STATIONARY ACTION PRINCIPLE FOR VEHICLE SYSTEM WITH DAMPING

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Abstract: The aim of this note is to show possible consequences of the principle of stationary action formulated for non-conservative systems. As an example, linear models of vibratory system with damping and with one and two degrees of freedom are considered. This kind of models are frequently used to describe road and rail vehicles. There are vibrations induced by road profile. The appropriate action functional is proposed with the Lagrangian density containing: the kinetic and potential energies as well as dissipative one. Possible variations of generalized coordinates are introduced together with a noncommutative rule between operations of taking variations of the coordinates and their time derivatives. The stationarity of the action functional leads to the Euler-Lagrange equations.

Key words: Non-Conservative System, Principle of Stationary Action, Lagrangian, Non-Commutative Rule, Damping

1. INTRODUCTION

Historically, the classical Lagrange and Hamilton's formalisms were formulated for the point mechanics problems. Accordingly, if a dynamical system is described by the vector-valued generalized coordinate $\boldsymbol{q}$ and the Lagrangian $L = T - V$, where T and V are, respectively, the kinetic and potential energy, then one formulates the variational principle of the dynamical system requiring that between all curves $\boldsymbol{q} = \boldsymbol{q}(t)$ in a configuration space $\mathcal{V}$ the actual path (i.e. the solution of the system) is that which makes the action integral

$$I[\boldsymbol{q}] = \int_{t_0}^{t_1} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)dt \tag{1}$$

stationary. Taking the first variation $\delta\boldsymbol{q}$ subject to the conditions $\delta\boldsymbol{q}(t_0) = \delta\boldsymbol{q}(t_1) = 0$ the stationarity of the action requires $\delta I = 0$, which is equivalent to the Euler-Lagrange's equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0 \tag{2}$$

provided the classical commutative rule:

$$\delta\dot{\boldsymbol{q}} = \frac{d}{dt}\,\delta\boldsymbol{q}, \text{ or written as } \left[\left|\delta, \frac{d}{dt}\right|\right]\boldsymbol{q} = 0 \tag{3}$$

holds. Here the bracket $\left[\left|\delta, \frac{\mathrm{d}}{\mathrm{dt}}\right|\right]$ defines the difference between two compositions of operators $\delta\frac{\mathrm{d}}{\mathrm{dt}} - \frac{\mathrm{d}}{\mathrm{dt}}\delta$, not a vector.

It is well known that the governing equations for non-conservative, mechanical systems, i.e. when dissipative phenomena occur, at the present time cannot be derived from Hamilton's variational principle understood as the requirement: $\delta I[\boldsymbol{q}] = 0$ occupied with (3) (Schechter, 1967). In the present paper we will show that neglecting the commutativity law (3) governing equations of non-conservative system is possible to derive. First in Sec. 2 short review of different approaches is presented together with the main assumption about non-commutativity. As an example, linear models of vibratory system with damping and with one and two degrees of freedom are considered in Sec. 3. This kind of models are frequently used to describe road and rail vehicles under vibrations induced kinematically by road profile. The appropriate action functionals are proposed with the Lagrangian density containing: the kinetic and potential energies as well as dissipative one. Possible variations of generalized coordinates are introduced together with a non-commutative rule between operations of taking variations of the coordinates and their time derivatives. The stationarity of the action functional leads to the Euler-Lagrange equations.

2. NON-COMMUTATIVITY RULE

In order to derive governing equations describing irreversible phenomena using the variational technique some artificial restrictions must be made, concerning the basic rules of variational calculus. A good example is served by the variational principle formulation made in: Biot (1970), Chambers (1956), Kotowski (1989, 1992), Marsden and Hughes (1983), Prigogine and Glandsdorff (1965), Rosen (1954), Schechter (1967), Vujanović and Djuković (1972) and Yang (2010).

