EVALUATION OF THE ACCURACY OF THE SOLUTION TO THE HEAT CONDUCTION PROBLEM WITH THE INTERVAL METHOD OF CRANK-NICOLSON TYPE

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Abstract: The paper deals with the interval method of Crank-Nicolson type used for some initial-boundary value problem for the one-dimensional heat conduction equation. The numerical experiments are directed at a short presentation of advantages of the interval solutions obtained in the floating-point interval arithmetic over the approximate ones. It is also shown how we can deal with errors that occur during computations in terms of interval analysis and interval arithmetic.

Key words: Interval Finite Difference Method of Crank-Nicolson Type, Interval Arithmetic, Computational Errors

1. INTRODUCTION

Interval methods belong to a large class of numerical methods first introduced by Sunaga (1958), Moore (1966) and Moore et al. (2009) that enable a result verification. Growing interest in interval methods over a few past decades results from the fact that interval solutions obtained with such methods include the exact solution of the problem. Furthermore, their computer implementation in the floating-point interval arithmetic (Jankowska, 2006, 2009a, b, 2010; Marciniak, 2008, 2009, 2012), together with the representation of the initial data in the form of machine intervals, let us achieve interval solutions that contain all possible numerical errors.

The one-dimensional heat conduction equation considered in the paper belongs to a group of initial-boundary value problems for partial differential equations that occur very often in many scientific fields. For some of such problems the appropriate interval methods were proposed by Jankowska and Marciniak (in press), Manikonda, Berz and Makino (2005), Marciniak (2008), Nakao (2001), Nagatou et al. (2007) and Watanabe et al. (1999).

The paper deals with the interval finite difference method of Crank-Nicolson type. The interval counterpart of the conventional Crank-Nicolson method for the one-dimensional heat conduction equation with the boundary conditions of the first kind were proposed by Marciniak (2012). Jankowska extended his work taking into account the same equation but with the mixed boundary conditions (2012). The interval method proposed enables to include in the interval solutions obtained the local truncation error of the conventional method that is normally neglected. Note that in practice it is not easy to satisfy all the assumptions made in the theoretical formulation of the method given in Jankowska (2012). Nevertheless, the appropriate techniques for the approximation of endpoints of the error term intervals in each step of the method are described in Jankowska (2011). Several numerical tests performed by the author confirmed their effectiveness and usefulness.

The interval method of Crank-Nicolson type described in Jankowska (2011, 2012) is used to solve some initial-boundary value problem for the heat conduction equation formulated in Section 3. The numerical results presented in Section 4 are directed at giving the interval solutions such as they contain the exact solution of the problem. We show how to estimate the errors caused by an inaccuracy of the initial data obtained from the physical experiment, an inexact representation of some real values in a set of all floating-point numbers and the rounding errors that occur during computations. Finally, some conclusions given in Section 5 brings the paper to the end.

2. SOME REMARKS ON INTERVAL METHOD OF CRANK-NICOLSON TYPE

In this section we shortly present the main idea of the interval method of Crank-Nicolson type. Before that it is necessary to introduce fundamentals of interval arithmetic (Moore, 1966; Moore et al., 2009) and its machine implementation (Jankowska, 2009a, b, 2010; Marciniak, 2009). Finally, the well-known sources of errors that can occur during computations are listed and we also explain how to handle them in terms of interval arithmetic.

2.1. Basics of interval arithmetic

A real interval covers the range of real numbers between two bounds and can be defined as follows.

A *real interval* or just an *interval* is a closed and bounded subset of the real numbers R, i.e.

$$X = [\underline{x}, \overline{x}] := \{ x \in \mathrm{R} : \underline{x} \le x \le \overline{x} \}, \tag{1}$$

where $\underline{x}$, $\bar{x}$ denote the lower and upper bounds (infimum and supremum) of the interval. The set of all real intervals is denoted by IR.

Degenerate intervals such as $\underline{x} = \bar{x}$ are called thin or point intervals. They are equivalent to real numbers and we may write *x* instead of *X* or [*x*, *x*].

Since intervals are sets, we can define the standard symbols $=, \in, \subset, \cap, \cup, /$, in the usual sense of set theory. Moreover, we consider the empty set ø as an interval. In this way we ensure that the set of intervals is closed with respect to intersection.

The *diameter* (*width*), *radius* and *midpoint* of an interval X are defined as follows:

$$d(X) := \operatorname{diam}(X) = \overline{x} - \underline{x}, \tag{2}$$

$$r(X) := \operatorname{rad}(X) = \frac{\overline{x} - \underline{x}}{2}, \tag{3}$$

$$m(X) := \operatorname{mid}(X) = \frac{\underline{x} + \overline{x}}{2}. \tag{4}$$

The *distance* $q(X,Y)$ between two intervals X and Y is defined as:

$$q(X,Y) := \max\left\{\left|\underline{x} - \underline{y}\right|, \left|\overline{x} - \overline{y}\right|\right\}. \tag{5}$$

From the above it follows that the distance equals 0 if and only if $X = Y$. It does not depend on the order of its arguments, i.e. $q(X,Y) = q(Y,X)$, and the triangle inequality holds. Hence, q is a metric and (IR, q) is a metric space. Moreover, the concepts of convergence and continuity may be introduced in the usual manner and it can be shown that (IR, q) is a complete metric space.

The *interval arithmetic* is an extension of real arithmetic for elements of IR. Let us denote by $\circ$ one of the following elementary operators $+, -, \cdot, /$. Then we define elementary arithmetic operations on intervals by:

$$X \circ Y = \left\{x \circ y : x \in X, y \in Y\right\}, \tag{6}$$

and we assume that $0 \notin Y$ for the definition of X/Y.

