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A structural optimisation viewpoint on growth phenomena

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Abstract. Evolutionary solid bodies undergoing changes of mass, of properties, and of shapes are considered in models of growth and adaptation and similarily in structural optimisation. A fundamental separation of different growth phenomena and a subsequent parametrisation using independent design variables for the amount of substance as well as for molar mass and molar volume facilitates an efficient formulation of the design space. Thus, the effects of design variations, i.e. change of amount of substance, on the variations of the structural response, i.e. the deformation in physical space, can be clearly described. Overall, a novel treatment of growth processes based on an evolution of the amount of substance is outlined. The parallelism of variations in physical and design space are highlighted and compared with the multiplicative decomposition of the deformation gradient into a growth and an elastic part incorporating an incompatible intermediate configuration. This drawback is overcome by a compatible manifold based on material points modelling the amount of substance outside of any geometrical space.

Key words: computational mechanics, structural optimisation, sensitivity analysis, configurational mechanics, growth.

1. Introduction

Models of growth and adaptation are similar to models of structural optimisation because *design* variables, say s, in addition to the mechanical and thermomechanical state variables parameterised by time t are involved. The difference of growth compared with structural optimisation is the fact that growth is a physical phenomenon and interacts with the deformation of the body. In case of growth, design variables are treated by evolution laws $s = s(t)$ as sketched in Fig. 1.

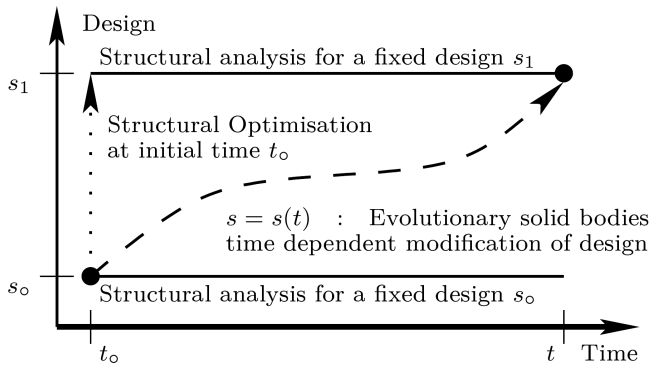

Fig. 1. Interaction of design and deformation

The solid lines represent the structural analysis paths for fixed designs s_0 and s_1. The dotted line describes structural optimisation as a time independent process being able to jump from one design s_0 to an improved design s_1 outside of physics. The dashed curve highlights the interaction of a design modification (growth) and the physical deformation process. An illustrative example linking structural optimisation and growth is given in Sec. 2.

Growth theories have been developed from different scientific points of view with increasing complexity and for various biomechanical applications over the last decades, see e.g. [1] for an overview, appreciation and further literature. A few aspects are highlighted in more detail.

The multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}_e\mathbf{F}_g$ into an elastic and a growth part, introduced in [2], allows a general treatment of growth processes. An incompatible intermediate configuration is introduced, see [3] for a comparison with similar theories.

A geometric approach for material inhomogeneities and their evolution has been formulated in [4]. The incompatible intermediate configuration can be considered as a non-Euclidean material manifold, see [5]. A domain variation technique, similarly used in structural optimisation, see [6,7] for details, has been applied to volumetric and surface growth in [8,9]. The term *material space* is used in Eshelbian mechanics for what is called *design space* in structural optimisation. The notion *design* is used highlighting the structural optimisation viewpoint.

A structural optimisation viewpoint on evolution can be determined using optimality principles instead of evolution laws, see [10]. Thus, design space models and variational design sensitivity analysis based on an improved formulation of continuum mechanics contribute significantly to an enhanced formulation, see [11,12].

The proposed novel concept consists of a *separation of all fundamental quantities*, a *reformulation of continuum mechanics* as well as a *framework for growth theories* generated from the first two steps. These items are outlined here. The missing information, i.e. the constitutive laws, the consistent linearisation technique and the computational scheme, are postponed to a forthcoming paper.

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2. An illustrative example

The author's structural optimisation viewpoint on growth phenomena sketched in Fig. 1 is outlined based on the evolution of a cantilever beam from an initial design with height $s_\circ$ to an optimised layout with height s_1, see Fig. 3.

