POSITIVE REALIZATIONS FOR DESCRIPTOR DISCRETE-TIME LINEAR SYSTEMS

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Abstract: Conditions for the existence of positive realizations for descriptor discrete-time linear systems are established. A procedure for computation of positive realizations for improper transfer matrices is proposed. The effectiveness of the method is demonstrated on numerical example.

Key words: Positive, Realization, Procedure, Descriptor, Linear Systems

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2002). Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. The positive fractional linear systems have been addressed in Kaczorek (2008a, 2009a, 2011d).

An overview on the positive realization problem is given in Benvenuti and Farina (2004), Farina and Rinaldi (2000), Kaczorek (2002, 2009b). The realization problem for positive continuous-time and discrete-time linear systems has been considered in Kaczorek (2004, 2006a, c, 2011a, c) and the positive minimal realization problem for singular discrete-time systems with delays in Kaczorek (2005). The realization problem for fractional linear systems has been analyzed in Kaczorek (2008b, 2011b, d) and for positive 2D hybrid systems in Kaczorek (2008c). A method based on the similarity transformation of the standard realizations to the desired form has been proposed in Kaczorek (2011c).

Positive stable realizations problem for continuous-time standard and fractional linear systems has been addressed in Kaczorek (2011a, b) and computation of realizations of discrete-time cone systems in Kaczorek (2006b).

In this paper a method for computation of positive realizations of descriptor discrete-time linear systems will be proposed.

The paper is organized as follows. In section 2 the positive realization problem for standard discrete-time linear systems is recalled. The positive realization problem for descriptor discrete-time linear systems is formulated and solved in section 3. The proposed procedure for computation of positive realizations of a given improper transfer matrix is illustrated by numerical example in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, $\Re^{p\times m}(z)$ – the set of $p\times m$ rational matrices in z with real coefficients, $\Re^{p\times m}[z]$ – the set of $p\times m$ polynomial matrices in z with real coefficients, I_n – the $n\times n$ identity matrix.

2. PRELIMINARIES AND POSITIVE REALIZATION PROBLEM FOR STANDARD SYSTEMS

Consider the standard discrete-time linear system:

$$x_{i+1}=Ax_i+Bu_i,\ i\in Z_+=\{0,1,...\} \tag{2.1a}$$

$$y_i=Cx_i+Du_i \tag{2.1b}$$

where: $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 2.1. The system (2.1) is called (internally) positive if $x_i\in\Re_+^n$, $y_i\in\Re_+^p$, $i\in Z_+$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u_i\in\Re_+^m$, $i\in Z_+$.

Theorem 2.1. The system (2.1) is positive if and only if (Farina and Rinaldi, 2000; Kaczorek, 2002):

$$A\in\Re_+^{n\times n},\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n},\ D\in\Re_+^{p\times m}. \tag{2.2}$$

The transfer matrix of the system (2.1) is given by:

$$T(z)=C[I_nz-A]^{-1}B+D. \tag{2.3}$$

The transfer matrix $T(z)\in\Re^{p\times m}(z)$ is called proper if and only if:

$$\lim_{z\to\infty}T(z)=K\in\Re^{p\times m} \tag{2.4}$$

and it is called strictly proper if $K=0$. Otherwise the transfer matrix is called improper.

Definition 2.2. Matrices (2.2) are called a positive realization of transfer matrix $T(z)$ if they satisfy the equality (2.3).

Different methods for computation of a positive realization (2.2) for a given proper transfer matrix $T(z)$ have been proposed in Kaczorek (2002, 2011a, b, d).

3. POSITIVE REALIZATION PROBLEM FOR DESCRIPTOR SYSTEMS

Consider the descriptor discrete-time linear system:

$$Ex_{i+1} = Ax_i + Bu_i,\; i \in Z_+ = \{0,1,...\} \tag{3.1a}$$

$$y_i = Cx_i \tag{3.1b}$$

where: $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

It is assumed that $\det E = 0$ and the pencil of (E, A) is regular, i.e.:

$$\det[Ez - A] \neq 0 \text{ for some } z \in \mathrm{C}$$
(3.2)

where: C is the field of complex numbers.

Definition 3.1. The descriptor system (3.1) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

If the nilpotency index μ of the matrix E is greater or equal to 1 (Kaczorek, 1992) then the transfer matrix of (3.1) is improper and given by:

$$T(z) = C[Ez - A]^{-1}B \in \Re^{p\times m}(z). \tag{3.3}$$

The improper matrix (3.3) can be always written as the sum of strictly proper part $T_{sp}(z)$ and the polynomial part $P(z)$, i.e.

$$T(z) = T_{sp}(z) + P(z) \tag{3.4a}$$

where:

$$P(z) = D_0 + D_1 z + ... + D_q z^q \in \Re^{p\times m}[z],\; q \in N = \{1,2,...\} \tag{3.4b}$$

and $q = \mu - 1$.