The result of an elementary interval operation is the set of real numbers obtained from combining any two numbers in X and in Y. Since the corresponding real operations are continuous, the right-hand side of (6) is an interval.

Closed intervals can be considered as sets (on which standard set operations apply), or as couples of elements of R on which an arithmetic can be build. Therefore the operations $\circ \in \{+,-,\cdot,/\}$ determined by (6) can be redefined as operations on the bounds of intervals as follows:

$$X + Y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \tag{7}$$

$$X - Y = [\underline{x} - \overline{y}, \overline{x} - \underline{y}], \tag{8}$$

$$\begin{aligned} X \cdot Y = [&\min\{\underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \overline{x}\cdot\overline{y}\}, \\ &\max\{\underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \overline{x}\cdot\overline{y}\}], \end{aligned} \tag{9}$$

$$X / Y = X \cdot [1/\overline{y}, 1/\underline{y}], \text{ if } 0 \notin Y \tag{10}$$

Note that in terms of the above definition of interval and the way that the interval arithmetic is constructed, a definition of interval function with intervals as variables can be specified. Furthermore, we can build such interval function corresponding to a real-value one in a quite simple way (see e.g. Moore (1966), Moore et al. (2009).

The interval arithmetic can be implemented in most high-level programming languages. However before that some additional definitions of a floating-point interval, a set of all floating-point intervals and basic arithmetic operations on floating-point intervals have to be given.

Let us denote by $\mathfrak{R}$ the set of all floating-point numbers, i.e. of all real numbers that can be represented in a given real number format.

A *floating-point interval* is a closed and bounded subset of the real numbers R, i.e.

$$X = [\underline{x}, \overline{x}] := \{x \in \mathrm{R} : \underline{x} \le x \le \overline{x},\ \underline{x}, \overline{x} \in \mathfrak{R}\}, \tag{11}$$

whose endpoints are floating-point numbers.

The set IR of all floating-point intervals over $\mathfrak{R}$ is denoted by:

$$\mathrm{IR} = \{[\underline{x}, \overline{x}] : \underline{x} \in \mathfrak{R} \wedge \overline{x} \in \mathfrak{R},\ \underline{x} \le \overline{x}\}. \tag{12}$$

Let us note that the floating-point interval $[\underline{x}, \overline{x}] \in$ IR is a connected subset of R, i.e. though $\underline{x}$, $\overline{x}$ are elements of $\mathfrak{R}$, the interval $[\underline{x}, \overline{x}]$ contains not only every floating-point number lying between its bounds, but also every real number within this range.

The rounding $\circ : \mathrm{R} \to \mathfrak{R}$, which maps a real number to a floating-point number, is defined by the following conditions:

$$\forall_{x \in \mathfrak{R}},\ \circ x = x, \tag{13}$$

$$\forall_{x,y \in \mathrm{R}},\ x \le y \Rightarrow \circ x \le \circ y, \tag{14}$$

where $\circ \in \{\square, \nabla, \Delta\}$, and the symbol $\square$ denotes ‘rounding to the nearest’, ∇ – ‘rounding down’ (or ‘toward $-\infty$’), and Δ – ‘rounding up’ (or ‘toward $+\infty$’).

2.2. Sources of errors during computations

When we are solving a problem in computer we have to be aware that a final result is only an approximation of an exact value. The errors during computations can arise because of three main sources.

First of all, some initial data to a problem can be just experimentally measured quantity that is known only with a limited accuracy. Hence, we use for computations the inexact initial data that can influence the final result very much. Secondly, there are two kinds of errors caused by floating-point arithmetic. Since a set of all floating-point numbers is finite and discrete, some real values (i.e. irrational numbers or real numbers represented by an infinite decimal or binary fraction) cannot be represented exactly in a given floating-point format. If such a real value has to be approximated by a finite fraction, the representation error occurs. On the other hand, even if operands of some arithmetic operator are two. floating-point numbers, the result of operation can be a real value that has to be rounded before it is stored in a given floating-point format. Such errors are called rounding errors and because of complexity of modern computational tasks they are of great importance. Finally, we consider errors of a method applied to solve a problem. If we use a direct method, then after a finite sequence of steps (in absence of rounding errors), we are given an exact solution. On the contrary to an approximate method, that with some necessary simplifications, produces only the approximate solution. An example of approximate methods are finite difference methods for solving an initial-boundary value problems for partial differential equations. An error of these methods is called a truncation error. It is defined as a difference between the partial differential equation and the finite difference approximation to it.

In terms of interval analysis and interval arithmetic we can deal with the errors considered above. Inexact initial data can be enclosed in an appropriate interval which endpoints depend on the measurement uncertainties. For a real number that cannot be represented exactly in a given floating-point format, we can always find an interval that include such number inside. Furthermore, its left and right endpoints are two neighboring machine numbers. Rounding errors are enclosed in a final interval value, if computations are performed in the floating-point interval arithmetic. Finally, for the interval method we assume that the error term of the conventional method (which is normally neglected) is also included in the final interval solution.

2.3. Interval method of Crank-Nicolson type

The interval finite difference method of Crank-Nicolson type (ICN method) concerns the heat conduction equation with the initial-boundary conditions of the following form:

$$\frac{\partial u}{\partial t}(x,t)-\alpha^2\frac{\partial^2 u}{\partial x^2}(x,t)=0,\quad 0<x<L,\ t>0, \tag{15}$$

$$u(x,0)=f(x),\quad 0\le x\le L, \tag{16}$$

$$\frac{\partial u}{\partial x}(0,t)-Au(0,t)=\varphi_1(t),\quad t>0, \tag{17}$$

$$\frac{\partial u}{\partial x}(L,t)+Bu(L,t)=\varphi_2(t),\quad t>0. \tag{18}$$

The ICN method is based on the conventional Crank-Nicolson method (see: Jain (1984), Marciniak et al. (2000)). Its theoretical formulation proposed in Jankowska (2012) assumes that we can enclose values of some partial derivatives of unknown function $u(x, t)$ at midpoints in the appropriate intervals that occur in the error term of the method. With such assumption we can prove that the exact solution of (15) with (16)-(18) at mesh points belongs to the interval solutions obtained. In practice it is impossible to find the endpoints of the intervals considered exactly. We can just approximate them in the best possible way (Jankowska, 2011). The numerical experiments show that such approximation is sufficiently good and the exact solution belongs to the interval solutions obtained.

