NULL-CONTROLLABILITY OF LINEAR SYSTEMS ON TIME SCALES

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Abstract: The purpose of the paper is to study the problem of controllability of linear control systems with control constrains, defined on a time scale. The obtained results extend the existing ones on any time domain. The set of values of admissible controls is a given closed and convex cone with nonempty interior and vertex at zero or is a subset of R^m containing zero.

Key words: Linear Control System, Control Constrains, Null-Controllability, Time Scale, Delta Derivative

1. INTRODUCTION

The paper deals with linear control systems $x^{\Delta}(t) = A(t)x(t) + B(t)u(t)$ defined on a time scale T. We assume that $u: T \to U$, where U is a subset of R^m. In systems theory linear systems have (by definition) $U = R^m$, but in many practical situations the set U should be bounded, see for example Abel (2010). A restriction on controls brings some difficulties with controllability conditions. For example (The example comes from Sontag (1998)), let us take a system $dx(t)/dt = -\mathrm{x} + \mathrm{u}$ and let $\mathrm{u}: [0, +\infty) \to (-1,1)$. It is easy to see that the pair (A, B) is controllable, but the system with restricted controls is not, since it is impossible to transfer the state $\mathrm{x}_0 = 0$ to $\mathrm{x}_\mathrm{f} = 2$ (we have $dx(t)/dt < 0$ whenever $x(t) \in (1,2)$).

For continuous-time linear systems, the problem of controllability with control constrains has been sudied for example in Ahmed (1985), Chukwu and Lenhart (1991), Klamka (1991), Path et al., (2000), Schmitendorf and Barmish (1981), Sontag (1998). For discrete-time case – in Benzaid and Lutz (1980), Path et al., (2000).

Analysis on time scales is nowadays recognized as the right tool to unify and extend the existing results for continuous- and discrete-time dynamical systems to the nonhomegonous time domains, see for example Bartosiewicz and Pawłuszewicz (2006), Bartosiewicz and Pawłuszewicz (2008), Davis et al., (2009), DaCunha and Davis (2011), Gravagne et al., (2009), Ferreira and Torres (2010), Pawłuszewicz and Torres (2010).

A time scale is a model of time. Besides the standard cases of the whole real line (continuous-time case) and all integers (discrete-time case) there are many other models of time included that can be partially continuous and partially discrete, q-scales, quantum time scales (objects with non-uniform domains), and many others – see Bohner and Peterson (2001). However, discrete-time systems on time scales are based on the difference operator and not on the more conventional shift operator. One of the main concepts in the time scale analysis is the delta derivative, which is a generalization of the classical (time) derivative in the continuous time and the finite forward difference in the discrete time. Similarly, the integral of a real function defined on a time scale is an extension of the Riemann integral in the continuous time and the finite sum in the discrete time. As a consequence, differential equations as well as difference equations are naturally accommodated in this theory.

The goal of this paper is to study conditions under which a linear system defined on a time scale with control constrains is controllable. For this aim, in Section 2 gives general information about solution of considered class of systems. Section 3 is devoted to the investigation of the problem of null-controllability of time-varying systems with control constrains. It also presents the necessary and sufficient conditions for global null controllability for the systems with control constrains on homogenous time scale. In Section 4 linear time-invariant systems with control constrains are studied. The main result of this Section is that such a system is controllable if and only if the Kalmann rank condition is satisfied.

The necessary elements of delta-measurability and nonlinear theory on time scales are presented in Appendix. At this moment we only introduce the following notation: if $a, b \in T, a \leq b$, then $[a, b]_T$ denotes the intersection of the real closed interval $[a, b]$ with T. A similar notation is used for open, half-open, or infinite intervals.

2. LINEAR SYSTEMS ON TIME SCALES

Let T be any time scale and let $A \subset T$. Recall (see Cabada and Vivero (2005)) that a function $f: T \to R$ is absolutely continuous on a time scale T if and only if f is continuous and of bounded variation on T and f maps every Δ-null subset of T into a null set. Let L_{Δ}^{p} denote spaces linked to the Lebesgue Δ-measure and absolutely continuous function on arbitrary closed interval of time scale T. We say that $f \in L_{\Delta}^{p}(E)$ provided that $\int |f(t)|^p \Delta t < \infty$ if $p \in R, p < \infty$, Agrawal et al (2006), Cabada and Vivero (2006).

