FACTORIZATION OF NONNEGATIVE MATRICES BY THE USE OF ELEMENTARY OPERATION

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Abstract: A method based on elementary column and row operations of the factorization of nonnegative matrices is proposed. It is shown that the nonnegative matrix $A \in \Re_+^{n\times m}$ $(n \geq m)$ has positive full column rank if and only if it can be transformed to a matrix with cyclic structure. A procedure for computation of nonnegative matrices $B \in \Re_+^{n\times r}$, $C \in \Re_+^{r\times m}$ $(r \leq \text{rank}\,(n,m))$ satisfying $A = BC$ is proposed.

Keywords: Factorization, Nonnegative Matrix, Positive Rank, Procedure, Computation

1. INTRODUCTION

The factorization problem can be stated as: given nonnegative matrix $A \in \Re_+^{n\times m}$, find two nonnegative matrices $B \in \Re_+^{n\times r}$ and $C \in \Re_+^{r\times m}$ such that $A = BC$. The problem has been considered in many papers (de Almeida, 2011; Cichocki and Zdunek, 2006; Cohen and Rothblum, 1993; Donoho and Stodden, 2004; Lin, 2007; Lee and Seung, 2001) and it arises in many problems for example signal processing, quantum mechanics, combinational optimization etc. (de Almeida, 2011; Cohen and Rothblum, 1993). The factorization problem is closely related to the positive rank of nonnegative matrices (Cohen and Rothblum, 1993). The positive rank of nonnegative matrices plays important role in control system theory specially in the reachability problem of positive linear systems (Kaczorek, 2001).

In this paper a method based on elementary column and row operations of the factorization of nonnegative matrices will be proposed.

The paper is organized as follows. In section 2 the factorization problem is formulated and some basic definitions are recalled. The main result of the paper is presented in section 3, which is divided in three subsections. In the subsection 3.1 the elementary column and row operations and the elementary operation matrices are defined and their properties are formulated. Matrices with cyclic structures are introduced in subsection 3.2 and it is shown that the nonnegative matrix has positive full column rank if and only if it can be transformed to a matrix with cyclic structure. The proposed method of the factorization of nonnegative matrices is presented in subsection 3.3. The concluding remarks are given in section 4.The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n \times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – *the* $n \times n$ identity matrix.

2. PRELIMINARIES AND PROBLEM FORMULATION

Definition 2.1. (Cohen and Rothblum, 1993) The smallest nonnegative integer r is called the positive rank of the matrix $A \in \Re_+^{n\times m}$ and denoted by $\text{rank}_+ A$ if there exist $b_k \in \Re_+^n$, $k = 1, \dots, r$ $(r \leq m)$ such that each column $a_i \in \Re_+^n$, $i = 1, \dots, m$ of A is the linear combination:

$$a_i = \sum_{k=1}^{r} c_{k,i} b_k \quad \text{for } i = 1,\dots,m \tag{2.1}$$

with nonnegative coefficients $c_{ki} \geq 0$, $k = 1, \dots, r$; $i = 1, \dots, m$ of the vectors b_k.

From (2.1) it follows that if $\text{rank}_+ A = r$ then the matrix $A \in \Re_+^{n\times m}$ can be written in the form:

$$A = BC \tag{2.2a}$$

where:

$$B = [b_1 \;\; \dots \;\; b_r] \in \Re_+^{n\times r}, \quad C = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} \in \Re_+^{r\times m}, \tag{2.2b}$$
$$c_k = [c_{k,1} \;\; \dots \;\; c_{k,m}], \quad k = 1,\dots,r.$$

Definition 2.2. The representation of the matrix $A \in \Re_+^{n\times m}$ in the form (2.2) is called factorization of the matrix A.

The standard $\text{rank}\ A$ of $A \in \Re_+^{n\times m}$ and the positive $\text{rank}_+ A$ are related by (Cohen and Rothblum, 1993):

$$\text{rank}\ A \leq \text{rank}_+ A \leq \min(n,m) \tag{2.3}$$

For example the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \tag{2.4}$$

has the standard rank equal to 3 and the positive rank equal to 4. The problem under considerations can be stated as follows.

Given a nonnegative matrix $A \in \Re_+^{n\times m}$, find its factorization (2.2) i.e. the matrices $B \in \Re_+^{n\times r}$ and $C \in \Re_+^{r\times m}$ such that (2.2) holds.

In this paper the factorization problem will be solved by the use of the elementary column and row operations.