Let us set the maximum time $T_{\max}$ and choose integers n and m. We find the mesh constants h and k such as $h = L/n$ and $k = T_{\max}/m$. Hence, the grid points are (x_i, t_j), where $x_i = ih$ for $i = 0, 1, \dots, n$ and $t_j = jk$ for $j = 0, 1, \dots, m$.

For the interval method all initial values, i.e. α, A, B, L, $T_{\max}$ should be given in the form of appropriate intervals. Then, for the functions f, φ_1, φ_2, their interval extensions F, Φ_1, Φ_2 are created.

With the interval function F the appropriate interval values of temperature distribution at $t = 0$ are computed. We have:

$$U_{i,0}=F(X_i),\quad i=0,1,\dots,n, \tag{19}$$

where $U_{i,0}=U(x_i,t_0=0)$ and $u(x_i,t_0=0)\in U_{i,0}$.

The ICN method [8]-[9] is given in the matrix form:

$$CU^{(j+1)}=D^{(j)}U^{(j)}+E^{(j)},\quad j=0,1,\dots,m-1, \tag{20}$$

where $U^{(j)} = [U_{0,j}, U_{1,j}, \dots, U_{n,j}]^T$, $E^{(j)} = S^{(j)} + R^{(j)}$ is a vector such as a local truncation error of the conventional Crank-Nicolson method at each mesh point is enclosed in $R^{(j)}$, $R^{(j)} = [R_{0,j}, R_{1,j}, \dots, R_{n,j}]^T$, and C, $D^{(j)}$ and $S^{(j)}$ are matrixes and a vector of coefficients, respectively.

We denote by:

$U_{i,j}$ – an interval value of temperature distribution in (x_i, t_j), $i = 0, 1, \dots, n$, $j = 0, 1, \dots, m$, obtained with the ICN method (20);

$R_{i,j}$ – an interval value which endpoints are appropriate approximations such as a local truncation error $r_{i,j}$ of conventional method is located inside; let us note that we cannot guarantee that:

$$r_{i,j}\in R_{i,j},$$

but numerical experiments confirm that if we consider $R_{i,j}$ in (20), then we have:

$$u(x_i,t_j)\in U_{i,j},$$

$U_{i,j}^C$ – an interval value obtained from the realization of conventional Crank-Nicolson method in interval arithmetic, where all initial data are enclosed in intervals; since the error term of conventional method is neglected, we usually have:

$$u(x_i,t_j)\notin U_{i,j}^C.$$

If we generate $U_{i,j}^C$ in the floating-point interval arithmetic with initial data enclosed in intervals, then their widths provide us information about an influence of errors of inexact input data, rounding errors and representation errors on values of the final result.

3. INITIAL-BOUNDARY VALUE PROBLEM FOR THE HEAT CONDUCTION EQUATION

We consider an infinite plate of thickness L made of the brass (see Fig. 1).

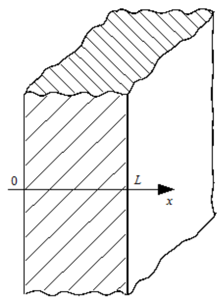

Fig. 1. An infinite plate of thickness L

We assume that it is homogeneous and there is no heat source inside. An initial temperature of the plate is equal to w_0. The external temperatures on the left and right sides of the plate are equal to w_1 and w_2, respectively and they are maintained constant over time.

Under the above assumptions, the distribution of temperature given by a function $w = w(x,t)$ depends on only one spatial variable x. Hence, it is described by the one-dimensional heat conduction equation of the form:

$$\frac{\partial w}{\partial t}(x,t)-\kappa\frac{\partial^2 w}{\partial x^2}(x,t)=0, \tag{21}$$

subject to the following initial and boundary conditions:

$$w(x,0)=w_0, \tag{22}$$

$$\lambda\frac{\partial w}{\partial x}(0,t)=\alpha_1\left(w(0,t)-w_1\right), \tag{23}$$

$$-\lambda\frac{\partial w}{\partial x}(L,t)=\alpha_2\left(w(L,t)-w_2\right), \tag{24}$$

where $\kappa = \lambda/(c\rho)$ [m²/s] is the thermal diffusivity of the material, λ [W/(m·K)] – the thermal conductivity, c [J/(kg·K)] – the specific heat, ρ [kg/m³] – the mass density and α_1, α_2 [W/(m²·K)] – the convection heat transfer coefficients.