Let I be the identity $n \times n$ - matrix and $Z \in R^{n \times n}$. Recall that the matrix Δ-differential system defined on time scale T:

$$X^{\Delta}(t) = Z(t)X(t) \qquad X(t_0) = I \tag{1}$$

for any $X \in R^n$, $\mathrm{t} \in [\mathrm{t}_0, \sup\mathrm{T})_\mathrm{T}$, has a unique solution $\mathrm{X}(t) = \Phi_{Z(t)}(t, t_0)$. Using the same arguments as in Bartosiewicz and

Pawłuszewicz (2006) for time-invariant case, we can show that for every $t, s, r \in T$ such that $t \leq s \leq r$ the following hold:
- $\Phi_0(t,s) = I, \Phi_{Z(t)}(t,t) = I;$
- If $Z(t)$ is an regressive matrix, i.e. if matrix $\big(I + \mu(t)\big)Z(t)$ is invertible, then $\Phi_{Z(t)}(t,s) = (\Phi_{Z(t)}(s,t))^{-1}$;
- $\Phi_{Z(t)}(t,s)\Phi_{Z(t)}(s,r) = \Phi_{Z(t)}(t,r)$.

If Z is time-invariant, the solution of the equation (1) is given by an exponential matrix function on time scale T $X(t) = e_Z(t,t_0)$, see: Bartosiewicz and Pawluszewicz (2006), Jackson (2007).

Let us consider a linear control system defined on T:

$$x^{\Delta}(t) = A(t)x(t) + B(t)u(t), \;\; x(t_0) = x_0 \tag{2}$$

where $A(t) \in R^{n\times n}$ and $B(t) \in R^{n\times m}$ are rd-continuous matrices on T, i.e. each entry of these matrices is an rd-continuous function on T. Also $x(t) \in \Sigma \subset R^n$ and $u(t) \in U \subset R^m$. Let us choose a control u. The trajectory of system (2) is a function $\psi(\cdot, t_0, x_0, u)\colon [t_0, supT)_T \to \Sigma$ that is the unique solution of (2), provided it is defined on $[t_0, supT)_T$ and for all $t \in [t_0, supT)_T$, $x(t) \in \Sigma$. This solution for all $t \in [t_0, supT)_T$ is given by (see Bartosiewicz and Pawłuszewicz (2006)):

$$x_f = \Phi_A(t_f,t_0)x_0 + \int_{t_0}^{t_f}\Phi_A\left(t_f,\sigma(s)\right)B(s)u(s)\Delta s \tag{3}$$

If A is a regressive matrix, i.e. if the matrix $\big(I + \mu(t)\big)A(t)$ is invertible, then (3) describes both forward and backward trajectories of (2).

We say that a control u is admissible for $x_0 \in R^n$ if there exists a trajectory of system (2) from x_0 corresponding to u. The set of all admissible controls (for x_0) will be denoted by U_{ad}.

Let $E = [t_0, t_f]_T$. Assume that the set of the values of admissible controls U is a given closed and convex cone with nonempty interior and vertex at zero. Thus the set of admissible controls U_{ad} for system (2) has the form $L^2_\Delta(E,U)$, i.e. is a Banach space endowed with the norm defined for every $u\colon E \to U$ as:

$$||u||_{L^2_\Delta} \coloneqq \left[\int_E |u|^2(t)\Delta t\right]^{1/2}$$

3. TIME-VARYING SYSTEMS WITH CONTROL CONSTRAINS

Let $\Sigma \subseteq R^n$. We say that system (2) is:
- *U*-controllable on a time interval $[t_0, t_f]_T$ if, for any $x_0 \in \Sigma$ and any x_f there exists a control $u \in L^2_\Delta(E,U)$ such that $\psi\big(t_f, t_0, x_0, u\big) = x_f, x_f \in \Sigma$.
- *U-controllable* if it is U-controllable on every time interval $[t_0, t_f]_T$.
- locally *U-controllable* on $[t_0, t_f]_T$ if, for the given trajectory $\psi(\cdot, t_0, x_0, u) = x(\cdot)$ of (2) with $u_0 \in L^2_\Delta(E,U)$ and $x(t_0) = x_0 \in \Sigma$ there exists a neighborhood V_{x_0} of x_0 such that, for any $z \in V_{x_0}$ there exists an admissible control u_0 such that $\psi\big(t_f, t_0, x_0, u\big) = x_f \in V_{x_0}$.

If $x_f = 0$, then we have respectively null *U*-controllability on a time interval $[t_0, t_f]_T$ null *U*-controllability, local null *U*-controllability.