3. PROBLEM SOLUTION

3.1. Elementary operations

To solve the factorization problem the following elementary column and row operations will be used:

- Multiplication of the ith column (row) by positive number c. This operation will be denoted by $\mathrm{R}[i \times c]$ $(\mathrm{L}[i \times c]$.
- Addition to the ith column (row) of the jth column (row) multiplied by negative number $-c$ $(c > 0)$. This operation will be denoted by $\mathrm{R}[i + j \times (-c)]$ $(\mathrm{L}[i + j \times (-c)]$.
- Interchange of the ith and jth columns (rows). This operation will be denoted by $\mathrm{R}[i, j]$ $(\mathrm{L}[i, j])$.

Let $R_m[i, c]$, $R_a[i, j, -c]$ and $R_i[i, j]$ be the elementary column operations matrices obtained by applying the elementary column operations $R[i \times c]$, $R[i + j \times (-c)]$ and $R[i, j]$ to the identity matrices respectively. Similarly, are defined the elementary row operations matrices $L_m[i, c]$, $L_a[i, j, -c]$ and $L_i[i, j]$. The elementary column operations are performed by post-multiplication of the matrix by the elementary column operations matrices and the elementary row operations are performed by premultiplication of the matrix by the elementary row operations (Kaczorek, 1993).

It is easy to prove the following lemmas.

Lemma 3.1. The inverse matrices $R_m^{-1}[i, c]$, $R_a^{-1}[i, j, -c]$, $R_i^{-1}[i, j]$ of $R_m[i, c]$, $R_a[i, j, -c]$, $R_i[i, j]$ and the inverse matrices $L_m^{-1}[i, c]$, $L_a^{-1}[i, j, -c]$, $L_i^{-1}[i, j]$ of $L_m[i, c]$, $L_a[i, j, -c]$, $L_i[i, j]$ satisfies the equalities:

$$R_m^{-1}[i,c] = R_m\left[i, \frac{1}{c}\right],\ R_a^{-1}[i,j,-c] = R_a[i,j,c],\ R_i^{-1}[i,j] = R_i[i,j] \quad (3.1a)$$

$$L_m^{-1}[i,c] = L_m\left[j, \frac{1}{c}\right],\ L_a^{-1}[i,j,-c] = L_a[i,j,c],\ L_i^{-1}[i,j] = L_i[i,j] \quad (3.1b)$$

Lemma 3.2. The elementary column operations $R[i \times c]$, $R[i + j \times (-c)]$, $R[i, j]$ and elementary row operations $L[i \times c]$, $L[i + j \times (-c)]$, $L[i, j]$ do not change the positive $\mathrm{rank}_+\mathrm{A}$ of the matrix $\mathrm{A} \in \Re_+^{n \times m}$.

Remark 3.1. It is assumed that after performance of any of the elementary column and row operations on a nonnegative matrix $\mathrm{A} \in \Re_+^{n \times m}$ the obtained matrix is also nonnegative.

For example performing on the matrix (2.4) the following elementary operations $L[2 \times 1/2]$, $L[4 \times 1/3]$, we obtain:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R[2,4]} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{L[3,4]} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R[2,3]} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (3.2)$$

The matrices (2.4) and (3.2) have the same positive rank equal to 4.

Let e_i be the ith column of the $n \times n$ identity matrix. The column ae_i for $a > 0$ is called the monomial column (Kaczorek, 2001). The nonnegative matrix consisting of $m (m \leq n)$ linearly independent monomial columns has full column positive rank. The positive rank and standard rank of this matrix are the same as the matrix (2.4).

3.2. Matrices with cyclic structure

Definition 3.1. A nonnegative matrix:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1,n} \\ \vdots & \dots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix},\ a_{ij} \geq 0,\ i, j = 1,\dots,n \quad (3.3)$$

is called the matrix with cyclic structure if:

$$a_{ii} \geq a_{i+1,i} \geq \dots \geq a_{n,i} \geq a_{1,i} \geq \dots \geq a_{i-1,i} \quad i = 1,\dots,n. \quad (3.4)$$

For example the matrix (3.2) has cyclic structure.

Theorem 3.1. [9] The system of linear algebraic equations:

$$a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n = 1,\ a_{i,j} \geq 0,\ i, j = 1,\dots,n \quad (3.5)$$

has a nonnegative solution $x_i \geq 0, i = 1, \dots, n$ if and only if its coefficient matrix has the cyclic structure.

