DESIGN OF FRACTIONAL ORDER CONTROLLER SATYSFYING GIVEN GAIN AND PHASE MARGIN FOR A CLASS OF UNSTABLE PLANT WITH DELAY

Tomasz NARTOWICZ*

*Białystok University of Technology, Faculty of Electrical Engineering, ul. Wiejska 45 D, 15-351 Białystok, Poland

tomek.nartowicz@gmail.com

Abstract: The paper describes the design problem of fractional order controller satisfying gain and phase margin of the closed loop system with unstable plant with delay. The proposed method is based on using Bode's ideal transfer function as a reference transfer function of the open loop system. Synthesis method is based on simplify of the object transfer function. Fractional order of the controllers is relative with gain and phase margin only. Computer method for synthesis of fractional controllers is given. The considerations are illustrated by numerical example and results of computer simulation with MATLAB/Simulink.

Key words: Fractional Order Controller, Stability, Delay, Bode's Ideal Transfer Function

1. INTRODUCTION

In recent years considerable attention has been paid to fractional calculus and its application in many areas in science or engineering (see, e.g. (Kilbas et al., 2006; Das, 2008; Ostalczyk, 2008; Kaczorek, 2011)).

In control system fractional order controllers are used to improve the performance of the feedback control loop. One of the most developed approaches in science to design robust and fractional order controllers is CRONE control methodology (French acronym of "Commande Robuste d'Ordre Non Entier" - non-integer order robust control; Oustaloup, 1991, 1995, 1999).

The fractional order PID controllers, namely $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controllers, where λ integrator order and μ differentiator order were proposed in (Podlubny, 1994, 1999). Several design methods based on the mathematical description of the process of tuning the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controllers were presented in (Monje et al., 2004; Valerio, 2005; Valerio and Costa, 2006).

Also knows in science are approaches based on optimalization methods (Monje et al., 2004), and classic Zieglera-Nicholsa metod (Valerio and da Costa, 2006). Methods based on the first order-plant with time delay, is the most frequently used model for tuning fractional and integral controllers (O'Dwyer, 2003).

In this paper a simple method of determining the fractional order controller satisfying given gain ang phase margin of the closed loop system with unstable plant with delay is given.

Transfer function of the controller follows from the use of Bode's ideal transfer function as a reference transfer function for the open loop system (Barbosa et al., 2004; Busłowicz and Nartowicz, 2009, Nartowicz 2010). Approach submit In the paper was proposed in (Barbosa et al., 2004) for a class of natural order controller, and (Busłowicz and Nartowicz, 2009) for a fractional order controller synthesis.

The considerations are illustrated by numerical example and results of computer simulation with MATLAB/Simulink.

2. METHOD

Consider the feedback control system show In Fig. 1 in which the process to be controlled is described by (1):

$$G(s) = \frac{k}{1 - s\tau} e^{-sh} \tag{1}$$

where: k, π, h are positive real numbers, and C(s) is fractional order controller.

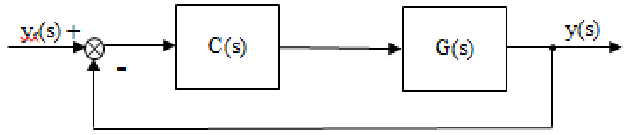

Fig. 1. Feedback control system structure

The paper presents the simple synthesis method of the fractional order controller satisfying given A_m gain and $\phi_m$phase margin of the of the closed loop system with unstable plant with delay. Transfer function of the controller follows directly from the use of Bode's ideal transfer function as a reference transfer function for the open loop system:

$$K(s) = \left(\frac{\omega_c}{s}\right)^{\beta} \tag{2}$$

where ω_c is gain crossover frequency ($|K(j\omega_c)| = 1$) and β is real number. Transfer function (2) describe derivative plant for $\beta < 0$ and integral plant for $\beta > 0$. The open loop system (2) has constant value of phase margin $\phi_m = (1 - 0.5\beta)\pi$ hence such a system is unsensitive to gain changes in open loop system. For a detailed analysis of the considered system, including time domain, see paper (Ostalczyk P., 2008).