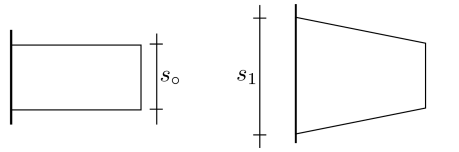

Fig. 2. Initial and optimal cantilever beam

2.1. The structural optimisation viewpoint. In structural optimisation, an initial design $s_\circ$ within a design space is generated and a gradient based nonlinear programming approach, e.g. the sequential quadratic programming (SQP) method, is applied to update these design variables s iteratively. An objective function f, say the structural compliance, and constraint functions g, say limits on displacements, stresses and on the amount of mass, must be considered. Overall, a stationary point of the Lagrangian function $L = f - \mu\, g$ is computed. All design modifications are modelled outside of physics, i.e. no growth theory is encountered to step from some design $s_\circ$ to another design s_1, see the dotted line in Fig. 1. The responses are always computed from scratch for all modified designs, see the solid lines in Fig. 1, and sensitivity analysis yield the desired gradients. Overall, the design variables s are not linked with physical time t.

2.2. The growth viewpoint. Alternatively, growth theories consider balance laws for mass within thermodynamics to end up with evolution laws for different physical properties. Well-known ideas from structural optimisation such as *design space* with *design variables* and *design velocitiy fields*, are missing. Moreover, an evolution of the material body is not directly available in classical theories. This is not astonishing due to the fact that continuum mechanics, see [13], has been outlined for a unique material body.

2.3. The benefits of joining both viewpoints. Thus, structural optimisation and growth theory, i.e. the parameterised design space as well as evolution laws for mass, should be linked. Unfortunately, the standard layout of structural optimisation using discrete design variables is not fully appropriate to be applied to growth. Alternatively, a more rigorous field description for an evolution of material bodies using tensor notation and manifolds (usually not used in structural optimisation textbooks) should be applied, see [11,12]. The author's approach to link design modifications and growing mass is outlined in detail in the sequal. As a consequence, the classical assumption of a fixed material body is no longer valid.

3. Observations from biological growth

The classical continuum mechanical theory assumes a fixed material body $\mathfrak{B}$ with a given reference placement $\mathcal{K} \equiv \Omega_{\mathrm{R}}$ and a constant mass m, see [13]. These assumptions are no longer valid for evolutionary solid bodies considered in structural optimisation and growth problems.

3.1. Basics on the amount of substance. The basics from natural sciences defined in the *International System of Units* (SI) are summarised as follows.

- The *amount of substance* n of a sample or system is a physical quantity which is proportional to the number of elementary entities present. The SI unit of *amount of substance* is a *mole* with unit symbol *mol*.
- *Elementary entities* may be atoms, molecules, ions, electrons, or particles, the choice of which is dependent upon context and must be stated.
- The *amount of substance* n is related to the *number of elementary entities* $N = N_A \cdot n$ with *Avogadro constant* $N_A = 6.022045 \cdot 10^{23}\ 1/mol$.

The basic variable is the *number of elementary entities* $N \in \mathbb{N}$ or equivalently the *amount of substance* $n \in \mathbb{N}$. Thus, growth is the evolution of these integer variables.

3.2. Basics on molar mass and molar volume Additionally, the two fundamental material properties of any existing and fixed amount of substance may change as well. These are the mass and the volume per amount of substance, i.e. the *molar mass* M_m, measured in $[kg/mol]$, and the *molar volume* V_m, measured in $\left[m^3/mol\right]$. The material property *mass density* ϱ is a derived quantity of molar mass and molar volume, i.e.

$$\varrho = \frac{\text{mass}}{\text{volume}} = \frac{\frac{\text{mass}}{\text{amount of substance}}}{\frac{\text{volume}}{\text{amount of substance}}} = \frac{\text{molar mass}}{\text{molar volume}} = \frac{M_m}{V_m}.$$

3.3. Growth phenomena in nature. The separation into three basic ingredients (amount of substance, molar mass, molar volume) allows a refined description of the growth phenomena compared to the traditional viewpoint of a mass density ϱ of material points $\mathfrak{s}$ of a material body $\mathfrak{B}$.

- The amount of biological material is changed during growth, i.e. cells occur or vanish. An initial open gap is filled with new cells without changing the existing cells. The design dependent amount of substance $n = n(s)$ enables an evolution of amount of substance $\partial n/\partial s$, i.e. elementary entities are created or destroyed. This is achieved using internal source and external flux terms.
- Independently, the amount of mass can be modified. For example, an existing cell has the ability to absorb or to segregate mass into its microstructure. The molar mass $M_m(s)$ yields the evolution $\partial M_m/\partial s$ caused by internal source or external flux terms.
- Similarly, the microstructure of any existing material may have the ability to rearrange its internal structure such that the occupied volume in space is modified. Thus, the design dependent molar volume $V_m = V_m(s)$ yields an evolution of the material constant $\partial V_m/\partial s$.

