EXACT SOLUTION OF A DIELECTRIC CRACK OF MODE III IN MAGNETO-ELECTRO-ELASTIC HALF-SPACE

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Abstract: This paper investigated the fracture behaviour of a piezo-electro-magneto-elastic material subjected to electro-magneto-mechanical loads. The PEMO-elastic medium contains a straight-line crack which is parallel to its poling direction and loaded surface of the half-space. Fourier transform technique is used to reduce the problem to the solution of one Fredholm integral equation. This equation is solved exactly. The semi-permeable crack-face magneto-electric boundary conditions are utilized. Field intensity factors of stress, electric displacement, magnetic induction, crack displacement, electric and magnetic potentials, and the energy release rate are determined. The electric displacement and magnetic induction of crack interior are discussed. Strong coupling between stress and electric and magnetic field near the crack tips has been found.

1. INTRODUCTION

Due to the growth in applications as smart devices, the mechanical and fracture properties of two-phase magnetostrictive/piezoelectric composites are becoming more and more important, see: Huang and Kuo (1997), Pan (2001), Buchanan (2004), Chen et al. (2005), Annigeri et al. (2006), Lee and Ma (2007), Calas et al. (2008) and Hou et al. (2009), among other, have been published the papers on this field. The fracture mechanics of PEMO-elastic materials also have attracted much attention and many research papers have been published; see e.g. Liu et al. (2001), Sih and Song (2003), Gao et al. (2003), Zhou et al. (2004), Hu and Li (2005), Wang and Mai (2006b), Li and Kardomateas (2007), Feng et al. (2007), Tian and Rajapakse (2008), Zhan and Fan (2008); among others.

For the fracture analysis of a magneto-electro-elastic solid of much interest are the effects of magneto-electric boundary conditions at the crack surfaces on the crack growth as well as the choosing of fracture criteria (Wang and Mai, 2006b, Wang et al., 2006a). As an approximation to a real crack, the magneto-electrically permeable and impermeable crack face boundary conditions are prevail in the above stated-works. However these two ideal crack models are only the limiting cases of real dielectric crack (Wang and Mai; 2006b; Rogowski, 2007). However, the above-mentioned works associated with semi-permeable crack problems are only limited to an infinite magneto-electro-elastic solid with cracks. Additionally, the numerical procedures are used to obtain the results. Motivated by this consideration this paper investigates a PEMO-elastic half-space with an electrically and magnetically conducting crack under anti-plane mechanical and in-plane electro-magnetic loadings to shown exact solution in simple analytical form.

2. BASIC EQUATIONS

For a linearly magneto-electro-elastic medium under anti-plane shear coupled with in-plane electric and magnetic fields there are only the non-trivial anti-plane displacement w:

$$u_x = 0\,,\; u_y = 0\,,\; u_z = w(x,y) \tag{1}$$

strain components γ_{xz} and γ_{yz}:

$$\gamma_{xz} = \frac{\partial w}{\partial x}\,,\; \gamma_{yz} = \frac{\partial w}{\partial y} \tag{2}$$

stress components τ_{xz} and τ_{yz}, in-plane electrical and magnetic potentials ϕ and ψ, which define electrical and magnetic field components E_x, E_y, H_x and H_y:

$$E_x = -\frac{\partial \phi}{\partial x}\,,\; E_y = -\frac{\partial \phi}{\partial y}\,,\; H_x = -\frac{\partial \psi}{\partial x}\,,\; H_y = -\frac{\partial \psi}{\partial y} \tag{3}$$

and electrical displacement components D_x, D_y and magnetic induction components B_x, B_y with all field quantities being the functions of coordinates x and y.

The generalized strain-displacement relations (2) and (3) have the form:

$$\gamma_{\alpha z} = w_{,\alpha}\,,\; E_\alpha = -\phi_{,\alpha}\,,\; H_\alpha = -\psi_{,\alpha} \tag{4}$$

where $\alpha = x, y$ and $w_{,\alpha} = \partial w / \partial \alpha$.