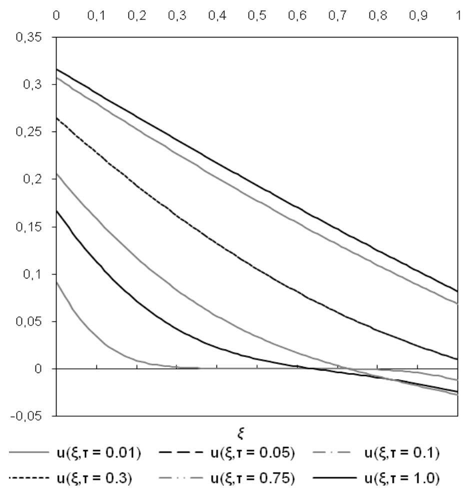

Fig. 2. Temperature distribution for selected values of time τ

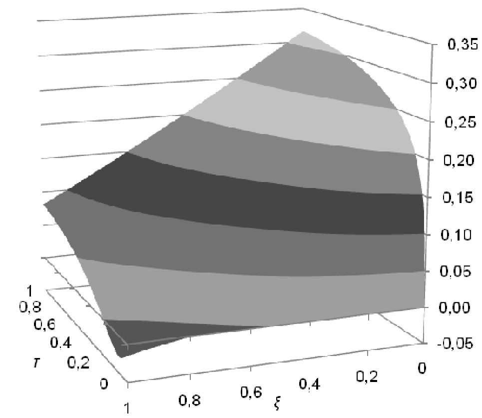

Fig. 3. Temperature distribution for $\tau \in [0,1]$

The initial-boundary problem (21)-(24) can be transformed to the non-dimensional form:

$$\frac{\partial u}{\partial \tau}(\xi,\tau)-\frac{\partial^2 u}{\partial \xi^2}(\xi,\tau)=0, \tag{25}$$

subject to the initial and boundary conditions:

$$u(\xi,0)=0, \tag{26}$$

$$\frac{\partial u}{\partial \xi}(0,\tau)-\mathrm{Bi}_1\, u(0,\tau)=g_1, \tag{27}$$

$$\frac{\partial u}{\partial \xi}(1,\tau)+\mathrm{Bi}_2\, u(1,\tau)=g_2, \tag{28}$$

where $\xi = x/L$, $\tau = \kappa t/L^2$ and $w(x, t) = w_0\, u(\xi, \tau) + w_0$. The Biot numbers are denoted and defined by $\mathrm{Bi}_1 = (\alpha_1 L)/\lambda$, $\mathrm{Bi}_2 = (\alpha_2 L)/\lambda$. Furthermore, we introduce the notation $g_1 = -(w_1/w_0 - 1)\ \mathrm{Bi}_1$ and $g_2 = (w_2/w_0 - 1)\ \mathrm{Bi}_2$.

The analytical solution of (25) with (26)-(28) can be derived and is given in the following form:

$$u(\xi,\tau)=f_0(\xi)+2\sum_{n=1}^{\infty}\frac{f_1(\mu_n,\xi)}{f_2(\mu_n)}\exp\left(-\mu_n^2\tau\right), \tag{29}$$

where:

$$f_0(\xi)=\frac{\mathrm{Bi}_2(w_2-w_0)+\mathrm{Bi}_1\left((\mathrm{Bi}_2+1)(w_1-w_0)+\mathrm{Bi}_2(w_2-w_1)\xi\right)}{(\mathrm{Bi}_1+\mathrm{Bi}_2+\mathrm{Bi}_1\mathrm{Bi}_2)w_0}, \tag{30}$$

$$\begin{aligned}f_1(\mu_n,\xi)=&\ \mathrm{Bi}_2(w_0-w_2)\mu_n\cos(\mu_n\xi)\\&+\mathrm{Bi}_1\left((w_0-w_1)\mu_n\cos(\mu_n-\mu_n\xi)+\right.\\&+\mathrm{Bi}_2\left.\left((w_0-w_2)\sin(\mu_n\xi)+(w_0-w_1)\sin(\mu_n-\mu_n\xi)\right)\right),\end{aligned} \tag{31}$$

$$\begin{aligned}f_2(\mu_n)=&\ w_0\mu_n\left(-\left(\mathrm{Bi}_1+\mathrm{Bi}_2+\mathrm{Bi}_1\mathrm{Bi}_2-\mu_n^2\right)\cos(\mu_n)\right.\\&+\left.(\mathrm{Bi}_1+\mathrm{Bi}_2+2)\mu_n\sin(\mu_n)\right).\end{aligned} \tag{32}$$

Note that μ_n, n = 1, 2, ... in (29), (31)-(32) are positive roots of the equation:

$$-\arctan\left(\frac{-\mathrm{Bi}_1\mathrm{Bi}_2+\mu_n^2}{(\mathrm{Bi}_1+\mathrm{Bi}_2)\mu_n}\right)+\frac{\pi}{2}=\mu_n. \tag{33}$$

The temperature distribution in the plate given by (29) with (30)-(32) for selected values of time τ is presented in Fig. 2. Similarly, in Fig. 3 we can see $u = u(\xi, \tau)$, $0 \le \xi \le 1$, $0 \le \tau \le 1$.

4. NUMERICAL RESULTS

We take the infinite plate of the thickness L = 0.05 [m] and the brass-specific quantities

λ = 110.6 [W/(m·K)], c = 377 [J/(kg·K)], ρ = 8520 [kg/m³]. (34)

The initial values of external temperatures and convection heat transfer coefficients are given by exact values or with some tolerance of accuracy depending on the experiment considered.

4.1. Numerical experiment 1

We set

w_0 = 280 [K], w_1 = 400 [K], w_2 = 250 [K], (35)

α_1 = 5000 [W/(m²·K)], α_2 = 2500 [W/(m²·K)]. (36)

Hence, the dimensionless quantities g_1, g_2 and the Biot numbers are represented by the intervals given as follows

$g_1 \in$ [−9.687419271506070E−1,−9.687419271506070E−1],
$g_2 \in$ [−1.210927408938258E−1,−1.210927408938258E−1],
$d(g_1) \approx$ 5.42E−19, $d(g_2) \approx$ 1.01E−19,

$\mathrm{Bi}_1 \in$ [+2.260397830018083E+0,+2.260397830018083E+0],
$\mathrm{Bi}_2 \in$ [+1.130198915009041E+0,+1.130198915009041E+0],
$d(\mathrm{Bi}_1) \approx$ 6.51E−19, $d(\mathrm{Bi}_2) \approx$ 3.25E−19.