To conclude, any existing material with given and fixed amount of substance, say cells for example, may change its

mass and volume. Additionally, the amount of substance of the material body may change. Furthermore, the overall constitutive behaviour of the material body can be designed. Overall, the described phenomena lead to *design variables* s, which parameterise the *design space* and can be interpreted as an *evolution time* or a *material deformation process time* in context of Eshelbian mechanics. This viewpoint enhances the phenomena in the *physical space* parametrised by *physical time* t.

4. Preliminaries on continuum mechanics

All physical and biological phenomena are discrete. There are finite numbers of atoms, molecules and cells involved in the observed structure. Alternatively, the mathematical theory smears the particles over a finite volume. Thus, only densities can be described in continuum mechanics. Thus, the concepts of *manifolds* and of *homogenisation* must be utilised for the detailed description.

4.1. The role of manifolds and charts. The manifold idea has been introduced to continuum mechanics at its early days, see [13,15]. Roughly speaking, *differentiable manifolds* are those sets which can be covered by an *atlas* using a finite number of *charts*. Each chart $(\mathfrak{U}_\mathfrak{s}, \Psi_\vartheta)$ describes the material manifold $\mathfrak{B}$ in a local environment $\mathfrak{U}_\mathfrak{s}$ of a chosen material point $\mathfrak{s}$ using coordinates. In detail, the ball $\mathrm{D}_\vartheta \subset \mathcal{B}$ at point $\vartheta \in \mathcal{B}$ yields the coordinate representation of the material body, see Fig. 3.

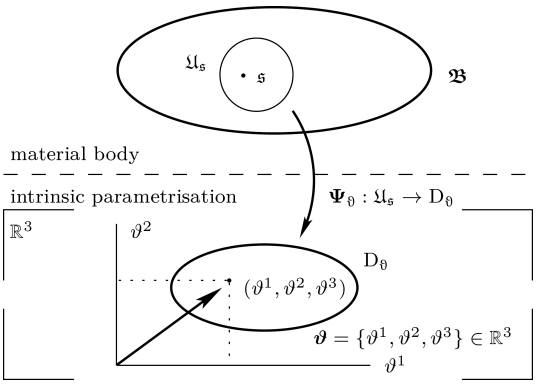

Fig. 3. Manifolds and coordinate systems

The properties of the manifold are not effected by the choice of coordinate system and the change of coordinate system is sufficiently smooth. The abstract manifold perspective is more general and rigorous. The computational formulations always rely on the special choice of the coordinate system and are more intuitive.

The points and manifolds are denoted by Fraktur letters. The corresponding points and charts with respect to coordinate systems are written using Calligraphic letters. The mappings between manifolds and corresponding charts are denoted by Ψ. Details are given in the figures.

4.2. Homogenisation of the discrete amount of substance. There are discrete particles in nature but only continuous values in continuum mechanics. The discrete values must be smeared over some set of points, i.e. the *reference volume element* (rve), in the so-called *homogenisation* process. Thus, only real valued densities ν of the amount of substance with respect to the different coordinate systems are considered for each point instead of integer values of the *amount of substance* $n \in \mathbb{N}$ or the *number of elementary entities* $N \in \mathbb{N}$. Therefore, the fundamental variable is either the *density of substance* or the corresponding *density of basic entities* defined on the considered set.

The densities ν are related to coordinate volume measures which are in general no physical volume because the manifolds may have a geometry-free meaning.

The different pictures of the material body under consideration must be separated in order to describe and highlight the different effects most clearly. This must be done in line with the field character of continuum mechanics. Growth, described here as a modification of the amount of substance within the material (design) space, must be separated from the placement into the Euclidean space and the physical motion in Euclidan space.

5. Elements of an enhanced framework

The proposed continuum mechanical setting for different but fixed bodies is visualised in Fig. 4 using coordinates.