Theorem 3.1. Let the matrices:

$$A \in \Re_+^{n\times n},\; B \in \Re_+^{n\times m},\; C \in \Re_+^{p\times n} \tag{3.5}$$

be a positive realization of the strictly proper transfer matrix $T_{sp}(z)$. Then there exists a positive realization of $T(z) \in \Re^{p\times m}(z)$ of the form:

$$\bar{E} = \begin{bmatrix} I_n & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ 0 & I_m & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & I_m & 0 \end{bmatrix} \in \Re_+^{\bar{n}\times\bar{n}},$$

$$\bar{A} = \begin{bmatrix} A & B & 0 & ... & 0 & 0 \\ 0 & I_m & 0 & ... & 0 & 0 \\ 0 & 0 & I_m & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & I_m \end{bmatrix} \in \Re_+^{\bar{n}\times\bar{n}},\quad \bar{B} = -\begin{bmatrix} 0 \\ I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re_+^{\bar{n}\times m},$$

$$\bar{C} = [C \quad D_0 \quad D_1 \quad ... \quad D_q] \in \Re_+^{p\times\bar{n}},\quad \bar{n} = n + (q+1)m \tag{3.6}$$

if and only if:

$$D_k \in \Re_+^{p\times m} \text{ for } k = 0,1,\ldots,q. \tag{3.7}$$

Proof. If the matrices (3.5) are a positive realization of $T_{sp}(z)$ then the standard system:

$$x_{i+1} = Ax_i + Bu_i \tag{3.8a}$$

$$y_i = Cx_i \tag{3.8b}$$

is positive and $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$. Defining the new state vector:

$$\bar{x}_i = \begin{bmatrix} x_i \\ u_i \\ u_{i+1} \\ \vdots \\ u_{i+q} \end{bmatrix} \in \Re^{\bar{n}} \tag{3.9}$$

and using (3.6) we obtain:

$$\bar{E}\bar{x}_{i+1} = \bar{A}\bar{x}_i + \bar{B}\bar{u}_i \tag{3.10a}$$

$$\bar{y}_i = \bar{C}\bar{x}_i \tag{3.10b}$$

From (3.10) if follows that $\bar{x}_i \in \Re_+^{\bar{n}}$ and $\bar{y}_i \in \Re_+^p$ for $i \in Z_+$ if and only if (3.7) holds since $x_i \in \Re_+^n$ and $u_i \in \Re_+^m$ for $i \in Z_+$. Using (3.6), (3.3) and (3.4) it is easy to verify that:

$$\bar{C}[\bar{E}z - \bar{A}]^{-1}\bar{B} = [C \quad D_0 \quad D_1 \quad ... \quad D_q]$$

$$\times \begin{bmatrix} I_n z - A & -B & 0 & ... & 0 & 0 \\ 0 & -I_m & 0 & ... & 0 & 0 \\ 0 & I_m z & -I_m & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & I_m z & -I_m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= [C \quad D_0 \quad D_1 \quad ... \quad D_q] \begin{bmatrix} [I_n z - A]^{-1}B \\ I_m \\ I_m z \\ \vdots \\ I_m z^q \end{bmatrix} \tag{3.11}$$

$$= C[Ez - A]^{-1}B + D_0 + D_1 z + ... + D_q z^q$$
$$= T_{sp}(z) + P(z) = T(z).$$

The positive realization problem for the descriptor system can be stated as follows. Given an improper rational matrix $T(z) \in \Re^{p\times m}(z)$, find its positive realization (3.6).

If the conditions of Theorem 3.1 are satisfied then the desired positive realization (3.6) of $T(z)$ can be computed by the use of the following procedure.

Procedure 3.1.

Step 1. Decompose the given matrix $T(z)$ into the strictly proper part $T_{sp}(z)$ and the polynomial part $P(z)$ satisfying (3.4).

Step 2. Using one of the well-known methods (Kaczorek, 2002, 2011 a, b) find the positive realization (3.5) of $T_{sp}(z)$.

Step 3. Knowing the realization (3.5) and the matrices $D_k \in \Re_+^{p\times m}$, $k = 0,1,\ldots,q$ of (3.4b) find the desired realization (3.6).

4. EXAMPLE

Find a positive realization (3.6) of the transfer matrix:

$$T(z) = \begin{bmatrix} \frac{z^4 - 3z^3 + 3z^2 - 2z + 0.5}{z^2 - 3z + 2} & \frac{z^3 - 2z^2 - 4z + 4}{z^2 - 4z + 3} \\ \frac{3z^3 - 11z^2 + 6z + 0.5}{z^2 - 4z + 3} & \frac{2z^4 - 9z^3 + 8z^2 + 2z + 3.2}{z^2 - 5z + 6} \end{bmatrix}. \tag{4.1}$$

Using Procedure 3.1 we obtain the following.