We use the interval realization of the conventional Crank-Nicolson method and the ICN method with h = 1E–2, $k \approx$ 3.91E–5.

The widths of interval solutions $U_{i,j}^{C}$ and $U_{i,j}$ at the mesh points (ξ_i, τ_j), $i = 0, 1, \ldots, n$, $j = 1, 2, \ldots, m$ are given in Fig. 4 and Fig. 5. Furthermore, in Fig. 6 the widths of the intervals $R_{i,j}$ of the error term are presented.

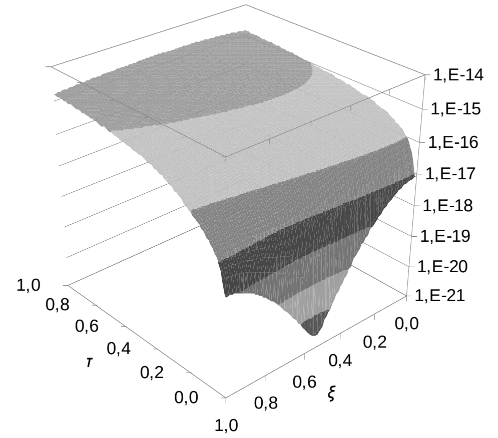

Fig. 4. Widths of the interval solutions $U_{i,j}^{C}$.

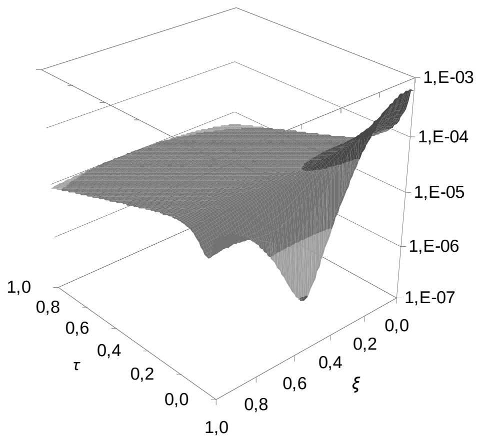

Fig. 5. Widths of the interval solutions $U_{i,j}$.

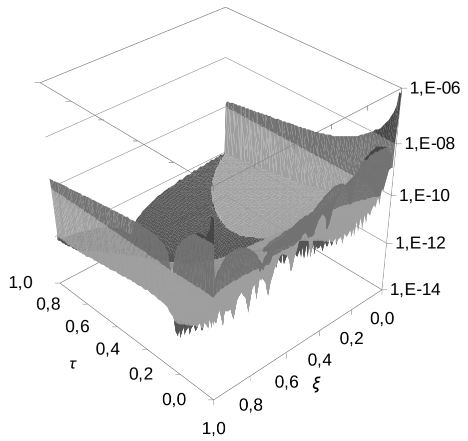

Fig. 6. Widths of the intervals $R_{i,j}$ of the error term of the conventional Crank-Nicolson method

In Tab. 1 and Tab. 2 we can also compare the exact solution $u(\xi, \tau)$ with the interval solutions $U^C(\xi, \tau)$ and $U(\xi, \tau)$ computed for $\tau = 1$ and selected values of ξ. Since in the interval solution $U^C(\xi, \tau)$ the local truncation error of the conventional method is neglected, then for all mesh points the exact solution is not included in the interval solution obtained. The difference between the exact solution and the left/right endpoint of the interval solution $U^C(\xi, \tau)$ is presented in Fig. 7.

Tab. 1. Values of the exact and interval solutions $U^C(\xi, \tau = 1)$ obtained with the interval realization of the conventional Crank-Nicolson method for h = 1E–2 and $k \approx$ 3.91E–5

ξ	$u(\xi, \tau = 1)$	$U^C(\xi, \tau = 1)$	width
0.0	3.159588E-1	[+3.15958712265990798E-1, +3.15958712265992277E-1]	1.477822E-15
0.1	2.906484E-1	[+2.90648170785494571E-1, +2.90648170785496338E-1]	1.766003E-15
0.2	2.656639E-1	[+2.65663537071720546E-1, +2.65663537071722517E-1]	1.970429E-15
0.3	2.410542E-1	[+2.41053828438518319E-1, +2.41053828438520416E-1]	2.096332E-15
0.4	2.168592E-1	[+2.16858708970163917E-1, +2.16858708970166073E-1]	2.155679E-15
0.5	1.931080E-1	[+1.93107500126792865E-1, +1.93107500126795024E-1]	2.157928E-15
0.6	1.698190E-1	[+1.69818449401430307E-1, +1.69818449401432420E-1]	2.112365E-15
0.7	1.469988E-1	[+1.46998275255881695E-1, +1.46998275255883720E-1]	2.024382E-15
0.8	1.246425E-1	[+1.24641999666420528E-1, +1.24641999666422430E-1]	1.901250E-15
0.9	1.027335E-1	[+1.02733072434454703E-1, +1.02733072434456455E-1]	1.751014E-15
1.0	8.124411E-2	[+8.12437841413371667E-2, +8.12437841413387475E-2]	1.580699E-15

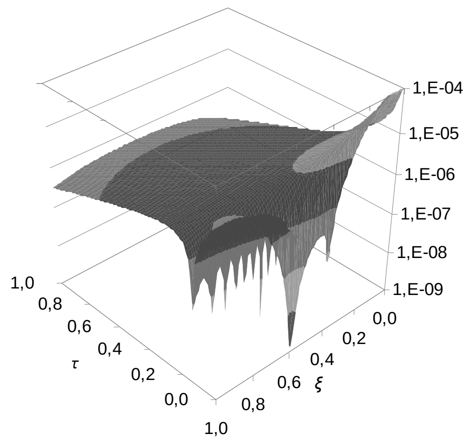

Fig. 7. The difference between the exact solution and the left/right endpoint of the interval solution $U^C(\xi, \tau)$

We can also examine the efficiency of the ICN method applied for solving the problem (25)-(28) with (34)-(36), if we compare widths of the interval solutions obtained for different values of stepsizes h and k. Such comparison is shown in Fig. 8 and Fig. 9. A decrease of stepsizes h and k contributes to the improvement of the interval solutions, i.e. we get the interval solutions of smaller widths. As numerical tests show, a good practice is to adjust the stepsize k in accordance to the stepsize h, such as $k \leq h^2/2$.