For linearly magneto-electro-elastic medium the coupled constitutive relations can be written in the matrix form

$$[\tau_{\alpha z}, D_\alpha, B_\alpha]^T = \mathbf{C}[\gamma_{\alpha z}, -E_\alpha, -H_\alpha]^T \tag{5}$$

where the superscript T denotes the transpose of a matrix and:

$$\mathbf{C}=\begin{bmatrix} c_{44} & e_{15} & q_{15} \\ e_{15} & -\varepsilon_{11} & -d_{11} \\ q_{15} & -d_{11} & -\mu_{11} \end{bmatrix} \tag{6}$$

where c_{44} is the shear modulus along the z-direction, which is direction of poling and is perpendicular to the isotropic plane (x, y), ε_{11} and μ_{11} are dielectric permittivity and magnetic permeability coefficients, ε_{15}, q_{15} and d_{11} are piezoelectric, piezo-magnetic and magneto – electric coefficients, respectively.

The mechanical equilibrium equation (called as Euler equation), the charge and current conservation equations (called as Maxwell equations), in the absence of the body force electric and magnetic charge densities, can be written as:

$$\tau_{z\alpha,\alpha}=0\,;\quad D_{\alpha,\alpha}=0\,;\quad B_{\alpha,\alpha}=0\,;\quad \alpha=x,y \tag{7}$$

Subsequently, the Euler and Maxwell equations take the form:

$$\mathbf{C}\left[\nabla^2 w,\nabla^2\phi,\nabla^2\psi\right]^T=\left[0,0,0\right]^T \tag{8}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\partial^2/\partial y^2$ is the two-dimensional Laplace operator.

Since $|\mathbf{C}|\neq 0$ one can decouple the equations (8):

$$\nabla^2 w=0\,;\quad \nabla^2\phi=0\,;\quad \nabla^2\psi=0 \tag{9}$$

If we introduce, for convenience of mathematics in some boundary value problems, two unknown functions:

$$\left[\chi-e_{15}w,\eta-q_{15}w\right]^T=\mathbf{C}_0\left[\phi,\psi\right]^T \tag{10}$$

where:

$$\mathbf{C}_0=\begin{bmatrix} -\varepsilon_{11} & -d_{11} \\ -d_{11} & -\mu_{11} \end{bmatrix} \tag{11}$$

Then:

$$\left[\phi,\psi\right]^T=\mathbf{C}_0^{-1}\left[\chi-e_{15}w,\eta-q_{15}w\right]^T \tag{12}$$

where:

$$\mathbf{C}_0^{-1}=\frac{1}{\varepsilon_{11}\mu_{11}-d_{11}^2}\begin{bmatrix} -\mu_{11} & d_{11} \\ d_{11} & -\varepsilon_{11} \end{bmatrix}=\begin{bmatrix} e_1 & e_2 \\ e_2 & e_3 \end{bmatrix} \tag{13}$$

The governing field variables are:

$$\begin{gathered} \tau_{zk}=\tilde{c}_{44}w_{,k}-\alpha D_k-\beta B_k \\ \phi=\alpha w+e_1\chi+e_2\eta \\ \psi=\beta w+e_2\chi+e_3\eta \\ D_k=\chi_{,k} \\ B_k=\eta_{,k}\,;\quad k=x,y \end{gathered} \tag{14}$$

$$\nabla^2 w=0\,;\quad \nabla^2\chi=0\,;\quad \nabla^2\eta=0 \tag{15}$$

where:

$$\begin{gathered} \tilde{c}_{44}=c_{44}+\alpha e_{15}+\beta q_{15} \\ \alpha=\frac{\mu_{11}e_{15}-d_{11}q_{15}}{\varepsilon_{11}\mu_{11}-d_{11}^2}=-\left(e_1e_{15}+e_2q_{15}\right) \\ \beta=\frac{\varepsilon_{11}q_{15}-d_{11}e_{15}}{\varepsilon_{11}\mu_{11}-d_{11}^2}=-\left(e_3q_{15}+e_2e_{15}\right) \end{gathered} \tag{16}$$

Note that $\tilde{c}_{44}$ is the piezo-electro-magnetically stiffened elastic constant.

Note also that:

$$\mathbf{C}^{-1}=\frac{1}{\tilde{c}_{44}}\begin{bmatrix} 1 & \alpha & \beta \\ \alpha & \alpha^2+\tilde{c}_{44}e_1 & \alpha\beta+\tilde{c}_{44}e_2 \\ \beta & \alpha\beta+\tilde{c}_{44}e_2 & \beta^2+\tilde{c}_{44}e_3 \end{bmatrix} \tag{17}$$

These material parameters will appear in our solutions.