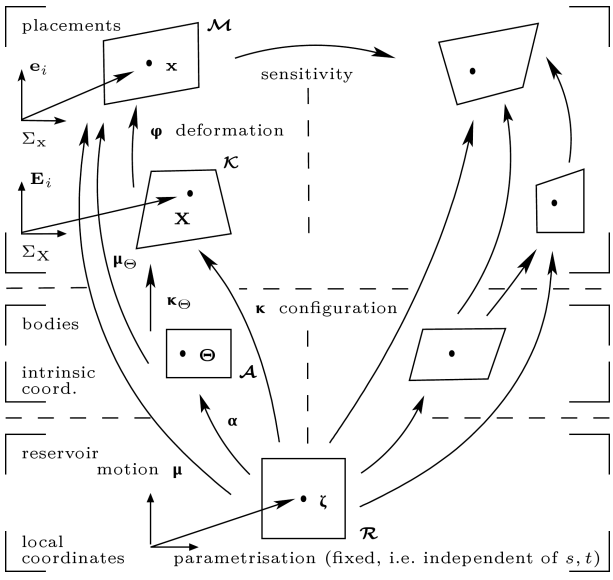

Fig. 4. Continuum mechanics applied to two different bodies

5.1. The reference and current placements. The reference placement $\mathcal{K}$ with Lagrangian coordinates $\mathbf{X}$ relative to observer Σ_X is mapped by $\boldsymbol{\varphi}$ to the current placement $\mathcal{M}$ with Eulerian coordinates $\mathbf{x}$ relative to observer Σ_x. Furthermore, the displacement vector $\mathbf{u} = \mathbf{x} - \mathbf{X}$ within the Euclidean space is utilised. This picture is widely used for theoretical and computational investigations.

Both placements $\mathcal{K}$ and $\mathcal{M}$ describe the embedding of the material body $\mathfrak{A}$ with set $\mathcal{A}$ of intrinsic coordinates into the Euclidean space given by mappings $\boldsymbol{\kappa}_\Theta$ and $\boldsymbol{\mu}_\Theta$. They can also be described using $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}$ defined on $\mathcal{R}$.

5.2. The initial material body with intrinsic coordinates The material body $\mathfrak{A}$ consists of the initial set of material points $\mathfrak{S}$. The *intrinsic coordinates* $\boldsymbol{\Theta}$, also termed *convected coordinates*, are inscribed to the body leading to a one-to-one mapping $\boldsymbol{\Psi}_\Theta : \mathfrak{A} \to \boldsymbol{\mathcal{A}}$ between the material points $\mathfrak{S} \in \mathfrak{A}$ and their (intrinsic) names $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{A}}$. Thus, the material body represents the homogenised material in mathematical terms with constant substance density, say $\nu_\Theta \equiv 1$. Every material point represents the same amount of substance and $\boldsymbol{\Theta}$ is the *name* of this (small) portion of homogenised material, i.e. $\mathrm{d}n \equiv \mathrm{d}V_\Theta$. The body grows with constant substance density, i.e. more names are needed in case of more material points, as outlined in Sec. 6.

5.3. The material reservoir with local coordinates The missing link in continuum mechanics is a technique to define the material body itself. The role of manifold and atlases, see Subsec. 4.1, can be converted, i.e. the material body is (locally) *defined* by the charts of an atlas, see [11]. This idea results in the concept of a *material reservoir* $\boldsymbol{\mathcal{R}}$ consisting of *generator points* $\mathfrak{z}$ defining material points and material bodies $\mathfrak{A}$. The choice of the coordinates is arbitrary and does not effect the properties. Thus, a second local coordinate system with fixed set $\boldsymbol{\mathcal{R}}$ independent of the choice of $\boldsymbol{\mathcal{A}}$ is introduced. Consequently, the substance density ν_ζ (reservoir) is no longer constant while the density ν_Θ (material body) is fixed, say $\nu_\Theta = 1$.

5.4. Gradient operators and tangent mappings. The domains are equipped with operators to be distinguished by subscripts indicating the coordinates. The mappings between the tangent spaces are derived from the point mappings using the introduced gradient operators. Pull-back and push-forward transformations for operators as well as for line, area and volume elements on different coordinate domains are available, see Fig. 5 for an overview.