Our goal is to show certain properties characterizing the null U-controllability. Let us assume that there exists a unique evolution operator ϑ defined as $\{\bar{\Phi}_A(t,s)\colon t_0 \leq s \leq t \leq t_f;\; t_0, s,\; t,\; t_f \in T\}$ and corresponding to the $\mathbb{A} = \{A(t)\colon t \in [t_0, supT)_T\}$ in (2). The ideas of proofs of next two propositions come from Chuwku and Lenhart (1991).

Proposition 1. Let us assume that system (2) is null U-controllable on $[t_0, t_f]_T$. Then there exists a bounded operator $H\colon \Sigma \to L^2_\Delta(E,U)$ such that, with the admissible control $u = Hx_0$, the solution of (2) satisfies $x\big(t_f\big) = \psi\big(t_f, t_0, x_0, Hx_0\big) = 0$.

Proof: For arbitrary initial state x_0, let $T_t\colon \Sigma \times U_{ad} \to \Sigma$ be a map defined as $T_t(x_0, u) \coloneqq \psi(t, t_0, x_0, u)$ for any $t \in [t_0, t_f]_T$. Then T_t is the continuous linear map with respect to u. Let us consider also a map $S_t\colon L^p_\Delta(E) \to \Sigma$ defined as:

$$S_t(u) \coloneqq \int_{t_0}^{t}\Phi_A\big(t,\sigma(s)\big)B(s)\Delta s$$

for any $t \in [t_0, t_f]_T$. Note that S_t is linear, bounded and $T_t(x_0, u) = \Phi_A(t, t_0)x_0 + S_t(u)$. Since for all $t_f \in T$ $\Phi_A\big(t_f, t_0\big)\Sigma \subset S_{t_f}\left(L^2_\Delta(E,U)\right)$ then, from definition, this condition is equivalent to the null *U*-controllability of (2).

Let us consider a map $\varsigma\colon N^\perp \to S_{t_f}\left(L^2_\Delta(E,U)\right)$, where $N^\perp$ denotes the orthogonal complement of the null space of S_{t_f}. Define $Hx_0 \coloneqq -\varsigma^{-1}\Phi_A(t,t_0)x_0$. Note that by Banach Theorem and closed graph theorem this operator is bounded (see Musielak (1989)). Moreover:

$$\psi\big(t_f, t_0, x_0, Hx_0\big) =$$
$$\Phi_A\big(t_f, t_0\big)x_0 + S_{t_f}(-\varsigma^{-1})\Phi_A\big(t_f, t_0\big)x_0 = 0$$

Proposition 2: Suppose that zero belongs to the interior of the set of admissible controls. If the system (2) is null U-controllable, then it is locally null *U*-controllable.

Proof follows from the fact that map H defined in Proposition 1 is continuous at 0. This implies the existence of an open set W_0 containing 0 and such that $H(W_0) \subset V \subset intU_{ad}$. Hence, the state 0 can be achieved from any $x_0 \in W_0$ using $u = Hx_0$.

Other conditions for null U-controllability can be obtained under exponential stability assumption. Recall that system (1) defined on unbounded time scale T with bounded graininess function $\mu\colon T \to R_+ \cup \{0\}$ is exponentially stable if there exists a constant $\alpha > 0$ such that for every $t_0 \in T$ there exists $K = K(t_0) \geq 1$ with

$$||\Phi_{Z(T)}(t,t_0)x(t_0)|| \leq Ke^{-\alpha(t-t_0)}||\, x(t_0)||$$

for any $t \in [t_0, \sup)_T$; $||.||$ denotes the classical Euclidian norm.

Proposition 3. If system (2) is null U-controllable on each time interval $[t_0, t + \sigma(t))_T$, $[t_0, supT)_T$, and the system $x^\Delta(t) = A(t)x(t)$, $x(t_0) = x_0$, is exponentially stable, then the system (2) is null U-controllable.

Proof: By Proposition 2, null U-controllable of the given system implies local null U-controllability of this system. Then there exists a neighborhood V_{x_0} of x_0 such that all states from V_{x_0} can be steered to 0 with $u \in U_{ad}$.