3. FORMULATION OF THE CRACK PROBLEM

Consider a PEMO-elastic half-space containing straight-line crack of length $2a$, parallel to the surface of a half-space which is subjected to electric, magnetic and mechanical loads. The crack is located along the x-axis from $-a$ to a at a depth h from the loaded surface with a rectangular coordinate system, as shown in Fig. 1. The PEMO-elastic half-space is poled in the z-direction.

To solve the crack problem in linear elastic solids, the superposition technique is usually used. Thus we first solve the magneto-electro-elastic field problem without the cracks in the medium under electric, magnetic and mechanical loads. This elementary solution is:

$$\begin{gathered} \tau_{yz}=\tau_0 \\ D_y=D=\begin{cases} D_0, & case\ I \\ \frac{e_{15}}{c_{44}}\tau_0+\left(\varepsilon_{11}+\frac{e_{15}^2}{c_{44}}\right)E_0+\left(d_{11}+\frac{e_{15}q_{15}}{c_{44}}\right)H_0, & case\ II \end{cases} \\ B_y=B=\begin{cases} B_0, & case\ I \\ \frac{q_{15}}{c_{44}}\tau_0+\left(d_{11}+\frac{e_{15}q_{15}}{c_{44}}\right)E_0+\left(\mu_{11}+\frac{q_{15}^2}{c_{44}}\right)H_0, & case\ II \end{cases} \end{gathered} \tag{18}$$

Then, we use equal and opposite values as the crack surface traction and utilize the unknowns d_0 and b_0 in the crack region. Thus, in this study, $-\tau_0$; $-(D-d_0)$, $-(B-b_0)$ are, respectively, mechanical, electrical and magnetic loadings applied on the crack surfaces (the so called perturbation problem).

The boundary conditions can be written as:

$$\begin{gathered} \tau_{zy}(x,h\pm)=-\tau_0\,,\ D_y(x,h\pm)=-D+d_0\,, \\ B_y(x,h\pm)=-B+b_0\,;\quad |x|<a \end{gathered} \tag{19}$$

$$\left\|\tau_{zy}\right\|=0\,,\ \left\|D_y\right\|=0\,,\ \left\|B_y\right\|=0\,,\ |x|<\infty\,,\ y=h \tag{20}$$

$$\left\|w\right\|=0\,,\ \left\|\phi\right\|=0\,,\ \left\|\psi\right\|=0\,,\ |x|\geq a,\ y=h \tag{21}$$

$$\tau_{zy}(x,0)=0\,,\ D_y(x,0)=0\,,\ B_y(x,0)=0\,;\ |x|<\infty \tag{22}$$

where the notation $[|f|] = f^+ - f^-$ and f^+ denotes the values for $h+$ while f^- for $h-$.

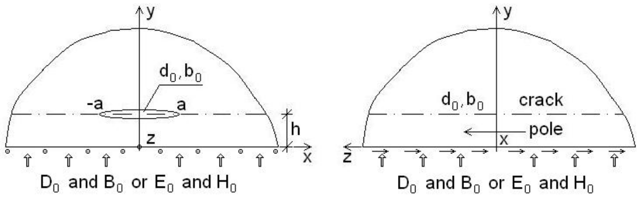

Fig. 1. The PEMO-elastic half-space with a crack parallel to its surface under an anti-plane mechanical and in-plane electric and magnetic loads. Inside the crack the unknown electromagnetic field appears (d_0 and b_0 are unknown to be determined)

Of course, in perturbation problem the surface of the half-space is free. The electric displacement d_0 and magnetic induction b_0 inside the crack are obtained from semi-permeable crack face boundary conditions (Rogowski (2007)). For two different magneto-electric media: PEMO-material and notch space we have continuity condition of electric and magnetic potential in both materials at interface. The semi-permeable crack-face magneto-electric boundary conditions are expressed as follows:

$$d_0 = -\varepsilon_c \frac{[|\phi|]}{2\delta(x)}, \quad b_0 = -\mu_c \frac{[|\psi|]}{2\delta(x)} \tag{23}$$

where $\delta(x)$ describes the shape of the notch and ε_c, μ_c are the dielectric permittivity and magnetic permeability of crack interior. If we assume the elliptic notch profile such that:

$$\delta(x) = (\delta_0/a)\sqrt{a^2 - x^2} \tag{24}$$

where δ_0 is the half-thickness of the notch at $x = 0$, we obtain:

$$\begin{aligned} 2d_0(\delta_0/\varepsilon_c)\sqrt{a^2 - x^2} &= -[|\phi|] \\ 2b_0(\delta_0/\mu_c)\sqrt{a^2 - x^2} &= -[|\psi|] \end{aligned} \tag{25}$$