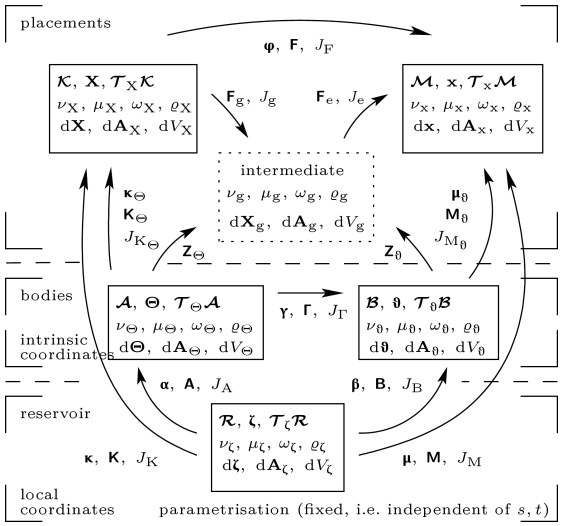

Fig. 5. Summary of properties of different domains

5.5. Amount of substance, mass and volume. There are four functions measuring the density of amount of substance ν, the mass density μ and the volume density ω with respect to the volume of the coordinates systems, where additional indices indicate the different domains. These densities are related by

$$\mu = M_m\,\nu, \quad \omega = V_m\,\nu \quad \text{and} \quad \varrho = \mu/\omega = M_m/V_m \tag{1}$$

with constants M_m, V_m, i.e. molar mass and volume.

5.6. Summary of properties. The term *manifold* indicates that the topology of the structure, i.e. the neighbourhood of material points, has been introduced. The evolution process with added substance is considered in Sec. 6.

On the other hand, no geometrical attributes have been introduced so far. The placements of the initial and grown bodies in Euclidean space and therefore the shapes of the initial and evolved structures are discussed in Sec. 7.

The physical and design spaces are parametrised by physical time t and design (evolution) s, respectively, and are coupled in Sec. 8. The derivatives are denoted by a superposed dot and a superscript prime, respectively.

6. Growth of material bodies

The considerations are restricted to a modification of the amount of substance ν keeping other properties fixed.

6.1. Initial and grown material bodies. All points and bodies are continuously generated by mappings on the reservoir $\boldsymbol{\mathcal{R}}$. The growth process starts from the initial material body $\mathfrak{A}$ to yield the grown material body $\mathfrak{B}(s)$. All abstracts quantities can be represented using their intrinsic and local coordinates. Furthermore, the mapping

$$\boldsymbol{\gamma} : \begin{cases} \boldsymbol{\mathcal{A}} & \to & \boldsymbol{\mathcal{B}} \\ \boldsymbol{\Theta} & \mapsto & \boldsymbol{\vartheta} := (\boldsymbol{\beta} \circ \boldsymbol{\alpha}^{-1})(\boldsymbol{\Theta}) \end{cases} \tag{2}$$

models the growth process with intrinsic coordinates $\boldsymbol{\vartheta}(s)$, see Subsec. 4.1 and Fig. 6.

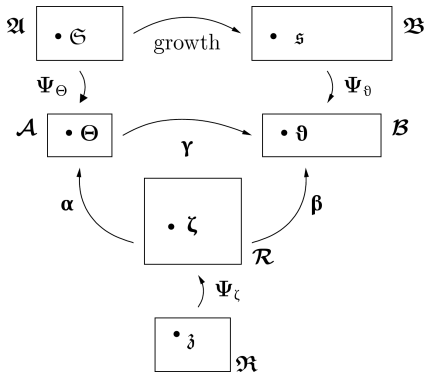

Fig. 6. Growth of material body

6.2. Kinematics of evolving bodies. The evolution of mapping $\boldsymbol{\beta} : \boldsymbol{\mathcal{R}} \to \boldsymbol{\mathcal{B}}$ and of its tangent $\mathbf{B} = \mathrm{grad}\boldsymbol{\beta}$ yields

$$\boldsymbol{\vartheta}' = \boldsymbol{\beta}'(\boldsymbol{\zeta}) \qquad \text{and} \qquad \mathbf{B}' = [\mathrm{grad}\boldsymbol{\beta}]' = \mathrm{grad}\boldsymbol{\beta}',$$

i.e. gradient and derivative can be interchanged on the fixed reservoir. Thus, the growth mapping $\boldsymbol{\gamma} : \boldsymbol{\mathcal{A}} \rightarrow \boldsymbol{\mathcal{B}}$ with $\vartheta = \boldsymbol{\gamma}(\boldsymbol{\Theta}) = (\boldsymbol{\beta} \circ \boldsymbol{\gamma}^{-1})(\boldsymbol{\Theta})$ evolves as well, i.e.

