# LIMITATION OF CAUCHY FUNCTION METHOD IN ANALYSIS OF ESTIMATORS OF FREQUENCY AND FORM OF NATURAL VIBRATIONS OF CIRCULAR PLATE WITH VARIABLE THICKNESS AND CLAMPED EDGES

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**Abstract:** In this paper the Berstein-Kieropian double estimators of basic natural frequency of circular plate with power variable thickness along the radius and clamped edges in diaphragm form were analyzed in a theoretical approach. The approximate solution of boundary problem of transversal vibration by means of Cauchy function and characteristic series method has been applied for chosen values of power indicator of variable thickness $m$ and material Poisson's ratio $v$ has been chosen which led to exact form solutions. Particular attention has been given to a singularity arising from the uncertainty of estimates of Bernstein-Kieropian. Improving this method has been obtained the general form of Cauchy function for arbitrary values of $m$ and $v$, which are physically justified. Therefore, the aim of the paper was to explore the reason why for a plate above a certain value $m = 3.97$ exact solution, which Conway couldn't receive (Conway, 1958a, b).

**Key words:** Vibration, Circular Plate, Boundary Value Problem, Cauchy Function, Bernstein-Kieropian's Estimators

## 1. INTRODUCTION

In a previous paper (Jaroszewicz, 2008) authors analyzed the use of simplest lower estimator to calculate the basic frequency of axi-symmetrical vibration of plates with variable thickness circular diaphragm type. The existence of the simplest estimator of the actual value of the parameter depending on the frequency rate of change characterized by thick plate ($m = 3.25$ to $5.999$) was analyzed. The accuracy of the method differed from the FEM and in order to improve the accuracy of the estimators it was decided to use a higher order, in this case double. Using the bilateral estimators the similar problem arose in the calculation of exact solution in a paper by Conway (Conway 1958a, 1958b). It seems that the problem lies in the fact that diaphragms for meters which is close to 4 in the center of symmetry have a very low rigidity and on a boundary value it creates a hole in the middle with a radius of $1\text{mm}$ to $23\text{mm}$. As a common known plate with a hole required 2 additional boundary conditions on the edge of the hole, the hole is a singularity which requires detailed analysis. It is widely known that clarification of the model leads to the complexity of solutions. The compromise between the possibilities of addressing coastal vibrations and stability of mechanical systems and simplifying gives the total allowable use of methods of the influence functions, and partial discretization characteristic series. Good results achieved in previous publication (Jaroszewicz et al., 2008), which included linear modeling of mechanical systems with discrete-continuous parameters encouraged authors to use the above-mentioned methods for studying vibration plates diaphragm type of ring. In this case, the influence of the function, which is the product of the Cauchy and Heaviside unit functions were applied. This function features the influence of derivatives, which are fundamental solutions of linear differential equations and can build their base of the integrated general solutions with various types of $\delta$ ratios in the Dirac. For the vibration test plate with varying parameters method of partial discretization has been used. It is based on the method of influence and has been previously proposed by Zoryj and Jaroszewicz to analyze vibration plates fixed and variable thickness with an additional mass focused (Jaroszewicz and Zoryj, 2005, 2006).The record for continuous or continuous-discrete mass distribution systems discrete can be replaced with one, two and n-degrees of freedom, which are characterized by the same function of stiffness. The plate's mass focuses on the rings with a certain radius. Total weight of the replacement is equal to its own weight plates. This procedure uses a universal characteristic of the equation.

## 2. FORMULATION OF THE PROBLEM

$R$ – radius circular plate having a clamped edge whose cross section presented in Fig.1 has been considered. Its thickness $h$ and flexural rigidity $D$ change in the following way (Tab. 1).