Let $z \in V_{x_0}$. Exponential stability of the system $x^\Delta(t) = A(t)x(t)$, $x(t_0) = z$, implies existence t_{f_1} such that the solution of this equation satisfies $x(t_{f_1}) = x_{f_1} \in V_{x_0}$. If we take as an initial data (t_{f_1}, x_{f_1}), then there exists $t_{f_2} \in (t_{f_1}, supT)_T$, such that, for some $\bar{u} \in U_{ad}$ holds $\psi\big(t_{f_1}, t_{f_1}, x_0, \bar{u}\big) = x_{f_1}$ and

$\psi(t_{f_2}, t_{f_1}, x_0, \bar{u}) = 0$. Taking as a control multifunction $v(t) := u(t)$ for $t \in [t_0, t_{f_1}]_T$ and $v(t) := \bar{u}(t)$ for $t \in [t_{f_1}, t_{f_2}]_T$ (with swithching time t_{f_1}), state z can be transferred to state 0 in time $t \in [t_0, t_{f_2}]_T$.

Let $X = \{x \in R^n : Lx = c\}$ where L is a given $p \times n$-matrix of the rank p and $c \in R^n$ is a given vector. For any vector $a \in R^p$ let $H_U(a) = \sup\{w^T L^T a : w \in U\}$ denote the support function of a set U.

Theorem 1: Let T be a time scale with a constant graininess μ. The system (2) is globally null U-controllable if and only if for every admissible control u holds

$$\int_{t_0}^{\infty} H_U\left(B^T(s)\Phi_A^T(t_0, \sigma(s))L^T a\right)\Delta s = +\infty \qquad (4)$$

where M^T denotes the transposition matrix of M.

Proof: For $T = R$ the theorem was proved in Schmitendorf and Barmish (1981), Klamka (1991) and, using more general approach, in Path et al., (2000). The proof for $T = hZ,\ h > 0$, mimics the one given in Path et al., (2000) for discrete-time systems.

In continuous-time case, relation (4) can be formulated in terms of the solution of the adjoint equation Ahmed (1985), Path et al., (2000). For time scale system (2) such reformulation requires an assumption that matrix A is regressive for all $t \in T^\kappa$.

4. LINEAR TIME-INVARIANT SYSTEMS WITH CONTROL CONSTRAINS

Let us consider a linear time-invariant control system defined on a time scale T:

$$x^\Delta(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \qquad (5)$$

where: $A \in R^{n\times n}$, $B \in R^{n\times m}$, $x(t) \in \Sigma \subset R^n$, $u(t) \in U$. As previous, we assume that the set of values of admissible controls U_{ad} is a given closed and convex cone with nonempty interior and vertex at zero. The matrix:

$$Q_{t_f} = \int_{t_0}^{t_f} e_A(t_0, s)BB^T e_A^T(t_0, s)\Delta s$$

is called the controllability gramian. If there exists $t_f \in T$ such that the matrix Q_{t_f} is nonsingular and A is a regressive matrix, then using control:

$$\bar{u}(t) = -B^T e_A^T(t_0, \sigma(s))\, Q_{t_f}^{-1}[e_A(t_f, t_0)x_0 \text{-} x_f]$$

every state $x_f = x(t_f)$ can be achieved from an initial state x_0.

Proposition 4. Let $0 \in intU_{ad}$. If the system (5) is controllable and matrix A is regressive, then it is locally null U-controllable.

Proof: If $t_f \in T$ is arbitrary, then there exists a control $\bar{u}(s) = -B^T e_A^T(t_0, \sigma(s))Q_{t_f}^{-1} e_A(t_f, t_0)x,\ s \in [t_0, t_f]_T$, such that state x can be steered to 0 in a finite time. Since map $t \to e_A(t, t_0)$ is rd-continuous, then there exists a constant K such that $\|\bar{u}(s)\| \le K\|x\|,\ s \in [t_0, t_f]_T$. Hence the thesis.

Let T be an unbounded time scale. Recall that system (5) is stabilizable (see Bartosiewicz et al.,, (2007)) if there exists a state feedback $u(t) = Fx(t)$, for $F \in R^{m\times n}$, such that the closed loop system $x^\Delta(t) = (A + BF)x(t)$ is exponentially stable. The set of exponential stability on a time scale T is defined as (see Pötzsche et al., (2003)):

$$S(T) := S_C(T) \cup S_R(T)$$

where: $S_C(T) = \left\{\lambda \in C : \lim_{t\to\infty} sup \frac{1}{\tau - t_0}\int_{t_0}^{\tau} \lim_{s\to\mu(\xi)} \frac{\log|1+s\lambda|}{s}\Delta\xi < 0\right\}$

$S_R(T) = \{\lambda \in R : \forall\tau \in T\ \exists\xi \in T, \xi > \tau : 1 + \mu(\xi)\lambda = 0\}$

For the arbitrary time scale T it holds that $S_C(T) \subseteq \{\lambda \in C : Re\lambda < 0\}$ and $S_R(T) \subset (-\infty, 0)$.