To obtain an open loop system in the form (2), simplify the plant transfer function:

$$G(s)=\frac{k}{s(1-s\tau)}e^{\ sh}\approx-\frac{k}{s^2\tau}e^{\ sh} \tag{3}$$

The controller transfer function should have a structure:

$$C(s)=-k_c s^{2\ \alpha} \tag{4}$$

where α is real number.

Open loop transfer function is given:

$$K(s)=C(s)G(s)=\frac{kk_c}{\tau}\frac{e^{\ sh}}{s^\alpha} \tag{5}$$

Note that open loop transfer function of control system shown in Fig.1. is different than Bode's ideal transfer function (2) with coefficient $\exp(-sh)$. It takes differences while Bode's diagram drawing.

Consider synthesis of the fractional order controller (4). For a given A_m gain and ϕ_m phase margins, the controller parameters k_c and real number α are searching.

Using $(j\omega)^\alpha=|\omega|^\alpha e^{j\alpha\pi/2}$ wrote gain and phase for a transfer function (5):

$$|K(j\omega)|=\frac{kk_c}{\tau}\frac{1}{\omega^\alpha}\qquad \phi(\omega)=\arg K(j\omega)=-h\omega-\alpha\frac{\pi}{2} \tag{6}$$

For a gain ω_g and ω_p phase crossover frequency terms can be written:

$$|K(j\omega_g)|=1\qquad \phi(\omega_p)=\arg K(j\omega_p)=-\pi \tag{7}$$

Using (6) the equations (7) can be rewritten as:

$$\frac{kk_c}{\tau\omega_g^\alpha}=1\qquad -h\omega_p-\alpha\frac{\pi}{2}=-\pi \tag{8}$$

By solving equations (8) gain ω_g and ω_p phase crossover frequency:

$$\omega_g^\alpha=\frac{kk_c}{\tau}\qquad \omega_p=\frac{(2-\alpha)\frac{\pi}{2}}{h} \tag{9}$$

Considering the second od (9) equations, we can said that ω_p is positive number while $\alpha<2$.

For a given A_m gain and ϕ_m phase margins:

$$\frac{kk_c}{\tau\omega_p^\alpha}=\frac{1}{A_m}\qquad \phi_m=\pi-h\omega_g-\alpha\frac{\pi}{2} \tag{10}$$

By solving (10) we can written:

$$\omega_p=\left(\frac{A_m kk_c}{\tau}\right)^{1/\alpha}\qquad \omega_g=\frac{(2-\alpha)\frac{\pi}{2}-\phi_m}{h} \tag{11}$$

Simply using first equations of (9) and (10):

$$A_m=\frac{\omega_p^\alpha}{\omega_g^\alpha} \tag{12}$$

Using second equations (9) and (11), and solving with (12) gain margin A_m is given by:

$$A_m=\left(\frac{(2-\alpha)\frac{\pi}{2}}{(2-\alpha)\frac{\pi}{2}-\phi_m}\right)^\alpha \tag{13}$$

Nonlinear equation (13) is handling A_m gain and ϕ_m phase margins and fractional order of the considerable controller (4) α.

Parameter α can be determined by solving equation (13) using computer methods.

By solving one of the first equations of (8) or (10):

$$k_c=\frac{\tau\omega_g^\alpha}{k}=\frac{\tau\omega_p^\alpha}{kA_m} \tag{14}$$

where k – gain of the transfer function (1).

Gain and phase fraquency crossover are determined from second equation of (9) or (11).

Note that fractional order of the controllers is relative with gain and phase margin only. Gain controller k_c is relative with gain or phase crossover frequency, gain k and time τ of the considerable object.

Method of the synthesis fractional order controller satisfying gain A_m and phase ϕ_m margin of the closed loop system with unstable plant with delay is given.

Synthesis method:

1. Solving nonlinear equation (13) for a given gain A_m and phase ϕ_m margins
   Real number α is given.
2. Solving phase crossover frequency from equation (9) or gain crossover frequency from equation (11)
   Parameter k_c of the controller is given with (14)
   Stability margin for a real object is smaller because of simplify used in (3).