Eqs (25) form two coupling linear equations with respect to d_0 and b_0 since $[|\phi|]$ and $[|\psi|]$ depends linearly on these quantities as show boundary conditions (19) and (21).

4. THE SOLUTION FOR HALF-SPACE WITH DISCONTINUITY AT y = h

Define the Fourier transform pair by equations:

$$\hat{f}(s) = \int_0^\infty f(x)\cos(sx)dx, \quad f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}(s)\cos(sx)ds \tag{26}$$

Considering the symmetry about y-axis the Fourier cosine transform is only applied in Eqs (15) resulting in ordinary differential equations and their solutions:

$$\begin{aligned} &\hat{w}(s,y) = A_1(s)e^{-sy} \\ &\hat{\chi}(s,y) = B_1(s)e^{-sy} \qquad y > h \\ &\hat{\eta}(s,y) = C_1(s)e^{-sy} \end{aligned} \tag{26a}$$

$$\begin{aligned} &\hat{w}(s,y) = A_2(s)e^{-sy} + A_3(s)e^{sy} \\ &\hat{\chi}(s,y) = B_2(s)e^{-sy} + B_3(s)e^{sy} \qquad 0 \le y < h \\ &\hat{\eta}(s,y) = C_2(s)e^{-sy} + C_3(s)e^{sy} \end{aligned} \tag{26b}$$

In the domain $y > h$ the solution has the form (26a) to ensure the regularity conditions at infinity.

The transforms of Eqs (14) yield:

$$\begin{aligned} &\hat{\varphi}(s,y) = e_1\hat{\chi}(s,y) + e_2\hat{\eta}(s,y) + \alpha\hat{w}(s,y) \\ &\hat{\psi}(s,y) = e_2\hat{\chi}(s,y) + e_3\hat{\eta}(s,y) + \beta\hat{w}(s,y) \\ &\hat{\tau}_{zy}(s,y) = \tilde{c}_{44}\hat{w}_{,y} - \alpha\hat{D}_y - \beta\hat{B}_y \\ &\hat{D}_y = \hat{\chi}_{,y}, \quad \hat{B}_y = \hat{\eta}_{,y} \end{aligned} \tag{27}$$

The unknown functions $A_i(s), B_i(s)$ and $C_i(s)$, $i = 1, 2, 3$, are obtained from the boundary conditions (20) and (22), which in transform domain are:

$$\begin{aligned} &[|\hat{\tau}_{zy}|] = 0; \quad [|\hat{D}_y|] = 0; \quad [|\hat{B}_y|] = 0 \\ &\hat{\tau}_{zy} = 0; \quad \hat{D}_y = 0; \quad \hat{B}_y = 0; \quad y = 0 \end{aligned} \tag{28}$$

where $[|\hat{f}|] = \hat{f}(s, h+) - \hat{f}(s, h-)$.

The result is:

$$\begin{aligned} &A_1(s) = \hat{f}(s)\left(e^{-sh} - e^{sh}\right) \\ &B_1(s) = \hat{g}(s)\left(e^{-sh} - e^{sh}\right) \\ &C_1(s) = \hat{h}(s)\left(e^{-sh} - e^{sh}\right) \\ &A_2(s) = A_3(s) = \hat{f}(s)\,e^{-sh} \\ &B_2(s) = B_3(s) = \hat{g}(s)\,e^{-sh} \\ &C_2(s) = C_3(s) = \hat{h}(s)\,e^{-sh} \end{aligned} \tag{29}$$