Theorem 2. (Pötzsche et al., (2003)) The following holds:

a) If (5) is exponentially stable then $spec(A) \subset S(T)$.
b) If A is diagonalizable, then (5) is exponentially stable if and only if $spec(A) \subset S(T)$

where: $spec(A)$ denotes the set of all eigenvalues of A.

Since the null U-controllability is a particular case of U-controllability, we can reformulate the result from Bartosiewicz et al., (2007) as follows:

Theorem 3. Assume that $\mu(t)$ is bounded. If system (5) is null U-controllable, then it is stabilizable.

Lemma 1. If system (5) is stabilizable then it is U-controllable.

Proof: The idea of the proof is based on Zabczyk (1995). Using classical arguments one can easily deduce that if the pair (A, B) is controllable, then there exists a matrix $F \in R^{m\times n}$ and a vector $v \in R^m$ such that the pair $(A + BF, Bv)$ is controllable.

Let $P \in R^{n\times n}$ be a nonsingular matrix such that $PAP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix}$, $PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ and the pair (A_{11}, B_1), $A_{11} \in R^{l\times l}$, $B_1 \in R^{l\times m}$, is controllable.

Since system (5) is stabilizable, then there is a matrix $F \in R^{m\times n}$, such that the closed loop system $x^\Delta(t) = (A + BF)x(t) + (Bv)u(t)$ is exponentially stable. The characteristic polynomial of $A + BF$ is of the form:

$$\begin{aligned} p_{A+BF}(\lambda) &= \det[\lambda I - (A + BF)] \\ &= \det(\lambda I - PAP^{-1} - PBFP^{-1}) \\ &= \det[\lambda I - (A_{11}P^{-1} + B_1 F)]\det(\lambda I \\ &\quad - A_{22}), \quad \lambda \in C. \end{aligned}$$

So, for any F, $spec(A_{22}) \subset spec(A + BF) \subset S(T)$ and, if there exists $\alpha > 0$ such that for every $t_0 \in T$ there is $K \ge 1$ then $\alpha \le -sup\{Re\lambda : \lambda \in spec(A_{22})\}$. Hence the contradiction with stabilizability of (5).

Exponential stability and Proposition 2 imply the following.

Proposition 5. If $0 \in U_{ad}$ system (5) is U-controllable and exponentially stable, then it is null U-controllable.

Let $A_{x_0,U_{ad}}(t_0, t_f)$ be a *reachability set* of system (2), i.e. a set of all points that can be reached at time t_f starting from $x_0 = x(t_0)$. The set of all points that can be reached from x_0 at t_0 in a finite time will be denoted as $A_{x_0,U_{ad}}(t_0)$. The image of the map $u \mapsto \psi(t_f, t_0, x_0, u)$, i.e. the set $A_{x_0,U_{ad}}(t_0, t_f)$ is a linear subspace of R^n and:

$$A_{x_0,U_{ad}}(t_0, t_f) = \Phi_A(t_f, t_0)x_0 + A_{0,U_{ad}}(t_0, t_f)$$

Using classicall arguments, similarly as in Sontag (1998) we can show the following:

– if U is convex, then $A_{o,U_{ad}}(t_0)$ is a convex subset of R^n;
– suppose that A is regressive. If system (5) is U-controllable and U_{ad} is a neighborhood of $0 \in R^n$ then $A_{o,U_{ad}}(t_0)$ is an open subset of R^n.

Collorary 1. Suppose that $0 \in intU$ system (5) is stabilizable and matrix A is diagonalizable. Then system (5) is null U-controllable.

For each eigenvalue λ of the matrix A, let $J_{k,\lambda} := \ker(\lambda I - A)^k$ and $J^R_{k,\lambda} := \{Rev: v \in J_{k,\lambda}\}$. Let $L = \bigcup_{Re\lambda \geq 0} J^R_{k,\lambda}$ and $M = \bigcup_{Re\lambda < 0} J^R_{k,\lambda}$. If C is an open convex subset of R^n, L is a subset of R^ncontained in C, then $C + T = C$, see Sontag (1998).

Lemma 2. Let A be an $n \times n$ −matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. If system (5) is U-controllable and U_{ad} is a neighborhood of 0, then $L \subseteq A_{o,U_{ad}}(t_0)$.