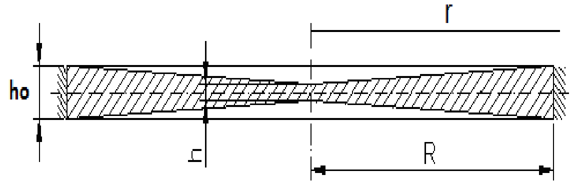
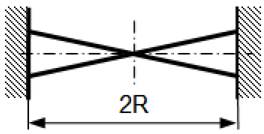



**Fig. 1.** Cross section of the plate of diaphragm type

$$h = h_0(\frac{r}{R})^{\frac{m}{3}}, D = D_0(\frac{r}{R})^m, 0 < r \le R, D_0 = \frac{Eh_0^3}{12(1-v^2)} \quad (1)$$

where: $D_0, h_0, m \ge 0$ are the constants, $r$ – denotes the radial coordinate, $E$ – Young's modulus, $v$ – denotes Poisson's ratio.

**Tab.1.** Variable thickness plates

| Diaphragm | Thickness | Schema |
|---|---|---|
| Linear variable thickness | $h = h_0(\frac{r}{R})^{\frac{m}{3}}$, $2 < m < 6$ |   |
| Power variable thickness | $h = h_0(\frac{r}{R})$ |   |

Investigation of free, axi-symmetrical vibrations of such a plate is reduced to an analysis of the boundary problem (Conway, 1958a, 1958b; Jaroszewicz and Zoryj, 2005):

$$L_0[u] - pr^{-\frac{2}{3}m}u = 0 \quad (2)$$

$$u(R) = 0,\ u'(R) = 0, \quad (3)$$

where:

$$L_0[u] = u^{IV} + \frac{2}{r}(m+1)u^{III} + \frac{1}{r^2}(m^2 + m + \upsilon m - 1) + \frac{1}{r^3}(m-1)(\upsilon m - 1)u, \quad (4)$$

$$p = \frac{\rho h_0}{D_0} R^{\frac{2}{3}m}\varpi^2,$$

$u = u(r)$ – denotes the amplitude of line of bending, $\rho$ – density of the material of the plate, $\omega$ – the parameter of frequency (angular velocity). Boundary value conditions corresponding to a clamped edge (3) have been defined as zero values of deflection and zero values of the angle of deflection for $r = R$ Additional conditions pertaining to the center of symmetry of a plate ($r = 0$) have limited values of deflection $u(0) < \infty$ and zero values of the angle of deflection $u'(0) = 0$. The value $m = 0$ refers to the plate with constant thickness; $m > 0$ to plates of the diaphragm type with thickness decreasing toward the axial center; $m < 0$ to disc type plates with thickness increasing toward the axial center (Woźniak, 2005; Jaroszewicz and Zoryj, 2005). The border of variation of the power index $m \geq 0$ has been determined, for which the most simple estimators of the basic frequency $\omega_1$ exist and therefore can be calculated, i.e. it has been searched for the lowest proper value of the border problem (2.÷.3). In the problem (2.÷.3), a limitation of solution for $r$ going to zero and their first up to the third derivatives, with respect to the independent variable $r$ is required (Conway, 1958b).

General form of Cauchy function was proposed by Zoryj and Jaroszewicz by mean following expression:

$$K_0(r,\alpha) = \frac{1}{(1-\upsilon)m}\left[\frac{1}{2\sqrt{D}}(r^{S_3}\alpha^{S_4+m+1} - r^{S_4}\alpha^{S_3+m+1}) - \frac{1}{2-m}(r^{2-m}\alpha^{m+1} - \alpha^3)\right], \quad (5)$$

where: $S_1, S_2, S_3, S_4$ – roots of characteristic equation, $D$ – determinant of square equation:

$$s^2 - (2-m)s - m(1-\upsilon), \quad (6)$$

$$D = (1 - \frac{m}{2})^2 + m(1-\upsilon). \quad (7)$$

Equation (6) received from characteristic equation:

$$s\{s^3 + 2(m-2)s^2 + [4 - 5m + m^2 + \upsilon m]s + m(2-m)(1-\upsilon)\}, \quad (8)$$

which roots present in next form:

$$s_1 = 0,\ s_2 = 2 - m,\ s_{3,4} = -\frac{m}{2} + 1 \mp \sqrt{D}. \quad (9)$$

So, for $L_0[u] = 0$ the fundamental system of solutions, in those cases is (according to Euler's theory of equations) as follows:

$$m = 0,\ 1,\ lnr,\ r^2,\ r^2 lnr,$$

$$m = 2,\ 1,\ lnr,\ r^{\sqrt{2-2\upsilon}},\ r^{-\sqrt{2-2\upsilon}}. \quad (10)$$