Finally, the solution for the half-space with dislocation density functions $f(s)$, $g(s)$ and $h(s)$ in the domain $y \ge 0$ $|x| < \infty$ is given by:

$$\begin{aligned} &w(x,y) = -\frac{2}{\pi}\int_0^\infty \hat{f}(s)\left[sgn(y-h)e^{-s|(y-h)|} - e^{-s(y+h)}\right]cos(sx)ds \\ &\chi(x,y) = -\frac{2}{\pi}\int_0^\infty \hat{g}(s)\left[sgn(y-h)e^{-s|(y-h)|} - e^{-s(y+h)}\right]cos(sx)ds \\ &\eta(x,y) = -\frac{2}{\pi}\int_0^\infty \hat{h}(s)\left[sgn(y-h)e^{-s|(y-h)|} - e^{-s(y+h)}\right]cos(sx)ds \end{aligned} \tag{30}$$

$$\tau_{zy}(x,y)=\frac{2}{\pi}\tilde{c}_{44}\int_0^\infty s\hat{f}(s)\left[e^{-s|(y-h)|}-e^{-s(y+h)}\right]cos(sx)ds+$$

$$-\frac{2}{\pi}\int_0^\infty s\left[\alpha\hat{g}(s)+\beta\hat{h}(s)\right]\left[e^{-s|(y-h)|}-e^{-s(y+h)}\right]cos(sx)ds$$

$$D_y(x,y)=\frac{2}{\pi}\int_0^\infty s\hat{g}(s)\left[e^{-s|(y-h)|}-e^{-s(y+h)}\right]cos(sx)ds$$

$$B_y(x,y)=\frac{2}{\pi}\int_0^\infty s\hat{h}(s)\left[e^{-s|(y-h)|}-e^{-s(y+h)}\right]cos(sx)ds$$

where $sgn(y-h)=+1$ or $y>h$ or $y<h$, respectively.

The potentials $\varphi(x,y)$ and $\psi(x,y)$ are obtained from Eqs (14).

5. FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

The unknown functions $f(s)$, $g(s)$ and $h(s)$ can be obtained from the mixed boundary conditions (19) and (21)which yield:

$$\frac{2}{\pi}\int_0^\infty s\hat{f}(s)\left[1-e^{-2sh}\right]\cos(sx)ds=-\frac{\tau_0+(D-d_0)\alpha+(B-b_0)\beta}{\tilde{c}_{44}}$$

$$\frac{2}{\pi}\int_0^\infty s\hat{g}(s)\left[1-e^{-2sh}\right]\cos(sx)ds=-(D-d_0) \tag{31a}$$

$$\frac{2}{\pi}\int_0^\infty s\hat{h}(s)\left[1-e^{-2sh}\right]\cos(sx)ds=-(B-b_0);\quad |x|<a$$

$$\int_0^\infty \hat{f}(s)cos(sx)ds=0$$

$$\int_0^\infty \hat{g}(s)cos(sx)ds=0 \tag{31b}$$

$$\int_0^\infty \hat{h}(s)cos(sx)ds=0\quad;\quad |x|\geq a$$

The integral equations (31a) may be rewritten as:

$$\frac{2}{\pi}\int_0^\infty\begin{Bmatrix}\hat{f}(s)\\ \hat{g}(s)\\ \hat{h}(s)\end{Bmatrix}\begin{Bmatrix}1-e^{-2sh}\\ 1-e^{-2sh}\\ 1-e^{-2sh}\end{Bmatrix}\sin(sx)ds=-\begin{Bmatrix}\frac{\tau_0+(D-d_0)\alpha+(B-b_0)\beta}{\tilde{c}_{44}}x\\ (D-d_0)x\\ (B-b_0)x\end{Bmatrix} \tag{32}$$

We introduce the integral representation of the unknown functions:

$$\begin{Bmatrix}\hat{f}(s)\\ \hat{g}(s)\\ \hat{h}(s)\end{Bmatrix}=-\begin{Bmatrix}\frac{\tau_0+(D-d_0)\alpha+(B-b_0)\beta}{\tilde{c}_{44}}\\ D-d_0\\ B-b_0\end{Bmatrix}\int_0^a\begin{Bmatrix}f(u)\\ g(u)\\ h(u)\end{Bmatrix}uJ_0(su)du \tag{33}$$