Proof: Without loss of generality we assume that U_{ad} is a convex neighborhood of 0. Using mathematical induction with respect to k, we show that $J^R_{k,\lambda} \subseteq A_{o,U_{ad}}(t_0)$.

For $k = 0$, the case is trivial. Let us assume that $J^R_{k-1,\lambda} \subseteq A_{o,U_{ad}}(t_0)$, $\lambda = \alpha + i\beta, \alpha, \beta \in R$, and take any $\bar{v} \in J_{k,\lambda}$, $\bar{v} = \bar{v}_1 + i\bar{v}_2$. Then for any $t \in [t_0, \sup)_T$, $e_\lambda(t, t_0) = e_\alpha(t, t_0)$. Since $0 \in \mathrm{int} U_{ad}$ we can choose any $\delta > 0$ such that $v_1 := \delta \bar{v}_1 \in A_{o,U_{ad}}(t_0)$. Moreover, since $v \in ker(\lambda I - A)^k$ and $e_A(t, t_0) = \sum_{k=0}^{\infty} A^k h_k(t, t_0)$, where $h_0(t, t_0) \equiv 1$, $h_{k+1}(t, t_0) = \int_{t_0}^{t} h_k(\tau, t_0)\Delta\tau$ (see Mozyrska and Pawłuszewicz (2008)), then:

$$e_A(t, t_0)v = \left[\sum_{k=0}^{\infty} A^k h_k(t, t_0)\right] v = v + w$$

with $w \in J_{k-1,\lambda}$. So,

$$\begin{aligned} e_\alpha(t, t_0) = e_\lambda(t, t_0) &= e_\lambda(t, t_0) e_{(A-\lambda I)}(t, t_0) v - e_\lambda(t, t_0) w \\ &= e_\lambda(t, t_0) e_{(A-\lambda I)}(t, t_0) v - e_\alpha(t, t_0) w \end{aligned}$$

Moreover, since $w = w_1 + iw_2$, then:

$$Re(e_\alpha(t, t_0)v_1) = e_A(t, t_0)v_1 + e_{\alpha I}(t, t_0)w_1 \in A_{x_0,U_{ad}}(t_0, t) + J^R_{k,\lambda} \subseteq A_{0,U_{ad}}(t_0)$$

L is the sum of the spaces $J^R_{k,\lambda}$ over all eigenvalues λ with the real part nonnegative, and each of these spaces is included $A_{o,U_{ad}}(t_0)$, so the sum of the L's is included in $A_{o,U_{ad}}(t_0)$.

The ideas of the next Lemma and next Theorem come from Sontag (1998).

Lemma 3. Let $supT = \infty$. If system (5) is U-controllable, U_{ad} is a convex, bounded neighborhood of 0, then there exists a set N such that $A_{o,U_{ad}}(t_0) = N + L$ and N is bounded, convex and open relative to M.

Proof: Note that (see Sontag (1998)):

$$(A_{o,U_{ad}}(t_0) \cap M) + L \subseteq A_{o,U_{ad}}(t_0) + L = A_{o,U_{ad}}(t_0)$$

and $A_{o,U_{ad}}(t_0) \supseteq (A_{o,U_{ad}}(t_0) \cap M) + L$

Let $N := A_{o,U_{ad}}(t_0) \cap M$. Then N is open and convex. Let $\pi: R^n \to R^n$, $\pi(x + y) = x$ for $x \in M$, $y \in L$. If $v = x + y$, $Ax \in M$, $Ay \in L$, then $PAv = Ax = APv$. Let $x \in A_{o,U_{ad}}(t_0) \cap M$. Since $x \in A_{o,U_{ad}}(t_0)$, then there exists an admissible control u and $t \geq t_0$ such that $x = \int_{t_0}^{t} e_A(t, \sigma(s))Bu(s)\Delta s$. On the other hand, since $x \in M$, $x = Px$, then:

$$x = Px = \int_{t_0}^{t} Pe_A(t, \sigma(s))Bu(s)\Delta = \int_{t_0}^{t} e_A(t, \sigma(s))x(s)\Delta s$$

where $x(s) = PBu(s) \in M \cap PB(U)$ for all $s \in T$. Since the restriction of A to M has all eigenvalues with a negative real part, then there are positive constants $\alpha, K > 0$ such that (see Pötzsche et al., (2003)): $|e_\lambda(t, t_0)| \cdot ||x|| = ||e_A(t, t_0)x|| \leq Ke^{-\alpha(t-t_0)}||x||$ for $t \geq t_0$ and $x \in M$. Since $PB(U)$ is bounded, there is a constant C such that if x is also in $PB(U)$, then:

$$|e_\lambda(t, t_0)| \cdot ||x|| = ||e_A(t, t_0)x|| \leq Ce^{-\alpha(t-t_0)}||x||, t \geq t_0$$