Based on those calculations, the following remarks can be formulated:

- all roots of the cubic equation (8), as well as those of the quadratic equation (6) are real numbers for any given physically justified values of $m$ – power indexes and values of Poisson's ratio $\upsilon$ ($\upsilon \in (0, 0.5)$);
- the equation (6) has no other multiple (repetitive) roots for $m \in (-\infty, +\infty)$ and $\upsilon \in (0, 0.5)$;
- the fundamental systems of solutions for Euler's differential equations $L_0[u] = 0$ possess logarithmic peculiarities only in cases $m = 0$, $m = 2$ and are determined by the formula (10); in all of the rest cases they have power character peculiarities.

## 3. THE BILATERAL ESTIMATORS FOR BASIC FREQUENCY IN PARTICULAR CASE $\nu = 1/m$

Take into account the series known Bernstein-Kieropian's estimators (Bernstein and Kieropian, 1960) with the following form can be applied:

$$(a_1^2 - 2a_2)^{-\frac{1}{2}} < \gamma_0 < \sqrt{2}(a_1 + \sqrt{a_1^2 - 4a_2})^{-\frac{1}{2}} \quad (11)$$

where: $\gamma_0$ – mean value of Bernstein-Kieropian estimators, $\gamma_- = (a_1^2 - 2a_2)^{-\frac{1}{2}}$ – lower estimator, $\gamma_+ = \sqrt{2}(a_1 + \sqrt{a_1^2 - 4a_2})^{-\frac{1}{2}}$ – upper estimator.

Coefficient $a_1$ scrutinized in previous work (Jaroszewicz 2008), where exact formula has been constructed:

$$a_1 = \frac{3^4}{(6-m)(6+m)(9+m)(12+m)}. \quad (12)$$

To develop formula for $a_2$ in a similar form formula in following form should be present:

$$A_1B_1 + A_2(1 - 2a) \equiv A_1 \cdot \frac{1}{a} \cdot F(m), \quad (13)$$

where:

$$F(m) = [(a+1)(b+1)(b+2)]^{-1} - 0.5[c(c+1)]^{-1} \quad (14)$$

Second addition in expression can be present in form:

$$B_2(1 + 2a) = \frac{1}{2}B_1 \cdot f(m) \quad (15)$$

where:

$$f(m) = [a(c+1)(c+2)]^{-1}. \quad (16)$$

Now exploit determination (14)÷(16) also formulas (Jaroszewicz 2008):

$$A_2 = \frac{1}{2}A_1[a(2a-1)c(c+1)]^{-1}, \quad (17)$$

$$B_2 = \frac{1}{2}B_1[a(2a+1)(c+1)(c+2)]^{-1} \quad (18)$$

we constructed coefficient $a_2$ record in form expression:

$$a_2 = \frac{1}{a^2(b+1)}\left[\frac{F(m)}{(a-1)b} + \frac{1}{2(a+1)(b+2)(c+1)(c+2)}\right]. \quad (19)$$

Take into account identity $m = 4.5 = 1.5(1-a)$ the first component sum (19) in form has been found:

$$\frac{F(m)}{(a-1)b} = -\frac{81(m-3)(m+24)}{2(15-2m)(9+m)(6+m)(12+m)(18-m)(21-m)}. \quad (20)$$

Considering also identity:

$$\frac{1}{2(a+1)(b+2)(c+1)(c+2)} = \frac{81}{2(12+m)(15-2m)(21-m)(24-m)}. \quad (21)$$

Finally from (19) general form second coefficients of characteristic series $a_2$ has been received:

$$a_2 = \frac{3^9 \cdot 5}{2(12+m)(6+m)(18-m)(21-m)(24-m)(6-m)^2(9+m)^2}. \quad (22)$$

Example results of calculation for changed cases $2 < m < 6$ and $\nu = 1/m$ obtain on the basis of formulas (11), (12) and (22) present in the Tab. 1, 2 and Fig. 2.

