study shows several possible effects of flexible parts introduction. In some cases, the system remains overconstrained, despite taking into account flexibility of selected bodies. In other cases, introduction of flexible bodies can be regarded as being equivalent to – arbitrary or random – redundant constraints elimination. In these cases, some deformations of elastic part, crucial for joint reaction calculations, are blocked by the rigid parts. In other cases, it is possible that, due to flexibility introduction, there are no redundant constraints (at least formally) and deformations of the flexible parts are allowed, however, the obtained reaction solution depends on the choice of parts modelled as flexible bodies. Finally, in some cases, taking into account flexibility of selected bodies leads to the desired, unique reaction solution.

In many technical problems deformations of mechanism parts are negligible when mechanism motion is concerned, thus rigid body models can be successfully applied. When modelling such mechanisms, the main reason for flexibility introduction can be the need for finding unique reaction solution. The results discussed in the paper show that if all reactions acting on a selected part can be uniquely determined using a rigid body model (methods described in [12] can be applied to check this), there is no point in taking into account flexibility of this part when solving for joint reactions. This would neither change the calculated reactions acting on the selected part nor the reactions acting on the other parts.

In this paper absolute Cartesian coordinates were utilised (with Euler angles used to parameterize rotation), however, it should be emphasised that uniqueness of joint reactions in rigid body model depends only upon the kinematic structure of the mechanism, thus – when solvability of joint reactions is considered – the type of coordinates is irrelevant. Similarly, it is irrelevant which multibody simulation software is used to perform calculations. Moreover, it does not matter which parameterization of rotation is used.

Acknowledgements. The Project was co-financed by the European Regional Development Fund within the framework of the 1. priority axis of the Innovative Economy Operational Programme, 2007–2013, through grant PO IG 01.02.01-00-014/08-00, and by the Institute of Aeronautics and Applied Mechanics statutory funds.

REFERENCES

[1] E.J. Haug, *Intermediate Dynamics*, Prentice Hall, New Jersey, 1992.

[2] J. Garcia de Jalon and E. Bayo, *Kinematic and Dynamic Simulation of Multibody Systems: the Real-Time Challenge*, Springer, Berlin, 1994.

[3] A.A. Shabana, *Computational Dynamics*, John Wiley & Sons, 2^{nd} edition, New York, 2001.

[4] K. Arczewski and J. Frączek, "Friction models and stress recovery methods in vehicle dynamics modelling", *Multibody System Dynamics* 14, 205–224 (2005).

[5] E.J. Haug, *Computer Aided Kinematics and Dynamics of Mechanical Systems*, Allyn and Bacon, Boston, 1989.

[6] P. Malczyk and J. Frączek, "Cluster computing of mechanisms dynamics using recursive formulation", *Multibody System Dynamics* 20, 177–196 (2008).

[7] P.E. Nikravesh, *Computer-Aided Analysis of Mechanical Systems*, Prentice Hall, London, 1988.

[8] L.J. Corwin and R.H. Szczarba, *Multivariable Calculus*, Marcel Dekker, New York, 1982.

[9] F. E. Udwadia and R. E. Kalaba, *Analytical Dynamics: a New Approach*, Cambridge University Press, Cambridge, 1996.

[10] E. Bayo and R. Ledesma, "Augmented lagrangian and mass-orthogonal projection methods for constrained multibody dynamics", *Nonlinear Dynamics* 9, 113–130 (1996).

[11] W. Blajer, "Augmented Lagrangian formulation: geometrical interpretation and application to systems with singularities and redundancy", *Multibody Systems Dynamics* 8, 141–159, (2002).

[12] M. Wojtyra, "Joint reactions in rigid body mechanisms with

dependent constraints", *Mechanism and Machine Theory* 44, 2265–2278 (2009).

[13] M. Wojtyra, "Joint reaction forces in multibody systems with redundant constraints", *Multibody System Dynamics* 14, 23–46 (2005).

[14] J. Frączek and M. Wojtyra "On the unique solvability of a direct dynamics problem for mechanisms with redundant constraints and Coulomb friction in joints", *Mechanism and Machine Theory* 46, 312–334 (2011).

[15] K. Mianowski and M. Wojtyra, "Virtual prototype of a 6-DOF parallel robot", *Eleventh World Congress in Mechanism and Machine Science* 1, 1604–1608 (2004).

[16] Z. Dziopa, I. Krzysztofik, and Z. Koruba, "An analysis of the dynamics of a launcher-missile system on a moveable base", *Bull. Pol. Ac.: Tech.* 58, 651–656 (2010).

[17] W. Ostapski, "Analysis of the stress state in the harmonic drive generator-flexspline system in relation to selected structural parameters and manufacturing deviations", *Bull. Pol. Ac.: Tech.* 58, 683–698 (2010).

[18] S.-M. Song and X. Gao, "The mobility equation and the solvability of joint forces/torques in dynamic analysis", *ASME J. Mechanical Design* 114, 257–262 (1992).

[19] G. Strang, *Linear Algebra and Its Applications*, Academic Press, New York, 1980.

[20] M. Wojtyra and J. Frączek, "Comparison of selected methods of handling redundant constraints in multibody systems simulations", *ASME J. Computational and Nonlinear Dynamics* 8, 021007 (1–9), DOI: 10.1115/1.4006958 (2013).

[21] A.A. Shabana, *Dynamics of Multibody Systems*, Cambridge University Press, Cambridge, 2010.

[22] R.R. Craig and M.C. Bampton, "Coupling of substructure in dynamic analysis", *American Institute of Aeronautics and Astronautics J.* 6, 1313–1319 (1968).

[23] *Theory of Flexible Bodies*, MD.ADAMS.R3 Documentation, 2008.