Different procedure was given by Vujanović (1971) and applied to governing hyperbolic equation of heat conduction (with finite wave speed), in which the new Lagrangian was proposed with an explicit dependence on time in the form of the exponential term $\exp t/\tau$ appearing as a factor. This term has the power t/τ, where τ is the thermal relaxation time. The corresponding transition to the case of infinite speed of thermal disturbance (parabolic equation) is performed by setting the relaxation time equal to zero.

After his first paper Vujanović has proposed in Vujanović (1974, 1975) the new method for deriving the class of equations describing some physical irreversible processes and based on the variational principle which has a Hamiltonian structure, and in most cases its form does not differ from that known for conservative systems. However, the crucial assumption of the new method is a non-commutative rule between operations of taking variations of the generalized coordinate (field) and their time derivatives.

In dissipative systems the loss of energy is a crucial effect. Because of this effect a physical process cannot be reversed without change in the environment. Such process is irreversible. Irreversibility means that variation of the quantity involved in description of irreversible process and the variation of its time derivative are related to dissipative mechanism governing the process considered, not the time differentiation. It means that the time derivative $\frac{d}{dt}\delta\boldsymbol{q}$ from one side must be different from the variation of the time derivative of the quantity, i.e. $\delta\frac{d}{dt}\boldsymbol{q}$. From the other side this variation is the dynamic quantity and it should depend on the non-conservative forces (according to Vuhanović (1975)) acting upon the system. We are not going to tamper with the usual notation of the variation of $\delta\boldsymbol{q}$ and the velocity of variation $\frac{d}{dt}\delta\boldsymbol{q}$. These two vectors are regarded as purely kinematic in nature. The vector $\delta\boldsymbol{q}$ means that we consider the *infinitesimal transformation* (i.e. first variation) replacing $\boldsymbol{q}(t,x)$, for example by $\boldsymbol{q}(t,x)+s\boldsymbol{h}(t,x)$, where $\boldsymbol{h}(t,x)$ is an arbitrary differentiable function of t and x, and s is a small parameter passing through zero. Then, from the definition:

$$\delta\boldsymbol{q}(t)=\frac{\partial}{\partial s}\big(\boldsymbol{q}(t)+\ s\boldsymbol{h}(t)\big)\delta s=\boldsymbol{h}(t)\delta s \tag{4}$$

Hence in this notation we have:

$$\frac{d}{dt}\delta\boldsymbol{q}=\frac{d}{dt}\boldsymbol{h}(t)\delta s \tag{5}$$

Since the vector $\delta\frac{d}{dt}\boldsymbol{q}$ has a purely dynamic character and its form depends on the nature of dissipative (non-conservative) phenomena and forces acting on the body, the infinitesimal transformation replaces $\frac{d}{dt}\boldsymbol{q}(t)$ by $\frac{d}{dt}\boldsymbol{q}(t)+s\boldsymbol{k}(t)$, where $\boldsymbol{k}(t)$ is not arbitrary differentiable function of t. This function, however, may differ from $\frac{d}{dt}\boldsymbol{h}(t,x)$ by a part that is related to the function $\boldsymbol{h}(t,x)$ via some relationship depending on the irreversible phenomena of the system. Hence we can write:

$$\delta\frac{d}{dt}\boldsymbol{q}=\frac{\partial}{\partial s}\left\{\frac{d}{dt}\boldsymbol{q}(t)+s\boldsymbol{k}(t)\right\}\delta s=\boldsymbol{k}(t)\delta s \tag{6}$$

together with:

$$\boldsymbol{k}(t)=\tfrac{d}{dt}\boldsymbol{h}(t)+\mathbf{H}\left(\boldsymbol{q}(t),\tfrac{d}{dt}\boldsymbol{q}(t),t\right)\boldsymbol{h}(t) \tag{7}$$

Notice that here we admit the tensor function $\mathbf{H}$. The appearing of the tensor $\mathbf{H}$ is a new fact (cf. Grochowicz and Kosiński (2011), Kosiński and Perzyna, 2012)).