Tab. 2. Values of the exact and interval solutions $U(\xi, \tau = 1)$ obtained with the ICN method for h = 1E-2 and $k \approx$ 3.91E-5

ξ	$u(\xi, \tau = 1)$	$U(\xi, \tau = 1)$	width
0.0	3.159588E-1	[+3.15954436738475349E-1, +3.15960387416827118E-1]	5.950678E-06
0.1	2.906484E-1	[+2.90643083764027751E-1, +2.90650297636329433E-1]	7.213872E-06
0.2	2.656639E-1	[+2.65657755235743023E-1, +2.65666052507040455E-1]	8,297271E-06
0.3	2.410542E-1	[+2.41047484227741547E-1, +2.41056658209754539E-1]	9,173982E-06
0.4	2.168592E-1	[+2.16851947574807204E-1, +2.16861769802251397E-1]	9,822227E-06
0.5	1.931080E-1	[+1.93100476191670666E-1, +1.93110702112374094E-1]	1,022592E-05
0.6	1.698190E-1	[+1.69811323517989626E-1, +1.69821698582011309E-1]	1,037506E-05
0.7	1.469988E-1	[+1.46991210327409063E-1, +1.47001476310586912E-1]	1,026598E-05
0.8	1.246425E-1	[+1.24635157247229580E-1, +1.24645058613463779E-1]	9,901366E-06
0.9	1.027335E-1	[+1.02726609076264561E-1, +1.02735899317649549E-1]	9,290241E-06
1.0	8.124411E-2	[+8.12378478407702735E-2, +8.12462956141347860E-2]	8,447773E-06

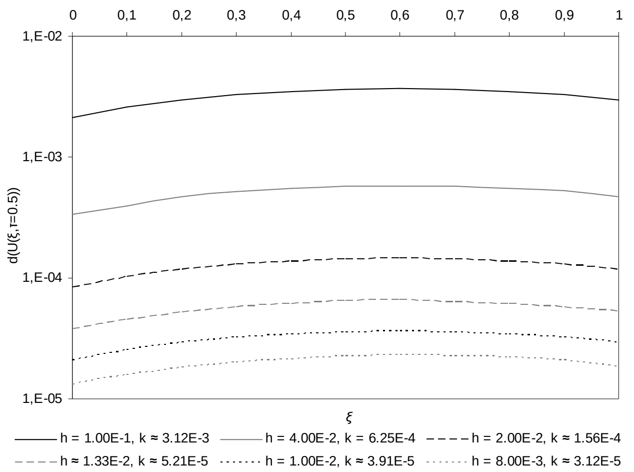

Fig. 8. Widths of the interval solution $U(\xi, \tau = 0.5)$ obtained with the ICN method for different values of stepsizes h and k

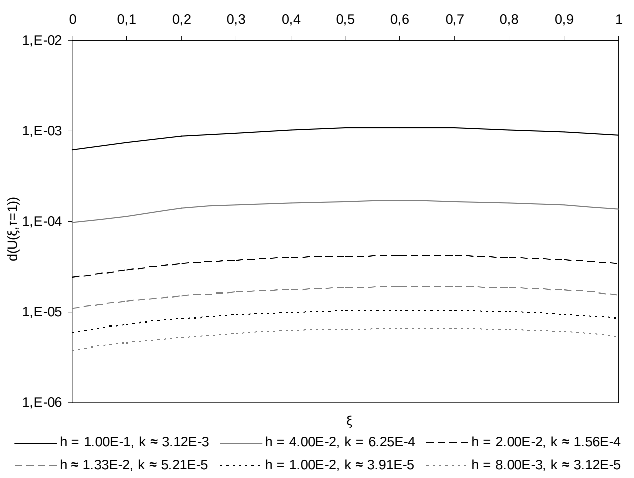

Fig. 9. Widths of the interval solution $U(\xi, \tau = 1)$ obtained with the ICN method for different values of stepsizes h and k

4.2. Numerical experiment 2

We set:

$$w_0 = 280 \text{ [K]},\ w_1 = 400 \pm 2 \text{ [K]},\ w_2 = 250 \pm 2 \text{ [K]}, \tag{37}$$

$$\alpha_1 = 5000 \text{ [W/(m}^2\cdot\text{K)]},\ \alpha_2 = 2500 \text{ [W/(m}^2\cdot\text{K)]}. \tag{38}$$

Hence, we take:

$w_1 \in [398.0, 402.0]$, $d(w_1) = 4.0$,
$w_2 \in [248.0, 252.0]$, $d(w_2) = 4.0$,

The dimensionless quantities g_1, g_2 and the Biot numbers are represented by the intervals given as follows:

$g_1 \in [-9.848876259364505E-1, -9.525962283647636E-1]$,
$g_2 \in [-1.291655902867476E-1, -1.130198915009041E-1]$,
$d(g_1) \approx 3.22E-02$, $d(g_2) \approx 1.61E-02$,

$Bi_1 \in [+2.2603978300180831E+0, +2.2603978300180831E+0]$,
$Bi_2 \in [+1.1301989150090415E+0, +1.1301989150090416E+0]$,
$d(Bi_1) \approx 6.51E-19$, $d(Bi_2) \approx 3.25E-19$.