## 4. DISCUSES OF RESULTS OF CALCULATIONS OF BASE FREQUENCY

Results of calculation where compared with previous paper (Jaroszewicz 2008) in Tab. 1 and Fig. 2 which were calculated by means Cauchy function and characteristic series method using simplest estimator and exact solution received on base Bessel special function by Conway, Hondkiewic and Kovalenko (Kovalenko, 1959; Hondkiewic, 1959).

Boundary values, for which the upper estimator does not exist can be settled on the basis of the investigation of under roots expression form change (23) $a_1^2 - 4a_2$ with $m = 3.97$ change sign form positive to negative with $m = 3.98$ so calculating accurate values $a_1$ and $a_2$ on the base of (19) we have respectively:

$$a_1^2 - 4a_2|_{m=3.97} = 7 \cdot 10^{-7} > 0,$$

$$a_1^2 - 4a_2|_{m=3.98} = -6 \cdot 10^{-7} < 0. \quad (23)$$

Under estimated values there for it cannot be even used in approximate application however the simplest lower estimator can be applied to preliminary engineering calculations for constant and for variable thickness plates when $0 \le m \le 4$.

It should be noticed that in the case of a constant thickness plate($m = 0$), the multiplier $\gamma = \gamma_0 = 10.2122$ is independent from $\nu$ (Kovalenko, 1959; Vasylenko and Oleksiejčuk, 2004). So the ratio of coefficients:

$$\frac{\gamma}{\gamma_0} = 1.1678. \quad (24)$$

It does agree with results of calculations obtained by Conway in (Conway, 1958a). Continuing in analogical fashion the basic frequency for other combinations of $m$ and $\nu$ has been calculated. The results of calculations are presented in Tab. 3.

In Tab. 3 are presented values received by Jaroszewicz J. and Zoryj L. cases for which Conway derive characteristic function on base special Bessel function. Ratio $\gamma/\gamma_0$ of natural frequency plate constants thickness and plate of variable thickness for Conway values $m, \nu$. This Tab. contain model value of solution with we compare approximate results.

In Tab. 3 Conway couldn't apply the exact method (Conway, 1958a), because the condition $\nu = \frac{2m-3}{9}$ was not fulfilled.

## 5. ESTIMATE OF RESULTS CALCULATION OF BASE FORM

As an example the current assumption was given $\mathrm{m} = 3$, $\nu = 1/3$ which considers linear thickness of plate $\mathrm{h} = \mathrm{h}_0 \frac{\mathrm{r}}{\mathrm{R}}$.

The way of calculation has been illustrated on example of equation (2), which will have following form:

$$L_0[u] - pr^{-2}u = 0.$$

$$L_0[u] \equiv u^{IV} + \frac{8}{r}u^{III} + \frac{12}{r^2}u^{II} \quad (25)$$

with boundary conditions (3).

