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On *k*-cyclic *SHn*-algebra

A. FERNANDEZ*

Universidad de Costa Rica, 2060 San José, Costa Rica

Abstract. In this work we consider a new class of algebra called k-cyclic SHn-algebra (A, T) where A is an SHn-algebra and T is a lattice endomorphism such that $T^k(x) = x$, for all x, k is a positive integer. The main goal of this paper is to show a Priestley duality theorem for k-cyclic SHn-algebra.

Key words: De Morgan algebra, Łukasiewicz algebra, Heyting algebra, Lattices and duality.

1. Introduction

As an application to the study of switching circuits, Gr.C. Moisil introduced in [1] the symmetric Boolean algebra, that is, Bolean algebras with an automorphism of period two and later, in [2], the cyclic Boolean algebras in which the automorphism of period two is replaced by an automorphism of period k. The symmetric Boolean algebras were also studied in A. Monteiro [3, 4]. M. Abad [5] has also investigated the action of a cyclic operation on Łukasiewicz algebra, and L. Iturrioz [6] gave complete description of the variety of involutive Heyting algebra. On the other hand, it has been shown that Post and Łukasiewicz algebra are both Heyting algebra with operators. In both Łukasiewicz and Post algebra a symmetry can be expressed in terms of the primitive operations. This led to the study of more general algebra, called Symmetrical Heyting Algebras of order n (or SHn-algebra) (see [7–10]).

A k-cyclic SH_n-algebra is a system (A, T), such that A is an SH_n-algebra and T is an endomorphism of A such that $T^k(x) = x$ for every $x \in A$. We usually use the same notation for a structure as for its universe. The class of k-cyclic SHn-algebra forms a variety. In this paper we give a Priestley-style duality for k-cyclic SHn-algebra.

2. Preliminaries

In this section, in order to simplify reading, we summarize the fundamental concepts we use.

Recall that V. Sofronie-Stokkermas [11] introduce the category **SHn** of SHn-algebra and SHn-homomorphisms, where an SHn-algebra is an algebra $\langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1, S_1, \cdots, S_{n-1}\rangle$ such that $\langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1\rangle$ is a symmetric Heyting algebra (see [12]) and $S_1, \cdots, S_{n-1}$ are unary operators defined on A fulfilling the following equalities:

(S1) $S_i(a \wedge b) = S_i a \wedge S_i b$,
(S2) $S_i(a \Rightarrow b) = (\bigwedge_{k=i}^{n} S_k(a) \Rightarrow S_k(a))$,
(S3) $S_i(S_j(a)) = S_j(a)$, for every $i, j = 1, \cdots, n-1$,
(S4) $S_1(a) \vee a = a$,
(S5) $S_i(\sim a) =\sim S_{n-i}(a)$, for $i = 1, \cdots, n-1$,
(S6) $S_1(a) \vee \neg S_1(a) = 1$, with $\neg a = a \Rightarrow 0$.

In addition, this author extended Esakia duality [13, 14] to the category **SHnSp** whose objects are SHn-spaces and whose morphisms are SHn-functions. Specifically, an SHn-space is a system $(X, \leq, \tau, s_1, \cdots, s_{n-1}, g)$ such that $(X, \leq, \tau)$ is an Esakia space and for every $x, y \in X$ the following conditions are satisfied:

(1) if $x \leq y$ then $g(y) \leq g(x)$,
(2) $g(s_i(x)) = s_{n-i}(g(x))$,
(3) $g(g(x)) = x$,
(4) $s_j(s_i(x)) = s_j(x)$,
(5) $s_1(x) \leq x$,
(6) $x \leq s_{n-1}(x)$,
(7) $s_i(x) \leq s_j(x)$ for every $i \leq j$,
(8) if $x \leq y$ then $s_i(x) = s_i(y)$,
(9) for all $i = 1, \cdots, n-1$, $x \leq s_i(x)$ or $s_{i+1}(x) \leq x$.

and an SHn-function from an SHn-space $(X, \leq, \tau, s_1, \cdots, s_{n-1}, g)$ into another $(X', \leq', \tau', s'_1, \cdots, s'_{n-1}, g')$ is an Esakia morphism (continuos bounded morphism) $f : X \to X'$ such that preserve the operations $s_1, \cdots, s_{n-1}, g$. Besides, he proved that **SHn** is dually equivalent to **SHnSp**.