We use the interval realization of the conventional Crank-Nicolson method and the ICN method with h = 1E–2, $k \approx$ 3.91E–5. Fig. 10, Fig. 11 and Tab.3 present the results of computations.

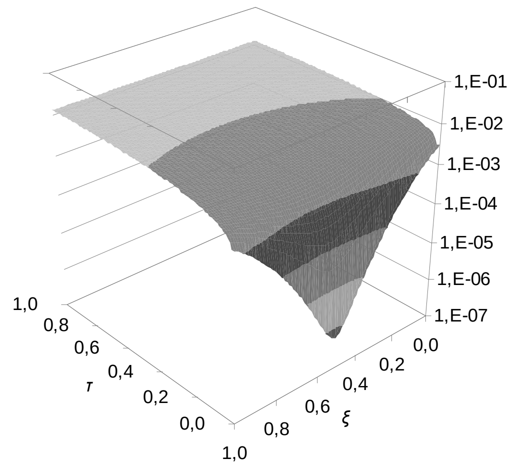

Fig. 10. Widths of the interval solutions $U_{i,j}^{C}$

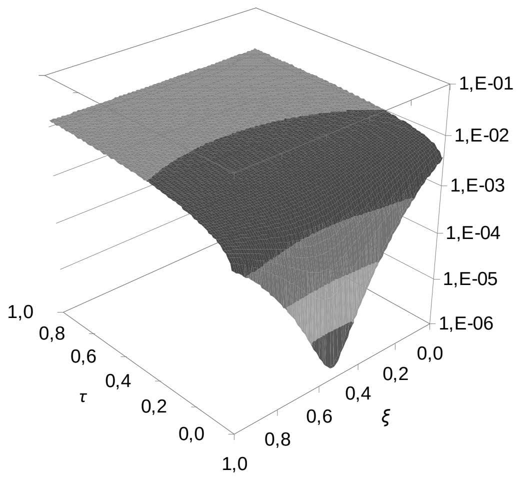

Fig. 11. Widths of the interval solutions $U_{i,j}$

Tab. 3. Values of the exact and interval solutions $U\ (\xi, \tau = 1)$ obtained with the ICN method for h = 1E-2 and $k \approx$ 3.91E-5

ξ	$u(\xi, \tau = 1)$	$U(\xi, \tau = 1)$	width
0.0	3.159588E-1	[+3.09184783437008590E-1, +3.22730040718293928E-1]	1.354526E-02
0.1	2.906484E-1	[+2.83952786874633136E-1, +2.97340594525724109E-1]	1.338781E-02
0.2	2.656639E-1	[+2.59035525184234897E-1, +2.72288282558548651E-1]	1.325276E-02
0.3	2.410542E-1	[+2.34480333446458798E-1, +2.47623808991037363E-1]	1.314348E-02
0.4	2.168592E-1	[+2.10325514492529920E-1, +2.23388202884528759E-1]	1.306269E-02
0.5	1.931080E-1	[+1.86599383498977412E-1, +1.99611794805067427E-1]	1.301241E-02
0.6	1.698190E-1	[+1.63319561770483399E-1, +1.76313460329517616E-1]	1.299390E-02
0.7	1.469988E-1	[+1.40492537317271714E-1, +1.53500149320724339E-1]	1.300761E-02
0.8	1.246425E-1	[+1.18113503180345723E-1, +1.31166712680347713E-1]	1.305321E-02
0.9	1.027335E-1	[+9.61664774474748486E-2, +1.09296030946439333E-1]	1.312955E-02
1.0	8.124411E-2	[+7.46247020072574241E-2, +8.78594414476476994E-2]	1.323474E-02

4.3. Numerical experiment 3

We set:

$$w_0 = 280\ [\mathrm{K}],\ w_1 = 400\ [\mathrm{K}],\ w_2 = 250\ [\mathrm{K}], \tag{39}$$

$$\alpha_1 = 5000 \pm 250\ [\mathrm{W/(m^2 \cdot K)}],\ \alpha_2 = 2500 \pm 250\ [\mathrm{W/(m^2 \cdot K)}], \tag{40}$$

and hence, we take:

$\alpha_1 \in [4750.0, 5250.0]$, $d(\alpha_1) = 500.0$,
$\alpha_2 \in [2250.0, 2750.0]$, $d(\alpha_2) = 500.0$,

The dimensionless quantities g_1, g_2 and the Biot numbers are represented by the intervals given as follows:

$g_1 \in [-1.017179023508137\mathrm{E}{+}0, -9.203048307930767\mathrm{E}{-}1]$,
$g_2 \in [-1.332020149832084\mathrm{E}{-}1, -1.089834668044432\mathrm{E}{-}1]$,
$d(g_1) \approx 9.68\mathrm{E}{-}02$, $d(g_2) \approx 2.42\mathrm{E}{-}02$,

$Bi_1 \in [+2.1473779385171792\mathrm{E}{+}0, +2.373417721518987\mathrm{E}{+}0]$,
$Bi_2 \in [+1.0171790235081374\mathrm{E}{+}0, +1.2432188065099457\mathrm{E}{+}0]$,
$d(Bi_1) \approx 2.26\mathrm{E}{-}1$, $d(Bi_2) \approx 2.26\mathrm{E}{-}1$.