On the base of (5) and (8) Cauchy function has been received in form:

$$K_0(r,\alpha) = \frac{1}{6}(r\alpha^2 - r^{-2}\alpha^5) + \frac{1}{2}(r^{-1}\alpha^4 - \alpha^3) \quad (26)$$

and limited for $r = 0$ solutions of equation (2):

$$u = C_1U_1(r) + C_2U_2(r), \quad (27)$$

where:

$$U_i(r) = a_{0i} + pa_{1i} + p^2a_{ai} + \cdots \quad (i = 1,2), \quad (28)$$

$$a_{ij} = \int_o^r K_o(r,\alpha)\alpha^{-2}\alpha_{j-1,i}(\alpha)d\alpha, \quad (29)$$

$$a_{01} = 1,\ a_{02} = r,\ a_{11} = \frac{r^2}{24},\ a_{21} = \frac{r^4}{24 \cdot 360},$$

$$a_{12} = \frac{r^3}{120},\ a_{22} = \frac{r^5}{120 \cdot 840}. \quad (30)$$

Appropriate forms of natural vibration can be derived similarly as for constant thickness plate (Kovalenko, 1959; Hondkiewic, 1959). On the base of formulas (26-30) we are able to find for parameter of base frequency ($\gamma_0 = 8.75$) following values:

$$\frac{\gamma_0^2}{24} = 3.19, \quad \frac{\gamma_0^2}{120} = 0.64,$$

$$U_1(\chi) = 1 + 3.19\chi^2 + .., U_2(\chi) = \gamma(1 + 0.64\chi^2 + \cdots \quad (31)$$

where: $\chi = (\frac{\gamma}{R})^2$.

Then form (27) by using boundary condition $U(R) = 0$ we find:

$$\alpha = \frac{C_2}{C_1} = -\frac{U_1(R)}{U_2(R)} = -2.55. \quad (32)$$

Moreover basic form in first approximate can be describe by means of the following function:

$$-U(R) = U_1 + \alpha U_2 = 1.64\chi^3 - 3.19\chi^2 + 2.55\chi - 1 \quad (33)$$

**Tab. 2.** Results of calculation of base frequency

| No. | | 1. | 2. | 3. | 4. | 5. | 6. | 7. | 8. | 9. |
|---|---|---|---|---|---|---|---|---|---|---|
| Coefficient m | | 2 | 2.5 | 3 | 3.7 | 3.975 | 4.49 | 5 | 5.5 | 5.999 |
| Two first terms of characteristic series | $a_1$ | 1/61 | 1/61 | 1/60 | 1/55 | 1/52 | 1/43 | 3/97 | 1/18 | 250 |
| | $a_2$ | 2/58941 | 3/75050 | 1/2016 | 6/7956 | 1/10684 | 8/5017 | 19/559 | 13/100 | 31250.2 |
| Simplest lower estimator | $\gamma(m) = (a_1)^{-1/2}$ | 7.80 | 7.83 | 7.75 | 7.41 | 7.19 | 6.58 | 5.69 | 4.24 | 0.06 |
| The double sides estimator | $\gamma_-$ | 8.38 | 8.56 | 8.65 | 8.62 | 8.55 | 8.27 | 7.75 | 6.73 | 0.83 |
| | $\gamma_+$ | 8.45 | 8.66 | 8.84 | 9.19 | 10.16 | - | - | - | - |
| | $\gamma_0$ | 8.42 | 8.61 | 8.75 | 8.91 | 9.36 | - | - | - | - |
| Value of exact solution | $\gamma_{exact}$ | 8.46 | 8.60 | 8.75 | - | - | - | - | - | - |