Theorem 3.2. The nonnegative matrix $\mathrm{A} \in \Re_+^{n \times m}$, $n \geq m$ has positive full column rank:

$$\mathrm{rank}_+ A = m \quad (3.6)$$

if and only if it can be transformed to a matrix A with cyclic structure, i.e. (3.4) holds for , $i = 1, \dots, m$.

Proof. Consider the matrix equation:

$$\begin{bmatrix} a_{11} & \dots & a_{1,m} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{m,1} & \dots & a_{m,m} & 0 & \dots & 0 \\ a_{m+1,1} & \dots & a_{m+1,m} & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{n,1} & \dots & a_{n,m} & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad (3.7)$$

By Theorem 3.1 the matrix equation has the nonnegative solution $x = [x_1 \dots x_m \quad 1 \dots 1]^T \in \Re_+^n$ if and only if the matrix A has cyclic structure.

From Theorem 3.2 and Lemma 3.2 we have the following important corollary.

Corollary 3.1. The nonnegative matrix $A \in \Re_+^{n \times m}$ has positive full column rank $\mathrm{rank}_+\mathrm{A} = m$ if and only if it has cyclic structure or can be transformed to this cyclic structure by the elementary column and row operations.

For example the matrix (2.4) has not the cyclic structure but it has been transformed to the matrix (3.2) with cyclic structure by the use of the elementary row and column operations.

3.3. The proposed method

First the proposed method of the factorization of nonnegative matrices will be demonstrated on the following examples.

Examples 3.1. For the nonnegative matrix:

$$A=\begin{bmatrix}2&4&2\\1&4&5\\0&1&2\end{bmatrix} \tag{3.8}$$

find nonnegative matrices B and C satisfying the condition (2.2a). Using the elementary column operations to (3.8) we obtain:

$$\begin{bmatrix}2&4&2\\1&4&5\\0&1&2\end{bmatrix}\xrightarrow{R[2+1\times(-2)]}\begin{bmatrix}2&0&2\\1&2&5\\0&1&2\end{bmatrix}\xrightarrow[R[3+2\times(-2)]]{R[3+1\times(-1)]}\begin{bmatrix}2&0&0\\1&2&0\\0&1&0\end{bmatrix}=\overline{A} \tag{3.9a}$$

and:

$$\overline{A}=AR \tag{3.9b}$$

where:

$$R=R_a[2,1,-2]R_a[3,1,-1]R_a[3,2,-2]=\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-2\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-2&3\\0&1&-2\\0&0&1\end{bmatrix}. \tag{3.10}$$

From (3.9) and (3.10) we have:

$$A=\overline{A}R^{-1}=\begin{bmatrix}2&0&0\\1&2&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&-2&3\\0&1&-2\\0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}2&0&0\\1&2&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&1\\0&1&2\\0&0&1\end{bmatrix}=BC \tag{3.11}$$

where:

$$B=\begin{bmatrix}2&0\\1&2\\0&1\end{bmatrix},\quad C=\begin{bmatrix}1&2&1\\0&1&2\end{bmatrix}. \tag{3.12}$$

Using (3.1) and (3.10) we may compute the inverse matrix R^{-1} as follows:

$$R^{-1}=R_a[3,2,2]R_a[3,1,1]R_a[2,1,2]=\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&1\\0&1&2\\0&0&1\end{bmatrix}. \tag{3.13}$$

The same result we obtain using the elementary row operations to (3.8):

$$\begin{bmatrix}2&4&2\\1&4&5\\0&1&2\end{bmatrix}\xrightarrow[L[1+2\times(-2)]]{L[2+3\times(-2)]}\begin{bmatrix}0&0&0\\1&2&1\\0&1&2\end{bmatrix}=\hat{A} \tag{3.14a}$$

and

$$\hat{A}=LA \tag{3.14b}$$

where:

$$L=L_a[1,2,-2]L_a[2,3,-2]=\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-2\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-2&4\\0&1&-2\\0&0&1\end{bmatrix}. \tag{3.15}$$

From (3.14) and (3.15) we have:

$$A=L^{-1}\hat{A}=\begin{bmatrix}1&-2&4\\0&1&-2\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}0&0&0\\1&2&1\\0&1&2\end{bmatrix}=\begin{bmatrix}1&2&0\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\1&2&1\\0&1&2\end{bmatrix}=BC \tag{3.16}$$

where the matrices B, C are given by (3.12).