Comparing (4) and (6) with (7) we end up with the following non-commutative rule:

$$\left[\left|\delta,\tfrac{d}{dt}\right|\right]\boldsymbol{q}=\mathbf{H}\left(\boldsymbol{q}(t),\tfrac{d}{dt}\boldsymbol{q}(t),t\right)\delta\boldsymbol{q} \tag{8}$$

We can see that the case when the function $\mathbf{H}=0$ corresponds to a commutative rule. This non-commutative rule will be crucial in developing new variation principle for deformable body made of the dissipative material with internal state variables.

In the previous papers (Grochowicz and Kosiński (2011), Kosiński and Perzyna, 2012)) the simpler version of the variational technique developed here was applied to 3 continuous irreversible systems. They were: long-line (i.e. telegraph) equation, hyperbolic model of heat conduction as well as governing equations of deformable body with internal state variables. In the next paper we will generalize the last derivation to the case of thermomechanics.

3. VARIATIONAL PRINCIPLE FOR DYNAMIC SYSTEM WITH DAMPING

Let us consider two linear models of a vehicle (Grzyb, 2012) which moves on a road or on a track. We will consider its vertical movement characterized by the displacement $Z(t)$ and produced by its horizontal motion along the road (track) profile, see Fig. 1. The vertical motion will be induced by kinematic excitation described by function $u(t)$. The car body of the vehicle is modeled by a rigid body of mass m. Vibrations of the mass are forced by a spring of an elastic constant k. The damping of the vehicle is realized by a linear, viscose damper characterized by a viscosity constant B. The first model has one degree of freedom.

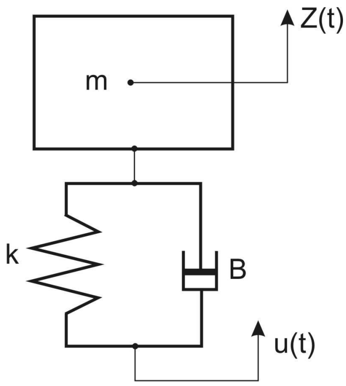

Fig. 1. Physical model of vehicle with one degree of freedom

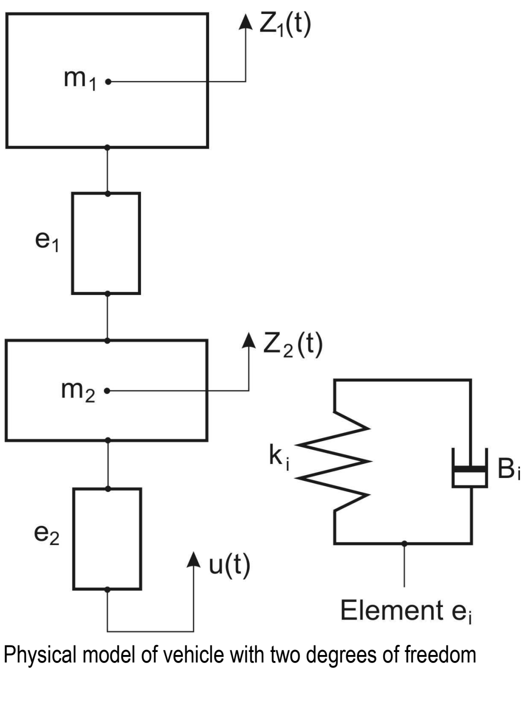

Fig. 2. Physical model of vehicle with two degrees of freedom

In order to take into account two parts of the vehicle, namely the vehicle body and the boogie, the model with two degrees of freedom, see Fig. 2, will be considered. The vibrations of the vehicle body of mass m_1 and the boogie of mass m_2 are characterized by displacements: $Z_1(t)$ and $Z_2(t)$, respectively. The vertical vibration of the both parts are induced by two separate springs with elastic constants: k_1 and k_2. The damping of considered vibrations is realized by linear dampers with viscosity constants B_1 and B_2, compare the element e_i on Fig. 2.