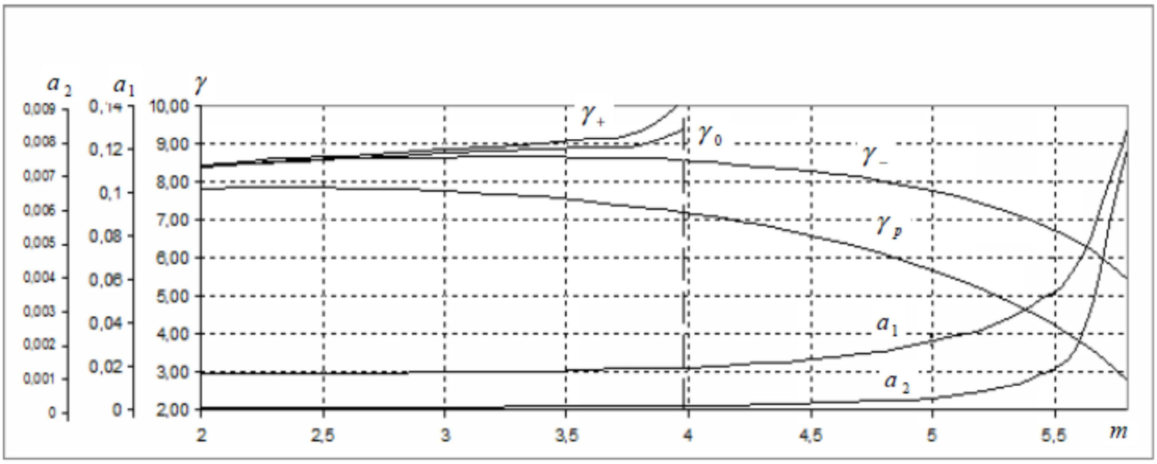



**Fig. 2.** The curve showing influence of the plate thickness index on the bilateral estimators of the frequency coefficients

**Tab. 3.** Results of calculations (Jaroszewicz and Zoryj, 2005)

| L.p. | $m$ | $\nu$ | $\frac{\gamma_0}{\gamma}$ | $\frac{T}{T_0}$ |
|---|---|---|---|---|
| 1 | 2 | $\frac{1}{9}$ | $\frac{10.2144}{9.4562} = 1.0802$ | $1.0824$ |
| 2 | $\frac{18}{7}$ | $\frac{5}{21}$ | $\frac{10.2144}{9.0777} = 1.1252$ | $1.1261$ |
| 3 | $\frac{18}{7}$ | $\frac{7}{18}$ | $\frac{10.2144}{8.6376} = 1.1825$ | Conway does not applying exact solution |

From condition $\mathrm{U(R)} = 0$ by extracting the solution $\chi = 1$ we have received square equation:

$$1.64\chi^2 - 1.56\chi + 1 = 0 \tag{34}$$

which does not possess real roots. On this base we can conclude, that in case $\mathrm{m} = 3, \nu = \frac{1}{3}$ basic form corresponds to the anodal form similar as for plates with constant thickness.

Obviously anodal forms do not exist in this specific case and wave parameter equals zero.

## 6. SUMMARY

On the basis of Fig. 2 it can be seen that the simplest estimator underestimates results of the value with the increase of the coefficient $m$. In contrast to double estimate $\gamma_-, \gamma_+$ gives accurate results, this follows from the analysis of the value of the denominator of the estimator $\gamma_0$ the value calculation is consistent with the exact solution for $m = 3$, is $8.75$ the same is true for $m = 2\ (8.40)$ and $m = 2.5\ (8.60)$. Deriving of the above mentioned formulas for the Cauchy function (5), as well as fundamental systems of function operator $L_0[u]$ allows to study of the convergence problem (velocity of convergence) of solutions of equation (4) in form of power series in respect to parameters of frequencies, depending on values of parameters $m$ and $\upsilon$.

Having the influence functions of operator $\mathrm{L_0[u]}$ corresponding solutions and use them for any given physically justified values of parameters $\mathrm{m}$ and $\upsilon$ $(\mathrm{m}\epsilon(-\infty, +\infty);\ \nu\epsilon(0, 0.5))$ can be consequently determined, when the exact solutions are unknown on base general form of Cauchy function (5). On the basis of quoted solutions, simple engineering formulas for frequencies estimators of circular plates, which are characterized by variable parameters distribution, can be derived and limits of their application can be identified. The bilateral estimators calculated using four first elements of the series, allows to credibly observe an influence material's constants: Young modulus – $E$, Poisson ratio – $\nu$, density – $\rho$, on the frequencies on axi-symmetrical vibrations of circular plates, which thickness or rigidity changes along the radius according to the power function. The bilateral estimator underrate values, application of bilateral estimator significantly improves the accuracy of calculations terms of exacts solutions.

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