3. SYNTHESIS METHOD FOR A UNSTABLE PLANT WITH INTEGRAL TERM WITH DELAY

Consider proposed synthesis method of the fractional controller in feedback control system shown in Fig. 1 in which the process to be controlled is described by transfer function:

$$G_1(s)=\frac{k}{s(1-s\tau)}e^{-sh} \tag{15}$$

To obtain an open loop system in the form (2), simplify the plant transfer function:

$$G(s)=\frac{k}{s(1-s\tau)}e^{\ sh}\approx-\frac{k}{s^2\tau}e^{\ sh} \tag{16}$$

The controller transfer function should have a structure:

$$C(s)=-k_c\frac{s^2}{s^\alpha}=-k_c s^{2-\alpha} \tag{17}$$

where α is real number.

Open loop transfer function is given:

$$K(s)=C(s)G(s)=\frac{kk_c}{\tau}\frac{e^{-sh}}{s^\alpha} \tag{18}$$

the same as given in (5), so for considered class main results are as given in. Determining controller parameters for a transfer function (15) we use equations (5-14) and given synthesis method.

4. SYNTHESIS METHOD FOR A SECOND ORDER UNSTABLE PLANT WITH DELAY

Consider proposed synthesis method of the fractional controller in feedback control system shown in Fig.1 in which the process to be controlled is described by transfer function:

$$G_1(s) = \frac{k}{(1-s\tau)(1+s\tau_2)} e^{-sh} \quad (19)$$

To obtain an open loop system in the form (2), simplify the plant transfer function:

$$G(s) = \frac{k}{(1-s\tau)(1+s\tau_2)} e^{-sh} \approx -\frac{k}{s\tau(1+s\tau_2)} e^{-sh} \quad (20)$$

The controller transfer function should have a structure:

$$C(s) = -k_c \frac{s(1+sT)}{s^{\alpha}} = -k_c s^{1-\alpha}(1+sT) \quad (21)$$

where α is real number, and while $\tau_2 = T$, we can said that open loop transfer function is given:

$$K(s) = C(s)G(s) = \frac{kk_c}{\tau} \frac{e^{-sh}}{s^{\alpha}} \quad (22)$$

the same as given in (5), so for considered class main results are as given. Determining controller parameters for a transfer function (23) we use equations (5-14) and given synthesis method.

5. RESULTS

Example 1:

Consider the feedback control system shown in Fig.1 in which the process to be controlled is described by transfer function:

$$G(s) = \frac{0.55}{1-62s} e^{10s} \quad (23)$$

Using synthesis method determine controller parameters for a given gain $A_m = 4$ (ab. 12dB) and phase $\phi_m = 55°$ (ab. 0.96 rad) margins for a closed loop system. In that case: $k = 0.55$, $\tau = 62, h = 10$.

Using synthesis method:

1. By solving equation (13) we get $\alpha = 1.13385$.
2. By solving (11) we get $\omega_g = 0.0401$.

Parameter k_c is given by (14): $k_c = 2.9358$

So transfer function of the controller (4) can be written:

$$C(s) = \frac{2.9358}{s^{0.13385}} \quad (24)$$

Fig. 2 shows step response for control system presented in Fig. 1, where the process to be controlled is described by (23) and fractional order controller (24). Step response is drawn for a few values of parameter k.

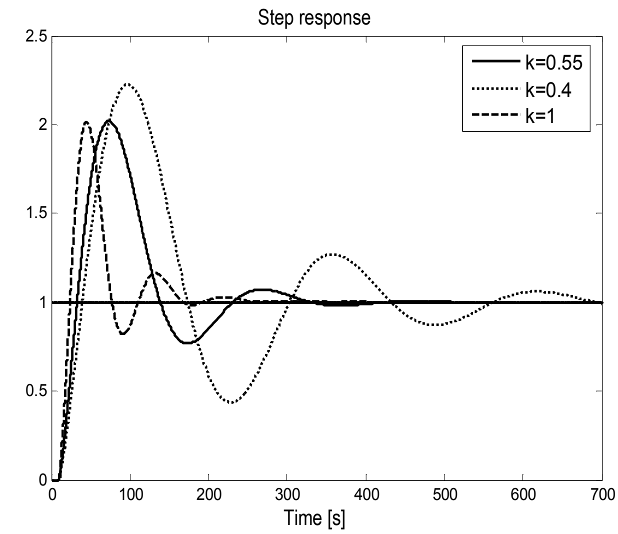

Fig. 2. Step response of the closed loop system with object (15) and controller (16) for a few different values of parameter k