So, (see Sontag (1998)):

$$||x|| \leq C \int_{t_0}^{t} Ce^{-\alpha(t-t_0)} ds \leq \frac{C}{\alpha}(1 - e^{-\alpha t}) \leq \frac{C}{\alpha}$$

Hence N is bounded.

Theorem 4. Let $supT = \infty$ and U_{ad} be bounded a neighborhood of zero. Then $A_{o,U_{ad}}(t_0) = R^n$ if and only if:

- system (5) is controllable;
- matrix A has no eigenvalues with a negative real part.

Proof: If $A_{o,U_{ad}}(t_0) = R^n$ then (i) is obvious (see Bartosiewicz and Pawłuszewicz (2006)). If (ii) doesn't hold then L should be a proper subspace of R^n and $K \neq 0$.We may assume that U_{ad} is convex and bounded. Lemma 3 implies that $R^n = A_{o,U_{ad}}(t_0)$ is a subset of $L + N$ and N is bounded, hence the contradiction. If (i) and (ii) hold, then by Lemma 2, $R^n = L \subseteq A_{o,U_{ad}}(t_0)$.

Theorem 4 and Kalman controllability rank condition imply the following.

Collorary 2. Let $supT = \infty$ and U_{ad} be a bounded neighborhood of zero. Then $A_{o,U_{ad}}(t_0) = R^n$ if and only if $rank[\mathrm{B}, \mathrm{AB}, \dots \mathrm{A}^{\mathrm{n}-1}\mathrm{B}] = \mathrm{n}$.

5. CONCLUSIONS

The paper extends the conditions for constrained relative controllability for linear time-varying and time-invariant systems to the systems defined on different time models, also on nonhomogenous time domains. A calculus on time scales is used to achieve this goal. The existing necessary and sufficient conditions for null controllability of time varying systems were unify. The Kalman rank condition for time-invariant systems with control constrains was extended on systems defined to any unbounded from above time scale.

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APPENDIX

A1. BASICS ON TIME SCALES CALCULUS

Let us recall that a *time scale* T is an arbitrary nonempty closed subset of the set R of real numbers. The standard cases comprise $T = R, T = Z$ and $T = hZ$ for $h > 0$. We assume that T is a topological space with the topology induced from R. For $t \in T$ we define the *forward jump operator* $\sigma: T \to T$ by $\sigma(t) := \inf\{s \in T: s > t\}$, the *backward jump operator* $\rho: T \to T$ by $\rho(t) := \sup\{s \in T: s < t\}$, the *graininess function* $\mu: T \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$. Using these operators we can classify the points of the time scale as follows:
– If $\sigma(t) > t$, then *t* is called *right-scattered* and if $\rho(t) < t$, then *t* is called *left-scattered*;
– if $t < supT$ and $\sigma(t) = t$, then t is called *right-dense* and if $t > infT$ and $\rho(t) = t$, then t is *left-dense*.

Function $f: T \to R$ is called *rd-continuous* provided it is continuous at right-dense points in T and its left-sided limits exist (finite) at left-dense points in *T*. Function f is called *regulated* provided its right-sided limits exist (finite) at all right-dense points of T and its left-sided limits exist (finite) at all left-dense points in T. Function f is *piecewise rd-continuous*, if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in T$.

Let $T^\kappa := T - \left((supT), supT\right]$ if $supT < \infty$ and $T^\kappa := \infty$ if $supT = \infty$.

Definition A1. Let $f: T \to R$ and $t \in T^\kappa$. The *delta derivative* of f at t, denoted by $f^\Delta(t)$, is the real number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood V of t such that:

$$|\left[f\left(\sigma(t)\right) - f(s)\right] - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in V$.

We say that f is delta differentiable on T^κ provided $f^\Delta(t)$ exists for all $t \in T^\kappa$. In general, the function σ may not be delta differentiable. Delta derivatives of higher order are defined in the standard way: $f^{[k]}(t) = f^\Delta(f^{\Delta^{k-1}}(t))$ for $k \geq 1$.