3. Priestley duality for k-cyclic SHn-algebra

In this section we study SHn-algebra endowed with a lattice endomorphism, more precisely:

Definition 1. An k-cyclic SHn-algebra is a pair (A, T) where A is an SHn-algebra and T is a lattice endomorphism which satisfies the following properties:

(K) $T^k(x) = x$, for every $x \in A$.

By **kSHn** we will denote the category of k-cyclic SHn-algebra and their corresponding homomorphims.

We indicate a Priestley-style duality for k-cyclic SHn-algebra.

*e-mail: arielfernandez@gmail.com

Definition 2. A k-space is a pair (X,t) where X is an SHn-space and the following conditions are satisfied:

(k1) $t : X \to X$ is an homeomorphism,
(k2) $t : X \to X$ is an order–isomorphism,
(k3) $t^k(x) = x$, for every $x \in X$.

Let **kSp** be the category whose objects are k-spaces and whose morphisms are SHn-functions such that preserve the operation t.

For every k-cyclic space (X,t), let $E(X)$ be the set of clopen order-filters of X. On $E(X)$ the following operations can be defined:

(a) $\vee : E(X)\times E(X) \to E(X)$ is defined by $U\vee V := U\cup V$.
(b) $\wedge : E(X)\times E(X) \to E(X)$ is defined by $U\wedge V := U\cap V$.
(c) $\Rightarrow: E(X) \times E(X) \to E(X)$ is defined by $U \Rightarrow V := \{x \in X | x \le y$ and $y \in U$ implies $y \in V\}$.
(d) $\sim: E(X) \times E(X) \to E(X)$ is defined for every $U \in E(X)$ by $\sim U := X \setminus g^{-1}(U)$.
(e) For every $i = 1, \cdots, n-1$, $S_i : E(X) \to E(X)$ is defined for every $U \in E(X)$ by $S_i(U) := s^{-1}(U)$.
(f) $T : E(X) \to E(X)$ is defined for every $U \in E(X)$ by $T(U) := t^{-1}(U)$.

For every morphism of k-cyclic space $\varphi : X \to X'$ let $E(\varphi) : E(X') \to E(X)$ be defined by $E(\varphi)(U) = \varphi^{-1}(U)$ for every $U \in E(X')$.

Proposition 1. The functor E : $\mathbf{kSp} \to \mathbf{kSHn}$ is well-defined, i.e. the following holds:

(i) The algebra $(E(X),T)$ is an k-cyclic SHn-algebra.
(ii) If $\varphi : X \to X'$ is a morphism of k-spaces, then $E(\varphi) : E(X') \to E(X)$ is a morphism of k-cyclic SHn-algebra.

Proof. We shall only prove that for all $U \in E(X)$, (t4) is satisfied. Indeed, the following conditions are equivalents:

(1) $x \in T^k(U)$,
(2) $t^k(x) \in U$,
(3) $x \in U$.

For every k-cyclic SHn-algebra (A,T) let $D(A) = (PF(A), \le, \tau, s_1, \cdots, s_{n-1}, g, t)$ where $(PF(A), \le, \tau, s_1, \cdots, s_{n-1}, g)$ is the Priestley space of the SHn-algebra A and $t : D(A) \to D(A)$ is defined for every $F \in D(A)$ by $t(F) = T^{-1}(F)$.

For every morphism of k-cyclic SHn-algebra $f : A \to A'$ let $D(f) : D(A') \to D(A)$ be defined by $D(f)(F) = f^{-1}(F)$ for every $F \in D(A')$.

Proposition 2. The functor D : $\mathbf{kSHn} \to \mathbf{kSp}$ is well-defined, i.e. the following holds:

(i) For every k-cyclic SHn-algebra (A,T), $(D(A),t)$ is an k-space.
(ii) If $f : A \to A'$ is a morphism of SHn-algebra, then $D(f) : D(A') \to D(A)$ is a morphism of k-spaces.

Proof. We can only prove that for all $F \in D(A)$, (K) is satisfied. Indeed, the following conditions are equivalents:

(1) $x \in t^k(F)$,
(2) $T^k(x) \in F$,
(3) $x \in F$.

From Proposition 1 and 2 and taking into account the results indicad in [11] we have.

Theorem 1. The categories **kSHn** and **kSp** are dually equivalent.

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