$$\vartheta' = \boldsymbol{\gamma}'(\boldsymbol{\Theta}) = (\boldsymbol{\beta}' \circ \boldsymbol{\alpha})(\boldsymbol{\Theta}) \qquad \text{and} \qquad \boldsymbol{\Gamma}' = \mathbf{B}' \, \mathbf{A}^{-1}.$$

The *growth velocity gradient* is given by

$$\mathrm{grad}_{\vartheta} \boldsymbol{\gamma}' = \boldsymbol{\Gamma}' \, \boldsymbol{\Gamma}^{-1} = \mathbf{B}' \, \mathbf{A}^{-1} \, \mathbf{A} \, \mathbf{B}^{-1} = \mathbf{B}' \, \mathbf{B}^{-1}$$

and corresponds to the evolution of intrinsic coordinates.

6.3. Balance of amount of substance and evolution laws. The substance densities with $\nu_{\vartheta} \equiv 1$ are coupled with the transformation between local and intrinsic coordinates

$$\nu_{\zeta} = \nu_{\vartheta} \, \det [\mathrm{grad} \boldsymbol{\beta}] = \nu_{\vartheta} \, \det \mathbf{B} = \nu_{\vartheta} \, J_{\mathrm{B}}. \tag{3}$$

The substance density $\nu_{\zeta}(\zeta, s)$ undergoes an evolution

$$\nu'_{\zeta} = \nu_{\zeta} \, \mathrm{div}_{\vartheta} \boldsymbol{\beta}' = \nu_{\zeta} \, (\mathbf{1} : \mathrm{grad}_{\vartheta} \boldsymbol{\beta}'), \tag{4}$$

which relates to the evolution of generator mapping $\boldsymbol{\beta}$. Thus, this type of growth yields compatible sets $\boldsymbol{\mathcal{B}}$.

The evolution of ν_{ζ} is derived from a local version of the *amount of substance balance equation* which is related to the mass balance equation available in all other growth theories, see e.g. [8,9]. Furthermore, the type of evolution can be specified in the *evolution law*, e.g. volumetric growth based on $\dot{\mathbf{F}}_{\mathrm{g}}$ pointing to an incompatible intermediate configuration, see Sec. 8 for an alternative.

7. Placements of growing bodies in space

The geometrical viewpoint is formulated by embedding the material body into the Euclidean space whereby the set of material points obtains its geometrical shape. This extension is possible because the material points are equipped with the *molar volume* V_m. Additionally, the motion of the material body in space can be described.

7.1. Placement and deformation of two different bodies The placement of one or two fixed material bodies in the Euclidean space is sketched in Fig. 5. A general *frame-free* viewpoint on continuum mechanics is given in [15,16].

The modifications between both bodies are introduced outside of any physical process. Thus, the complete continuum mechanical mechanism is applied twice on different bodies. The viewpoint of structural optimization is that the shape of the body, i.e. the reference placement of the material body, is the design function, see [11,12] using a presentation based on local coordinates on a fixed domain. Every new domain $\boldsymbol{\mathcal{K}}_s$ is immediately filled with the neccessary material in its unstressed form. Overall, the material body has no importance in this viewpoint, i.e. the overall story can be told by just using the reference and current placements. But, opposite to structural analysis, the reference and current placements are now dependent on a scalar valued design parameter s. This viewpoints can also be applied to configurational mechanics, see [14].

7.2. Properties of placement mappings in case of growth The outlined approach to growth phenomena benefits from the strict separation of growth of amount of substance and placements in Euclidean space. The placement mapping must fulfill the constraint that the occupied volume of the material body is consistent with the material properties, namely the molar volume. Thus, any physical volume element dV of the reference placement $\boldsymbol{\mathcal{K}}$ yields

$$\mathrm{d}V = \omega_{\mathrm{X}} \, \mathrm{d}V_{\mathrm{X}} = \omega_{\zeta} \, \mathrm{d}V_{\zeta} = V_m \, \nu_{\zeta} \, \mathrm{d}V_{\zeta} = V_m \, \mathrm{d}V_{\Theta} \tag{5}$$

linking the mapping $\boldsymbol{\kappa}_{\Theta}$ to the material property V_m via $J_{\mathrm{K}_{\Theta}} = \det \mathbf{K}_{\Theta} \equiv V_m$. Unfortunately, growth of the spatial volume element $\mathrm{d}v = \omega_{\mathrm{x}} \, \mathrm{d}V_{\mathrm{x}}$ due to modification of molar volume V'_m or growth of the substance density ν'_{ζ} cannot be separated from physical deformation $\boldsymbol{\mu}_{\vartheta}$.