Remark A2. [Bohner and Peterson (2001)] If $T = R$, then $f: R \to R$ is delta differentiable at $t \in R$ if and only if f is differentiable in the classical sense at t. If $T = Z$, then $f: Z \to R$ is always delta differentiable at every $t \in Z$ with $f^\Delta(t) = f(t + 1) - f(t)$.

Let $f: T \to R$ be a bounded function on $[a, b]_T$ and let P be a partition of $[a, b]_T$ such that $a = t_0 < t_1 < \cdots < t_n = b$. In each interval $[t_{i-1}, t_i)_T$, $i = 1, \ldots, n$, choose an arbitrary ξ_i and form the sum:

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$$

We say that f is *Riemann* Δ*-integrable* (or Δ*-integrable*) from a to b if there exists a number I with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$|S - I| < \varepsilon$$

for every S corresponding to any partition P of $[a, b]_T$ and independent of the choice of $\xi_i \in [t_{i-1}, t_i)_T$ ξ_i, $i = 1, \ldots, n$. Such a number I is unique, see Bohner and Peterson (2003).

A function $F: T \to R$ is called a Δ-*antiderivative* of $f: T \to R$ provided F is Δ-differentiable on T^κ and $F^\Delta(t) = f(t)$ for all $t \in T^\kappa$. *F* is called *a* Δ-*prederivative* of f provided F is Δ-predifferentiable with region of differentiation D and $F^\Delta(t) = f(t)$ for all $t \in D$.

Theorem A3. [Bohner and Peterson (2003)] Let f be a Δ-integrable function on $[a,b]_T$. If f has a Δ-prederivative $F: [a,b]_T \to R$ with region of differentiation D, then:

$$\int_s^t f(t)\Delta t := F(t) - F(s)$$

A.2. ELEMENTS OF Δ-MEASURES ON TIME SCALES

The notions of Δ-measurable set and Δ-measurable function are studied in Cabada and Vivero (2006), Deniz (2007). Let us consider a set $\mathrm{F} = \{[a,b)_T : a,b \in T, a \leq b\}$ The interval $[a,a)_T$ is understood as the empty set. Let $m_1: F \to [0,\infty)$ be a set of functions that assigns to each interval $[a,b)_T \in F$ its length: $m_1([a,b)_T) = b - a$. Using the pair (F, m_1) one can generate an outer measure m_1^* on the family of all subsets of T as follows. Let $E \subseteq T$. If there exists at least one finite or countable system of intervals $V_j \in F, j \in N$, such that $E \subset \bigcup_j V_j$, then we put $m_1^*(E) = inf \sum_j m_1(V_j)$, where the infinum is taken over all coverings of E by a finite or countable system of intervals $V_j \subseteq F$. If there is no such covering of E, then we put $m_1^*(E) = \infty$. A subset A of a time scale T is Δ-measurable if $m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (T - A))$ holds true for any $E \subset T$. Defining a family:

$$M(m_1^*) = \{\Lambda \subset T : \Lambda\ is\ \Delta\text{-measurable}\}$$

the *Lebesgue* Δ-*measure*, denoted by μ_Δ, is the restriction of $m_1^*$to $M(m_1^*)$. If set E is Lebesgue measurable, then set $E \cap T$ is Δ-measurable, see Deniz (2007).

A function $f: T \to [-\infty, \infty]$ is Δ-measurable if for every real α the set $f^{-1}([-\infty, \alpha)) = \{t \in T : f(t) < \alpha\}$ is Δ-measurable. If f is rd-continuous, then f is Δ-measurable, see Deniz (2007).

Properties of rd-continuous and continuous functions on a time scales implies that if f is a continuous function defined on T, then it is Δ-measurable. Moreover, if an rd-continuous function f is defined on a Δ-measurable set $E \subseteq T$, then f is a Δ -measurable function.

Proposition A4 [Deniz (2007)] Let f be defined on a Δ-measurable subset E of T. Function f is Δ -measurable if the set of all right-dense points of E, where f is discontinuous, is a set of Δ-measure zero.

Proposition A5 [Pawłuszewicz and Torres (2010)] Assume that $f: T \to [-\infty, \infty]$. Then f is Δ-measurable if and only if, given $\varepsilon > 0$, there is a rd-continuous function $\varphi: [a,b]_R \to R$ such that the Δ-Lebesgue measure of the set $\{x : f(x) \neq \varphi(x)\}$ is strictly less than ε.

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