Using (3.1) and (3.15) we may compute the inverse matrix L^{-1} as follows:

$$L^{-1}=L_a[2\quad 3\quad 2]L_a[1\quad 2\quad 2]=\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&0\\0&1&2\\0&0&1\end{bmatrix}. \tag{3.17}$$

In general cases let us consider the nonnegative matrix $A\in\Re_+^{n\times m}$ with $n\geq m$. If $\mathrm{rank}_+\mathrm{A}=m$ then the matrix has trivial factorization (2.2) with B positive full column rank, i.e. $\mathrm{rank}_+\mathrm{B}=m$ and any nonnegative elementary column operations matrix C.

From example for the matrix $A=\begin{bmatrix}0&4&2\\1&0&3\\0&0&1\end{bmatrix}$ we have:

$$B=\begin{bmatrix}0&2&0\\1&0&0\\0&0&1\end{bmatrix},\quad C=\begin{bmatrix}1&0&3\\0&2&1\\0&0&1\end{bmatrix}. \tag{3.18}$$

Let:

$$\mathrm{rank}_+A=r<\min(n,m)\,. \tag{3.19}$$

If $n>m$ the following elementary column operations procedure is recommended.

Procedure 3.1.

Step 1. Using a suitable sequence of elementary column operations reduce the matrix $A\in\Re_+^{n\times m}$ to the form:

$$\overline{A}=AR=[B\quad 0]\in\Re_+^{n\times m},\quad B\in\Re_+^{n\times r} \tag{3.20}$$

where:

$$R=R_1R_2...R_q \tag{3.21}$$

And $R_k, k=1,\dots,q$ are the elementary column operation matrices defined in 3.1.

Step 2. Performing the elementary column operations on the identity matrix I_n and using (3.1) compute the inverse matrix:

$$R^{-1} = R_q^{-1}...R_2^{-1}R_1^{-1} = \begin{bmatrix} \overline{R}_1 \\ \overline{R}_2 \end{bmatrix}, \quad \overline{R}_1 \in \Re_+^{r\times m}, \quad \overline{R}_2 \in \Re_+^{(m-r)\times m} \tag{3.22}$$

Step 3. Using (3.20) and (3.22) find the desired matrices $B \in \Re_+^{n\times r}$ and $C = \bar{R}_1 \in \Re_+^{r\times m}$ satisfying (2.2).

Justification of Procedure 3.1 follows from (3.20) and (3.22) since:

$$A = \overline{A}R^{-1} = [B \quad 0]\begin{bmatrix} \overline{R}_1 \\ \overline{R}_2 \end{bmatrix} = B\overline{R}_1 = BC . \tag{3.23}$$

Remark 3.2. If $n > m$ and $\mathrm{rank}_+ A = m$ then the elementary row operations procedure is recommended or we may apply Procedure 3.1 to the transpose matrix A^T and use the equality $A^T = (BC)^T = C^T B^T$.
Example 3.2. Find the factorization (2.2) of the nonnegative matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0.5 \\ 6 & 8 & 4 & 1 \end{bmatrix} \in \Re_+^{3\times 4} . \tag{3.24}$$

In this case we apply the elementary row operations approach since $m = 4 > n = 3$. Using Procedure 3.1 we obtain the following.
Step 1. Using the following row operations we obtain:

$$A \xrightarrow[L[3+2\times(-2)]]{L[3+1\times(-2)]} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \overline{A} = \begin{bmatrix} C \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0.5 \end{bmatrix} . \tag{3.25}$$

Step 2. Performing the elementary row operations $L[3 + 2 \times 2]$ $L[3 + 1 \times 2]$ on the identity matrix I_3 we obtain:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[L[3+1\times 2]]{L[3+2\times 2]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} = L^{-1} = [B \quad \overline{B}], \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{3.26}$$

Step 3. The desired matrices B and C satisfying $A = BC$ have the forms:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0.5 \end{bmatrix}. \tag{3.27}$$

4. CONCLUDING REMARKS

The factorization problem of nonnegative real matrices has been addressed. A method based on elementary column and row operations of the factorization of nonnegative matrices has been proposed. It has been shown that the nonnegative matrix $A \in \Re_+^{n\times m}$ $(n \geq m)$ has positive full column rank if and only if it can be transformed to a matrix A with cyclic structure (Theorem 3.2). A procedure based on the elementary operations for computation of nonnegative matrices $B \in \Re_+^{n\times r}$, $C \in \Re_+^{r\times m}$ $(r \leq rank(n, m))$ satisfying the condition (2.2a) has been proposed and illustrated by numerical examples.

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