where $J_0(su)$ is the Bessel function of the first kind and zero order and $f(u)$, $g(u)$, $h(u)$ are new auxiliary functions. This representation satisfies equations (31b) automatically and converts equations (32) to the Abel one integral equation, which can be solved explicitly. The result is the Fredholm integral equation of the second kind:

$$f(u)-\int_0^a f(v)K(u,v)\,dv=1 \tag{34}$$

with the kernel:

$$K(u,v)=v\int_0^\infty se^{-2sh}J_0(su)J_0(sv)ds \tag{35}$$

and:

$$f(u)=g(u)=h(u) \tag{36}$$

of course $\hat{f}(s)$, $\hat{g}(s)$ and $\hat{h}(s)$ are dissimilar since are proportional to $\tau_0+(D-d_0)\alpha+(B-b_0)\beta$; $D-d_0$ and $B-b_0$, respectively, and d_0, b_0 are dissimilar functions, defined by Eqs. (23).

6. THE SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

The kernel function $K(u,v)$ may be presented in more useful form. Using the Neumann's theorem (Watson, 1966):

$$J_0(su)J_0(sr)=\frac{1}{\pi}\int_0^\pi J_0(sR)d\alpha\,;\qquad R^2=u^2+r^2-2ur\cos\alpha \tag{37}$$

and the integral:

$$\int_0^\infty sJ_0(Rs)e^{-2sh}ds=\frac{2h}{\left[R^2+(2h)^2\right]^{3/2}} \tag{38}$$

the kernel function becomes:

$$K(u,v)=\frac{4hv}{\pi l^{3/2}}\int_0^{\pi/2}\frac{d\alpha}{\left(1-k^2\cos^2\alpha\right)^{3/2}}\qquad l^2=(u+v)^2+4h^2,\quad k^2=\frac{4uv}{l^2} \tag{39}$$

The kernel function is presented by means of elliptic integral. The integral equation (34) can be solved by iterative method.

The recurrence formula is:

$$f_i(u)=1+\int_0^a f_{i-1}(v)K(u,v)dv\,,\quad f_0(v)=1\,,\quad i=1,2,...n \tag{40}$$

The n-th approximation gives:

$$f(u)=1+\frac{a}{a+u}\left[1-\frac{4h}{\pi}\frac{K(k_0)}{l_0}\right]+ \\ +\left(\frac{a}{a+u}\right)^2\left[1-\frac{4h}{\pi}\frac{K(k_0)}{l_0}\right]^2+....+ \\ +\left(\frac{a}{a+u}\right)^n\left[1-\frac{4h}{\pi}\frac{K(k_0)}{l_0}\right]^n \tag{41}$$

where $K(k_0)$ is the elliptic integral of the first kind defined by:

$$K(k_0)=\int_0^{\pi/2}\frac{d\alpha}{(1-k_0^2\cos^2\alpha)^{1/2}} \\ l_0^2=(a+u)^2+4h^2\,,\quad k_0^2=\frac{4au}{l_0^2} \tag{42}$$

The sum of infinite geometric series converges to the solution as $n\to\infty$, giving:

$$f(u)=\left[1-\frac{a}{a+u}\left(1-\frac{2}{\pi}\frac{K(k_0)}{l_0/2h}\right)\right]^{-1}\qquad |u|\le a \tag{43}$$

The range of convergence is given by inequality:

$$\frac{2}{\pi}K(k_0)<\left(2+\frac{u}{a}\right)\frac{l_0}{2h}\qquad |u|\le a \tag{44}$$

and is satisfied for all of u and a/h.

For $h\to\infty$, $(2/\pi)K(k_0)\to 1$ and $l_0/2h\to 1$, while for $h\to 0$, we have the logarithmic singularity of $K(k_0)$ for $u=a$:

$$K(k_0)\sim\ln\frac{1}{1-\frac{2\sqrt{au}}{a+u}} \tag{45}$$

But $hK(k_0)/l_0$ tends to zero as $a/h\to\infty$.

Thus we have the values:

$$f\left(\frac{a}{h}\right)=2\left[1+\frac{2}{\pi}\frac{1}{\sqrt{1+\delta^2}}K\left(\frac{\delta}{\sqrt{1+\delta^2}}\right)\right]^{-1} \\ f(0)=\sqrt{1+\frac{\delta^2}{4}}\,,\quad \delta=\frac{a}{h} \tag{46}$$

The values of $f(a/h)$ changes from 1 to 2 for all of a/h and $f(u)$ is given explicitly by Eq. (43).