3.1. One degree of freedom model

In the first model the motion of the system (i.e. the car) is governed by the second order ODE (cf. Grzyb (2012)):

$$m\ddot{Z}(t) + B\dot{Z}(t) + kZ(t) = B\dot{u}(t) + ku(t) \tag{9}$$

According to the author of Grzyb (2012) this equation can be derived from the following Lagrange equation of the second kind:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}}\right) + \frac{\partial R}{\partial \dot{q}} + \frac{\partial E_p}{\partial q} = Q \tag{10}$$

with the appropriate forms of the kinetic energy E_k, the potential energy E_p, the dissipation function of the system R and the generalized force Q. In Grzyb (2012) in the derivation the author put $q = Z$, $Q = 0$ and suitable forms of other functions. It will be shown, that in the two degrees of freedom system the governing of equations are Euler-Lagrange equations derivable from a stationary action principle.

3.2. Two degree of freedom model

Having the derivation of the first model, let us start with the definition of the kinetic and potential energies for the system together with its dissipative part. Then, assuming the index 1 for the quantities describing the car body, we have the kinetic energy for the whole system: the car body and the boogie $T(\dot{Z}_1, \dot{Z}_2)$:

$$T(\dot{Z}_1, \dot{Z}_2) = \frac{1}{2}[m_1 {\dot{Z}_1}^2 + m_2 {\dot{Z}_2}^2] \tag{11}$$

the potential energy $V(Z_1, Z_2)$:

$$V(Z_1, Z_2, u) = \frac{1}{2}[k_1(Z_2 - Z_1)^2 + k_2(u - Z_2)^2] \tag{12}$$

and the dissipative part $D(\dot{u}, Z_2)$:

$$D(\dot{u}, Z_2) = B_2 \dot{u} Z_2 \tag{13}$$

Let us postulate the following non-commutative rule (cf. (8)):

$$\left[\left|\delta, \frac{d}{dt}\right|\right] \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{bmatrix} \begin{bmatrix} \delta Z_1 \\ \delta Z_2 \end{bmatrix} = \begin{bmatrix} -\frac{B_1}{m_1} & \frac{B_1}{m_1} \\ \frac{B_1}{m_2} & -\frac{B_1 + B_2}{m_2} \end{bmatrix} \begin{bmatrix} \delta Z_1 \\ \delta Z_2 \end{bmatrix} \tag{14}$$

Let us define the action functional with its Lagrangian density $L = T + D - V$:

$$I(Z_1, Z_2, u) = \int_{t_0}^{t_1} \left(T(\dot{Z}_1, \dot{Z}_2) + D(\dot{u}, Z_2) - V(Z_1, Z_2)\right) dt \tag{15}$$

Now we formulate the following stationary action principle.

Postulate: Let a system will be exited by the kinematic loading u. Along all curves $(Z_1(t), Z_2(t))$ in the configuration space the actual path is that which makes the action integral (15) stationary provided the both variations $\delta Z_1(t)$ and $\delta Z_2(t)$ vanish at the end points t_0 and t_1 and the non-commutativity load (14) holds.

Let us take the first variation of (15). We get:

$$\delta I(Z_1, Z_2, u) = \int_{t_0}^{t_1} \left(\frac{\partial T}{\partial \dot{Z}_1}\delta \dot{Z}_1 + \frac{\partial T}{\partial \dot{Z}_1}\delta \dot{Z}_2\right) dt - \int_{t_0}^{t_1} \left(\frac{\partial (V-D)}{\partial Z_1}\delta Z_1 + \frac{\partial (V-D)}{\partial Z_2}\delta Z_2\right) dt \tag{16}$$

Now we use the rule (14) to $\delta \dot{Z}_1$ and $\delta \dot{Z}_1$, to get:

$$\delta I = \int_{t_0}^{t_1} \left(\frac{\partial T}{\partial \dot{Z}_1}\left(\frac{d\delta Z_1}{dt}\right) + H^{11}\delta Z_1 + H^{12}\delta Z_2\right) dt + \int_{t_0}^{t_1} \left(\frac{\partial T}{\partial \dot{Z}_2}\left(\frac{d\delta Z_2}{dt}\right) + H^{21}\delta Z_1 + H^{22}\delta Z_2\right) dt - \int_{t_0}^{t_1} \left(\frac{\partial (V-D)}{\partial Z_1}\delta Z_1 + \frac{\partial (V-D)}{\partial Z_2}\delta Z_2\right) dt \tag{17}$$