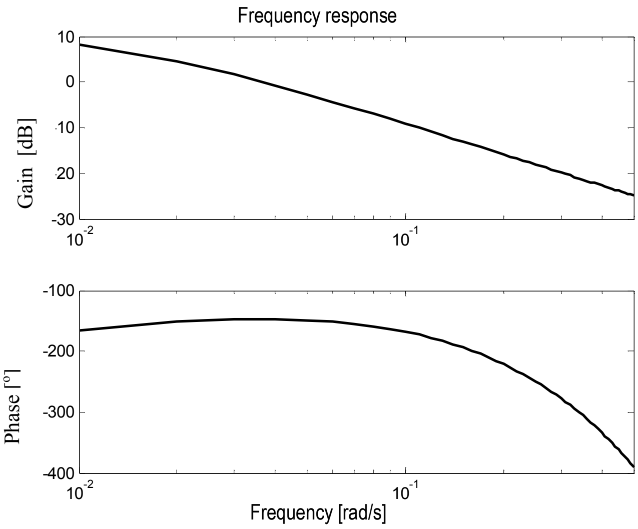

Fig. 3. Frequency response of the open loop system with object (15) and controller (16)

Overshoot of step response drawn for $k = 0.55$ is 100%. Overshoot is growing for a values less than 0.55. Drawing step response for a values k more that 0.55 (simulations were drawing to 1) no changing of overshoot were note.

Note that parameter k_c of fractional controller (24) is negative, so with simplify transfer function (3) we finally get negative feedback. Simulations in matlab are drawn for a original transfer function of the considerable transfer function, note that feedback is positive.

Fig. 3 shows Frequency response of the open loop system with object (15) and controller (16). Measured stability margin for a designed control system:

$$A_m = 3.5956, \phi_m = 32.9208°.$$

Stability margin measured is smaller because of simplify used in (3).

Example 2:

Consider the feedback control system shown in Fig.1 in which the process to be controlled is described by transfer function:

$$G(s) = \frac{0.55}{s(1-62s)} e^{-10s} \tag{25}$$

Using synthesis method determine controller parameters for a given gain $A_m = 4$ (ab. 12dB) and phase $\phi_m = 55°$ (ab. 0.96 rad) margins for a closed loop system.

In that case: $k = 0.55, \tau = 62, h = 10$.

Using synthesis method::

1. By solving equation (13) we get $\alpha = 1.13385$.
2. By solving (11) we get $\omega_g = 0.0401$.

Parameter k_c is given by (14): $k_c = 2.9358$.

So transfer function of the controller (4) can be written:

$$C(s) = -\frac{2.9358}{s^{0.13385}} \tag{26}$$

Fig. 4 shows step response for control system show in Fig. 1, where the process to be controlled is described by (25) and fractional order controller (26). Step response is drawn for a few values of parameter k.

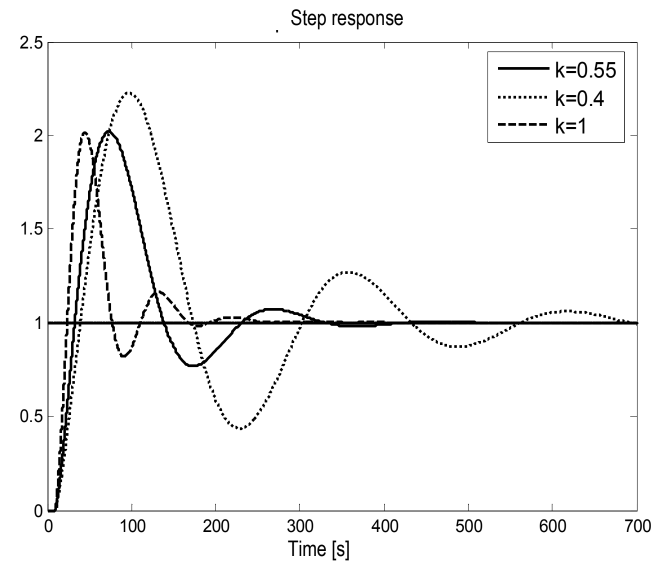

Fig. 4. Step response of the closed loop system with plant (21) and controller (22) for a few different values of parameters k

Overshoot of step response drawn for $k = 0.55$ is 100%. Overshoot is growing for a values less than 0.55. Drawing step response for a values k more that 0.55 (simulations were drawing to 1) no changing of overshoot were note. In that case we can also said that parameter k_c of fractional controller (26) is negative, and because of simplify (3) we finally get negative feedback. Simulations in matlab are drawn for a original transfer function of the considerable transfer function, note that feedback is positive.