7. FIELD INTENSITY FACTORS

The electric displacement, magnetic induction and shear stress outside of the crack surface can be expressed by:

$$\begin{Bmatrix}D_y(x,h\pm)\\B_y(x,h\pm)\end{Bmatrix}=-\frac{2}{\pi}\begin{Bmatrix}D-d_0\\B-b_0\end{Bmatrix}\int_0^a f(u)udu\int_0^\infty sJ_0(su)\left(1-e^{-2sh}\right)\cos(sx)ds \tag{47}$$

$$\tau_{zy}(x,h\pm)=-\frac{2}{\pi}\tau_0\int_0^a f(u)udu\int_0^\infty sJ_0(su)\left(1-e^{-2sh}\right)\cos(sx)ds$$

Using the integral:

$$\int_0^\infty e^{-2sh}\sin(sx)J_0(su)ds=\frac{\eta}{x(\xi^2+\eta^2)} \\ u^2=x^2(1+\xi^2)(1-\eta^2)\qquad 2h=x\xi\eta\;,\quad x\xi>0,\quad \eta>0 \tag{48}$$

equations (47) may be written as:

$$\begin{Bmatrix}D_y(x,h\pm)\\B_y(x,h\pm)\end{Bmatrix}=-\frac{2}{\pi}\begin{Bmatrix}D-d_0\\B-b_0\end{Bmatrix}\frac{d}{dx}\int_0^a f(u)udu\left[\frac{|x|}{x\sqrt{x^2-u^2}}-\frac{\eta}{x(\xi^2+\eta^2)}\right] \\ \tau_{zy}(x,h\pm)=-\frac{2}{\pi}\tau_0\frac{d}{dx}\int_0^a f(u)udu\left[\frac{|x|}{x\sqrt{x^2-u^2}}-\frac{\eta}{x(\xi^2+\eta^2)}\right] \tag{49}$$

The singular terms of these quantities ($|x|\to a^+$) are included in the first term in Eq. (49). Since the singular field near the crack tip exhibits the inverse square-root singularity we define the stress, electric displacement and magnetic induction intensity factors as follows:

$$\begin{Bmatrix}K_\tau\\K_D\\K_B\end{Bmatrix}=\lim_{|x|\to a^+}\sqrt{2(|x|-a)}\begin{Bmatrix}\tau_{zy}\\D_y\\B_y\end{Bmatrix} \tag{50}$$

The intensity factors are obtained as:

$$K_w=\frac{1}{\tilde{c}_{44}}\left(K_\tau+\alpha K_D+\beta K_B\right) \\ K_\phi=\alpha K_w+e_1K_D+e_2K_B \\ K_\psi=\beta K_w+e_2K_D+e_3K_B \\ K_\tau=\frac{2}{\pi}\tau_0 f(a)\sqrt{a} \\ K_D=\frac{2}{\pi}(D-d_0)f(a)\sqrt{a} \\ K_B=\frac{2}{\pi}(B-b_0)f(a)\sqrt{a} \tag{51}$$

The jumps of displacement, electric potential and magnetic potential of the crack surfaces can be expressed as:

$$[\![w]\!]=\frac{4}{\pi}\frac{\tau_0+(D-d_0)\alpha+(B-b_0)\beta}{\tilde{c}_{44}}\int_x^a\frac{f(u)udu}{\sqrt{u^2-x^2}} \\ [\![\phi]\!]=\frac{4}{\pi}\left[\frac{\alpha[\tau_0+(D-d_0)\alpha+(B-b_0)\beta]}{\tilde{c}_{44}}+e_1(D-d_0)+e_2(B-b_0)\right]\int_x^a\frac{f(u)udu}{\sqrt{u^2-x^2}} \\ [\![\psi]\!]=\frac{4}{\pi}\left[\frac{\beta[\tau_0+(D-d_0)\alpha+(B-b_0)\beta]}{\tilde{c}_{44}}+e_2(D-d_0)+e_3(B-b_0)\right]\int_x^a\frac{f(u)udu}{\sqrt{u^2-x^2}} \tag{52}$$