8. A novel framework for growth theories

The kinematics of growth is formulated in literature as a multiplicative split of the deformation gradient in form of

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} \, \mathbf{F}_{\mathrm{g}}. \tag{6}$$

Here, $\mathbf{F}_{\mathrm{g}}$ is the growth part from the known reference placement $\boldsymbol{\mathcal{K}}$ into an unknown and incompatible intermediate configuration. This means that the gradient mapping is not integrable into a real placement, i.e. there is no displacement field available. Afterwards, the intermediate configuration is mapped into the current deformed configuration $\boldsymbol{\mathcal{M}}$ via $\mathbf{F}_{\mathrm{e}}$, see Fig. 6. A short formulation of an established theory is given by [3]. The *spatial velocity gradient* $\mathbf{L} = \dot{\mathbf{F}} \, \mathbf{F}^{-1} = \mathrm{grad}_{\mathrm{x}} \dot{\mathbf{x}}$ can be split up into growth and elastic parts, i.e.

$$\mathbf{L} = \dot{\mathbf{F}}_{\mathrm{e}} \, \mathbf{F}_{\mathrm{e}}^{-1} + \mathbf{F}_{\mathrm{e}} \, \left[\dot{\mathbf{F}}_{\mathrm{g}} \, \mathbf{F}_{\mathrm{g}}^{-1} \right] \, \mathbf{F}_{\mathrm{e}}^{-1}. \tag{7}$$

The elastic material behaviour and the growth law are dependent on $\mathbf{F}_{\mathrm{e}}$ and $\dot{\mathbf{F}}_{\mathrm{g}}$, see e.g. [9] for details.

8.1. Reformulation based on compatible mappings. The starting point for the rearrangement is the decomposition

$$\boldsymbol{\varphi} = \boldsymbol{\mu} \circ \boldsymbol{\kappa}^{-1} = \boldsymbol{\mu}_{\vartheta} \circ \boldsymbol{\gamma} \circ \boldsymbol{\kappa}_{\Theta}^{-1} = \boldsymbol{\mu}_{\vartheta} \circ \boldsymbol{\beta} \circ \boldsymbol{\alpha}^{-1} \circ \boldsymbol{\kappa}_{\Theta}^{-1}. \tag{8}$$

Consequently, the deformation gradient takes the form

$$\mathbf{F} = \mathbf{M} \, \mathbf{K}^{-1} = \mathbf{M}_{\vartheta} \, \boldsymbol{\Gamma} \, \mathbf{K}_{\Theta}^{-1} = \mathbf{M}_{\vartheta} \, \mathbf{B} \, \mathbf{A}^{-1} \, \mathbf{K}_{\Theta}^{-1}. \tag{9}$$

Here, the tensors $\mathbf{F}_{\mathrm{e}}$ and $\mathbf{F}_{\mathrm{g}}$ can be recovered by inserting $\mathbf{Z}_{\vartheta}^{-1} \, \mathbf{Z}_{\vartheta}$ in the above representation, i.e.

$$\mathbf{F} = \left[\mathbf{M}_{\vartheta} \, \mathbf{Z}_{\vartheta}^{-1} \right] \, \left[\mathbf{Z}_{\vartheta} \, \boldsymbol{\Gamma} \, \mathbf{K}_{\Theta}^{-1} \right] = \left[\mathbf{M}_{\vartheta} \, \mathbf{Z}_{\vartheta}^{-1} \right] \, \left[\mathbf{Z}_{\vartheta} \, \mathbf{B} \, \mathbf{A}^{-1} \, \mathbf{K}_{\Theta}^{-1} \right],$$

and comparing with $\mathbf{F} = \mathbf{F}_{\mathrm{e}} \, \mathbf{F}_{\mathrm{g}}$ to yield

$$\mathbf{F}_{\mathrm{e}} = \mathbf{M}_{\vartheta} \, \mathbf{Z}_{\vartheta}^{-1} \tag{10}$$

and

$$\mathbf{F}_{\mathrm{g}} = \mathbf{Z}_{\vartheta} \, \boldsymbol{\Gamma} \, \mathbf{K}_{\Theta}^{-1} = \mathbf{Z}_{\vartheta} \, \mathbf{B} \, \mathbf{A}^{-1} \, \mathbf{K}_{\Theta}^{-1}, \tag{11}$$

see Fig. 6 for an overview.