Step 1. The transfer matrix (4.1) has the strictly proper part:

$$T_{sp}(z) = \begin{bmatrix} \frac{z-1.5}{z^2 - 3z + 2} & \frac{z-2}{z^2 - 4z + 3} \\ \frac{z-2.5}{z^2 - 4z + 3} & \frac{z-2.8}{z^2 - 5z + 6} \end{bmatrix} \tag{4.2}$$

and the polynomial part:

$$P(z) = \begin{bmatrix} z^2 + 1 & z + 2 \\ 3z + 1 & 2z^2 + z + 1 \end{bmatrix} = D_0 + D_1 z + D_2 z^2, \ (q = 2) \tag{4.3a}$$

where:

$$D_0 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}, \ D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \tag{4.3b}$$

Step 2. The strictly proper transfer matrix (4.2) can be rewritten in the form:

$$T_{sp}(z) = \frac{1}{(z-1)(z-2)(z-3)} \times \begin{bmatrix} (z-1.5)(z-3) & (z-2)^2 \\ (z-2.5)(z-2) & (z-2.8)(z-1) \end{bmatrix} \tag{4.4}$$

and the well-known Gilbert method can be applied to find its positive realization (Kaczorek, 2002, 2011a; Shaker and Dixon, 1977). Following Gilbert method we compute the matrices:

$$T_1 = \lim_{z \to z_1 = 1} (z - z_1) T_{sp}(z) = \begin{bmatrix} \frac{z-1.5}{z-2} & \frac{z-2}{z-3} \\ \frac{z-2.5}{z-3} & \frac{(z-1)(z-2.8)}{(z-2)(z-3)} \end{bmatrix}_{z=1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.75 & 0 \end{bmatrix}, \tag{4.5a}$$

$$r_1 = \text{rank}\, T_1 = 2,$$

$$T_2 = \lim_{z \to z_2 = 2} (z - z_2) T_{sp}(z) = \begin{bmatrix} \frac{z-1.5}{z-1} & \frac{(z-2)^2}{(z-1)(z-3)} \\ \frac{(z-2)(z-2.5)}{(z-1)(z-3)} & \frac{z-2.8}{z-3} \end{bmatrix}_{z=2} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}, \tag{4.5b}$$

$$r_2 = \text{rank}\, T_2 = 2,$$

$$T_3 = \lim_{z \to z_3 = 3} (z - z_3) T_{sp}(z) = \begin{bmatrix} \frac{(z-1.5)(z-3)}{(z-1)(z-2)} & \frac{z-2}{z-1} \\ \frac{z-2.5}{z-1} & \frac{z-2.8}{z-2} \end{bmatrix}_{z=3} = \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0.2 \end{bmatrix}, \tag{4.5c}$$

$$r_3 = \text{rank}\, T_3 = 2$$

$$\begin{aligned} &T_1 = C_1 B_1, \ C_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.75 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ &T_2 = C_2 B_2, \ C_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ &T_3 = C_3 B_3, \ C_3 = \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0.2 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{4.6}$$

and:

$$A = \text{blockdiag}[I_{r_1} z_1, I_{r_2} z_2, I_{r_3} z_3] = \text{diag}[1,1,2,2,3,3],$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.7}$$

$$C = [C_1 \quad C_2 \quad C_3] = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0 & 0 & 0.5 \\ 0.75 & 0 & 0 & 0.8 & 0.25 & 0.2 \end{bmatrix}.$$

Step 3. The desired positive realization of (4.1) has the form:

$$\begin{aligned} &\overline{E} = \begin{bmatrix} I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \end{bmatrix} \in \Re_+^{\overline{n} \times \overline{n}}, \\ &\overline{A} = \begin{bmatrix} A & B & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_2 \end{bmatrix} \in \Re_+^{\overline{n} \times \overline{n}}, \ \overline{B} = \begin{bmatrix} 0 \\ -I_2 \\ 0 \\ 0 \end{bmatrix} \in \Re_+^{\overline{n} \times 2}, \\ &\overline{C} = [C \quad D_0 \quad D_1 \quad D_2] \in \Re_+^{2 \times \overline{n}}, \\ &\overline{n} = n + (q+1)m = 6 + 3 * 2 = 12 \end{aligned} \tag{4.8}$$

and the matrices A, B, C, D_0, D_1, D_2 are given by (4.7) and (4.3b).

5. CONCLUDING REMARKS

A method for computation of positive realizations for descriptor discrete-time linear systems has been proposed. Conditions for the existence of positive realizations for given improper transfer matrices have been established. A procedure for computation of positive realizations has been proposed and illustrated by a numerical example. The proposed method can be easily extended to descriptor continuous-time linear systems and asymptotically stable descriptor discrete-time linear systems. An open problem is an extension of this method for fractional descriptor linear systems.

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