Taking the time derivative on some terms and using formula of product differentiation as well as grouping similar terms, we obtain:

$$\begin{aligned} \delta I = &\int_{t_0}^{t_1} \left(-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Z}_1}\right)\delta Z_1 - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Z}_2}\right)\delta Z_2\right) dt \\ &+ \left[\frac{\partial T}{\partial \dot{Z}_1}\delta Z_1 + \frac{\partial T}{\partial \dot{Z}_2}\delta Z_1\right]\Bigg|_{t = t_0}^{t = t_1} \\ &+ \int_{t_0}^{t_1} \left(\frac{\partial T}{\partial \dot{Z}_1}H^{11} + \frac{\partial T}{\partial \dot{Z}_2}H^{21}\right)\delta Z_1 dt \\ &+ \int_{t_0}^{t_1} \left(\frac{\partial T}{\partial \dot{Z}_1}H^{12} + \frac{\partial T}{\partial \dot{Z}_2}H^{22}\right)\delta Z_2 dt \\ &- \int_{t_0}^{t_1} \left(\frac{\partial (V-D)}{\partial Z_1}\delta Z_1 + \frac{\partial (V-D)}{\partial Z_2}\delta Z_2\right) dt \end{aligned} \tag{18}$$

Since our Postulate requires vanishing $\delta I[Z_1, Z_2, u] = 0$ and the variations of Z_1 and Z_2 vanish at the end points, then inside the interval grouping terms appearing in the front of the variations δZ_1 and δZ_2, we obtain, in view of their arbitrariness and independence, two ODE's:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{Z}_1} + \frac{\partial (V-D)}{\partial Z_1} - \frac{\partial T}{\partial \dot{Z}_1}H^{11} - \frac{\partial T}{\partial \dot{Z}_2}H^{21} = 0 \\ &\frac{d}{dt}\frac{\partial T}{\partial \dot{Z}_2} + \frac{\partial (V-D)}{\partial Z_1} - \frac{\partial T}{\partial \dot{Z}_1}H^{12} - \frac{\partial T}{\partial \dot{Z}_2}H^{22} = 0 \end{aligned} \tag{19}$$

The above equations form the Euler-Lagrange equations of our stationary action functional Postulate.

We substitute the expressions for the ***H***'s coefficients from (14). We obtain:

$$m_1\ddot{Z}_1 + B_1(\dot{Z}_1 - \dot{Z}_2) + k_1(Z_1 - Z_2) = 0$$
$$m_2\ddot{Z}_2 - B_1\dot{Z}_1 + (B_1 + B_2)\dot{Z}_2 - k_1(Z_1 - Z_2) + k_2Z_2 = k_2u + B_2\dot{u} \quad (20)$$

The above method can be applied to the previous model with one degree of freedom system (9) by putting:

$$V(Z) = \frac{1}{2}m\dot{Z}^2$$
$$V(Z,u) = \frac{1}{2}k(u - Z)^2 \quad (21)$$
$$D(\dot{u}, Z) = B\dot{u}Z$$

and the non-commutativity rule:

$$\left[\left|\delta, \frac{d}{dt}\right|\right] Z = -\frac{B}{M} \quad (22)$$

4. CONCLUSIONS

In this paper starting with the action functional in which kinetic energy, potential energy and dissipative part of energy related to both elastic and irreversible phenomena of the system appear, the principle of its stationarity is formulated. Making the first variation of the functional compatible with the assumed initial and boundary conditions and postulating the appropriate non-commutative laws between the variation and time differentiation operators, the consequence in the form of Euler-Lagrange equations is obtained. The derived equations form the governing system of equations of the model with two degree of freedom. The passage to the system with one degrees of freedom is also shown.

The formulated method may be very helpful in other derivations and the investigation of particular and approximate forms of solutions of non-conservative systems.

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