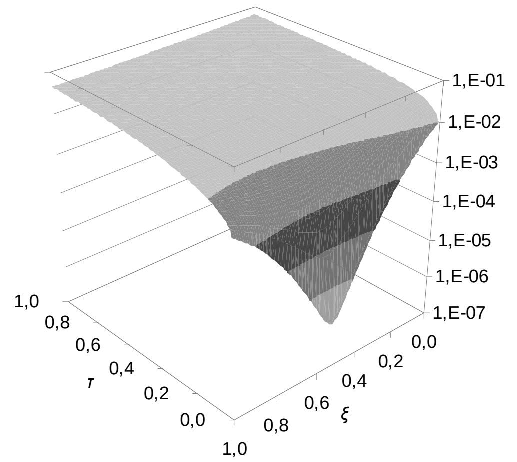

Fig. 12. Widths of the interval solutions $U^C_{i,j}$

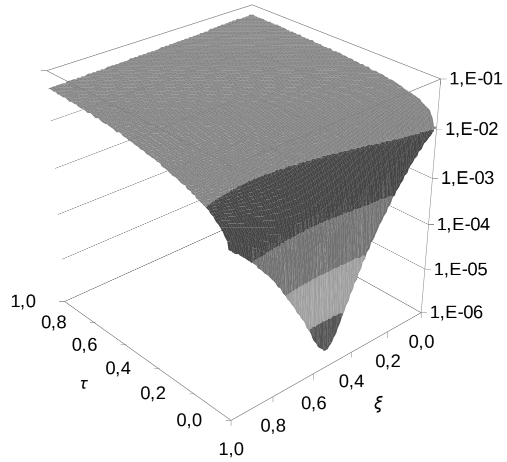

Fig. 13. Widths of the interval solutions $U_{i,j}$

Tab. 4. Values of the exact and interval solutions $U(\xi, \tau = 1)$ obtained with the ICN method for h = 1E-2 and $k \approx$ 3.91E-5

ξ	$u(\xi, \tau = 1)$	$U(\xi, \tau = 1)$	width
0.0	3.159588E-1	[+2.85981889581640435E-1, +3.48735827509592245E-1]	6.275394E-02
0.1	2.906484E-1	[+2.61941715297462713E-1, +3.22087348519591537E-1]	6.014563E-02
0.2	2.656639E-1	[+2.38161863649265124E-1, +2.95848727684306683E-1]	5.768686E-02
0.3	2.410542E-1	[+2.14682974454601979E-1, +2.70079238878866673E-1]	5.539626E-02
0.4	2.168592E-1	[+1.91537766753450941E-1, +2.44827119587005676E-1]	5.328935E-02
0.5	1.931080E-1	[+1.68750177753681014E-1, +2.20128436301078798E-1]	5.137826E-02
0.6	1.698190E-1	[+1.46334732916635882E-1, +1.96006236264017843E-1]	4.967150E-02
0.7	1.469988E-1	[+1.24296163758492568E-1, +1.72470005509110585E-1]	4.817384E-02
0.8	1.246425E-1	[+1.02629283427895288E-1, +1.49515445902885899E-1]	4.688616E-02
0.9	1.027335E-1	[+8.13191232548695133E-2, +1.27124576470833617E-1]	4.580545E-02
1.0	8.124411E-2	[+6.03413267279543483E-2, +1.05266156532657562E-1]	4.492483E-02

Similarly, as in the previous numerical experiments, we use the interval realization of the conventional Crank-Nicolson method and the ICN method with h = 1E−2 and $k \approx$ 3.91E−5. The results of computations are presented in Fig. 12, Fig. 13 and Tab. 4.

4.4. Discussion of results

In Section 4.1 we assumed that all input data, i.e. the material specific quantities, the temperatures and the convection heat transfer coefficients are known exactly. Hence, we only had to deal with the representation errors, rounding errors and the error of the conventional method. In Fig. 4 we see the widths of the interval solutions $U^C(\xi_i, \tau_j)$. The influence of rounding errors and the representation errors on the final results is not significant. It could be significant if we continue computations for greater values of time τ. On the other hand more important is in this case the contribution of the error term of the conventional method enclosed in $R_{i,j}$ (see Fig. 6) to the final intervals $U(\xi_i, \tau_j)$ obtained with the ICN method (see Fig. 5).

In Section 4.2 and 4.3 we introduced some kind of uncertainty in values of the external temperatures and the convection heat transfer coefficients, respectively. Fig. 10 and Fig. 12 show that the influence of inexactness of the initial values that are enclosed in intervals is probably greater (see also Fig. 4) then the rounding errors (for a given time τ). Furthermore, if we compare Fig. 10 with Fig. 11 (and Fig. 12 with Fig. 13), we conclude that the influence of the errors of the initial data together with rounding errors is greater than the error of the method.

In all example experiments the approximation of the endpoints of intervals $R_{i,j}$ that should include the error of the conventional method is good enough. Hence, as we see in Tab. 2, Tab. 3, Tab. 4, the exact solution belongs to the interval solutions $U(\xi_i, \tau_j)$ obtained with the ICN method.

The accuracy of the interval results in the first example experiment (see Tab. 2) is of order 1E–6. Its further improvement is possible if we set smaller stepsizes h and k. The numerical tests carried out in Section 4.2 and 4.3 indicate that due to the inexactness of some input quantities to the heat conduction problem, the accuracy of the interval solution is of order 1E–2.

5. CONCLUSIONS

The numerical experiments presented in Section 4 are conceived to show some advantages of the interval method of Crank-Nicolson type for solving the initial-boundary value problem for the heat conduction equation. Such interval method gives interval solutions that with a good approximation of the error term of the conventional Crank-Nicolson method include the exact solution of the problem. Furthermore, the implementation of the interval method considered in the floating-point interval arithmetic leads to the interval values that also contain errors of the inexact initial data, errors of representation and rounding errors.

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