8.2. Time derivatives. Next, it is shown that $\mathbf{Z}_\Theta$ and $\mathbf{Z}_\vartheta$ cancel out in all relevant evolution equations. The design variable is linked to time, i.e. $s = t$, and the derivative $(\cdot)'$ is replaced by the usual time derivative.

The time derivative of $\mathbf{F}_\mathrm{e}$ yields with fixed $\mathbf{A}$ and $\mathbf{K}_\Theta$

$$\dot{\mathbf{F}}_\mathrm{e} = \dot{\mathbf{M}}_\vartheta \, \mathbf{Z}_\vartheta^{-1} - \mathbf{M}_\vartheta \, \mathbf{Z}_\vartheta^{-1} \, \dot{\mathbf{Z}}_\vartheta \, \mathbf{Z}_\vartheta^{-1} \tag{12}$$

and thus

$$\dot{\mathbf{F}}_\mathrm{e} \, \mathbf{F}_\mathrm{e}^{-1} = \dot{\mathbf{M}}_\vartheta \, \mathbf{M}_\vartheta^{-1} - \mathbf{M}_\vartheta \, \mathbf{Z}_\vartheta^{-1} \, \dot{\mathbf{Z}}_\vartheta \, \mathbf{M}_\vartheta^{-1}. \tag{13}$$

Similarly, the time derivative of $\mathbf{F}_\mathrm{g}$ gives

$$\dot{\mathbf{F}}_\mathrm{g} = \dot{\mathbf{Z}}_\vartheta \, \boldsymbol{\Gamma} \, \mathbf{K}_\Theta^{-1} + \mathbf{Z}_\vartheta \, \dot{\mathbf{B}} \, \mathbf{A}^{-1} \, \mathbf{K}_\Theta^{-1} \tag{14}$$

and thus

$$\dot{\mathbf{F}}_\mathrm{g} \, \mathbf{F}_\mathrm{g}^{-1} = \dot{\mathbf{Z}}_\vartheta \, \mathbf{Z}_\vartheta^{-1} + \mathbf{Z}_\vartheta \, \dot{\boldsymbol{\Gamma}} \, \boldsymbol{\Gamma}^{-1} \, \mathbf{Z}_\vartheta^{-1}. \tag{15}$$

Overall, the spatial velocity gradient is given in (7) and we observe that the term $\mathbf{M}_\vartheta \, \mathbf{Z}_\vartheta^{-1} \, \dot{\mathbf{Z}}_\vartheta \, \mathbf{M}_\vartheta^{-1}$ cancels out. Thus, the resulting spatial velocity gradient, see also (7), decomposes additively to elastic and growth parts

$$\mathbf{L} = \dot{\mathbf{M}}_\vartheta \, \mathbf{M}_\vartheta^{-1} + \mathbf{M}_\vartheta \, \left[\dot{\mathbf{B}} \, \mathbf{B}^{-1} \right] \, \mathbf{M}_\vartheta^{-1}, \tag{16}$$

which is verified by direct computation using $\mathbf{F} = \mathbf{M} \, \mathbf{K}^{-1}$ and $\mathbf{M} = \mathbf{M}_\vartheta \, \mathbf{B}$. All tangent mappings are computed as gradients of the underlying point mappings, i.e. only compatible mappings are involved.

9. Conclusions

The proposed enhanced theory offers a deeper interpretation of the combined growth-deformation-process. A well-known concept from structural optimisation, i.e. a design space parameterisation using design variables, is applied to the general continuum mechanical framework. Thus, the classical concept of a unique and fixed material body is modified to allow an evolution of solid bodies.

In detail, a previously introduced concept for variational design sensitivity analysis in structural optimisation is enhanced to separate topology and geometry. Here, evolving sets of names, i.e. intrinsic coordinates belonging to different material bodies, are introduced and compared via local coordinates on a material reservoir. Is is crucial to observe that there is no geometry available in the space of intrinsic coordinates.

This novel viewpoint is compared with the classical growth theory using a multiplicative decomposition of the deformation gradient yielding an incompatible intermediate configuration. This approach can be considered as the pull-back of an intermediate configuration to the name space of intrinsic coordinates. Thus, there is no need to discuss about incompatibility because the sets of names are always compatible. Overall, a rigorous analysis of the design space separating geometrical and topological entities enables a theory using only compatible mappings.

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