Measured stability margin for a designed control system:

$$A_m = 3.5956, \phi_m = 32.9208°.$$

Stability margin measured is smaller because of simplify used in (3), and the same as given in example 1.

Example 3:

Consider the feedback control system shown in Fig.1 in which the process to be controlled is described by transfer function:

$$G(s) = \frac{0.55}{(1-62s)(1+36)} e^{\ 10s} \tag{27}$$

Using synthesis method determine controller parameters for a given gain $A_m = 4$ (ab. 12dB) and phase $\phi_m = 55°$ (ab. 0.96 rad) margins for a closed loop system.

In that case: $k = 0.55, \tau = 62, \tau_2 = 36, h = 10$.

Using synthesis method transfer function of the controller (25) can be written:

$$C(s) = -\frac{2.9358}{s^{0.13385}} - 105.6888s^{0.8661}. \tag{28}$$

Fig. 5 shows step response for control system show in Fig. 1, where the process to be controlled is described by (27) and fractional order controller (28). Step response is drawn for a few values of parameter k.

Overshoot of step response drawn for $k = 0.55$ is 100%. Overshoot is growing for a values less than 0.55. Drawing step response for a values k more that 0.55 (simulations were drawing to 1) no changing of overshoot were note.

Measured stability margin for a designed control system:

$$A_m = 3.5956, \phi_m = 32.9208°.$$

Stability margin measured is the same as in example 2 and example 2 because of transfer function of the open loop system given by the same transfer function

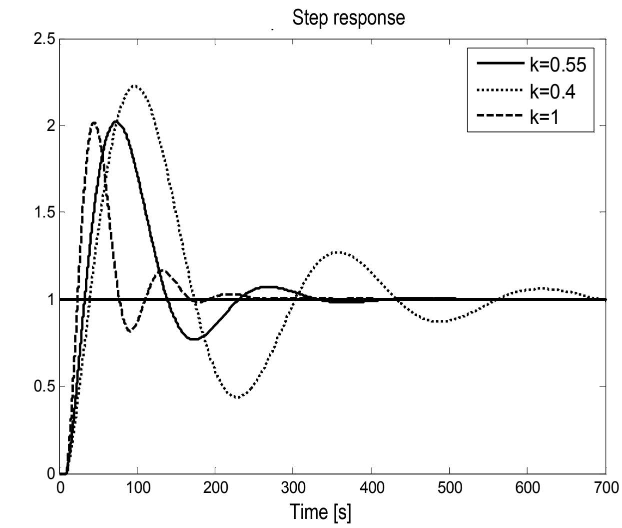

Fig. 5. Step response of the closed loop system with object (27) and controller (28) and a few different values of parameters k

6. CONCLUSIONS

The paper considers the design problem of fractional order controller satisfying gain and phase margin of the closed loop system. The proposed method is based on using Bode's ideal transfer function as a reference transfer function of the open loop system. Synthesis method is based on simplify of the object transfer function. Fractional order of the controllers is relative with gain and phase margin only. Method is based on using Bode's ideal transfer function as a reference transfer function of the open loop system. Synthesis method is based on simplify of the object transfer function. Fractional order of the controllers is relative with gain and phase margin only. Open loop transfer function of control system shown in Fig.1. is different than Bode's ideal transfer function (2) with coefficient $\exp(-sh)$. It takes differences while Bode's diagram drawing, phase and gain margin is different than given while synthesis the controller.

Parameter k_c of fractional controller is negative in each example considered in the paper, so with simplify transfer function (3) we finally get negative feedback. Simulations in matlab are drawn for a original transfer function of the considerable transfer function, the feedback is positive this example.

Computer method for synthesis of fractional controllers is given. The considerations are illustrated by numerical example and results of computer simulation with MATLAB/Simulink.

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