Substituting Eqs (52) into Eqs (25) and differentiating both obtained equations with respect to x and using the following rule of differentiation under integral sign:

$$\frac{d}{dx}\int_x^a \frac{f(u)du}{\sqrt{u^2-x^2}} = -\frac{xf(a)}{a\sqrt{a^2-x^2}} + x\int_x^a \frac{d}{du}\left(\frac{f(u)}{u}\right)\frac{du}{\sqrt{u^2-x^2}} \tag{53}$$

equations (25) may be converted to two equations in which singular terms at $x \to a-0$, appear. For the singularity to vanish at $x \to a-0$, it must be true that:

$$\begin{aligned} d_0 &= -\varepsilon_0 f(a)\left[\frac{\alpha}{\tilde{c}_{44}}[\tau_0+(D-d_0)\alpha+(B-b_0)\beta]+e_1(D-d_0)+e_2(B-b_0)\right] \\ b_0 &= -\mu_0 f(a)\left[\frac{\beta}{\tilde{c}_{44}}[\tau_0+(D-d_0)\alpha+(B-b_0)\beta]+e_2(D-d_0)+e_3(B-b_0)\right] \end{aligned} \tag{54}$$

where:

$$\varepsilon_0 = \frac{2}{\pi}\frac{a}{\delta_0}\varepsilon_c, \quad \mu_0 = \frac{2}{\pi}\frac{a}{\delta_0}\mu_c \tag{55}$$

Thus:

$$\begin{aligned} D-d_0 &= \left\{D\left[1-\mu_0 f\left(\frac{\beta^2}{\tilde{c}_{44}}+e_3\right)\right]+B\varepsilon_0 f\left(\frac{\alpha\beta}{\tilde{c}_{44}}+e_2\right)+\right. \\ &\quad \left.+\frac{\tau_0\varepsilon_0}{\tilde{c}_{44}} f\left[\alpha+\mu_0 e_{15}\left(e_1e_3-e_2^2\right)f\right]\right\}\times \\ &\quad \times\left\{1-f\left[\varepsilon_0\left(\frac{\alpha^2}{\tilde{c}_{44}}+e_1\right)+\mu_0\left(\frac{\beta^2}{\tilde{c}_{44}}+e_3\right)-\varepsilon_0\mu_0\frac{c_{44}}{\tilde{c}_{44}}\left(e_1e_3-e_2^2\right)\right]\right\}^{-1} \\ B-b_0 &= \left\{D\mu_0 f\left(\frac{\alpha\beta}{\tilde{c}_{44}}+e_2\right)+B\left[1-\varepsilon_0 f\left(\frac{\alpha^2}{\tilde{c}_{44}}+e_1\right)\right]+\right. \\ &\quad \left.+\frac{\tau_0\mu_0}{\tilde{c}_{44}} f\left[\beta+\varepsilon_0 q_{15}\left(e_1e_3-e_2^2\right)f\right]\right\}\times \\ &\quad \left\{1-f\left[\varepsilon_0\left(\frac{\alpha^2}{\tilde{c}_{44}}+e_1\right)+\mu_0\left(\frac{\beta^2}{\tilde{c}_{44}}+e_3\right)-\varepsilon_0\mu_0\frac{c_{44}}{\tilde{c}_{44}}\left(e_1e_3-e_2^2\right)\right]\right\}^{-1} \end{aligned} \tag{56}$$

where $f = f(a/h)$.

The electric and magnetic intensity factors are obtained by substitution of Eqs (56) into Eqs (51).

Furthermore we consider the behaviour of the jumps of the displacement, electric and magnetic potentials and define the following intensity factors:

$$\begin{Bmatrix} K_w \\ K_\phi \\ K_\psi \end{Bmatrix} = \lim_{|x|\to a^-} \frac{1}{2\sqrt{2(a-|x|)}}\begin{Bmatrix} [|w|] \\ [|\phi|] \\ [|\psi|] \end{Bmatrix} \tag{57}$$

In view of the results in Eq. (52), we have:

$$\begin{aligned} K_w &= \frac{1}{\tilde{c}_{44}}\left(K_\tau+\alpha K_D+\beta K_B\right) \\ K_\phi &= \alpha K_w + e_1K_D + e_2K_B \\ K_\psi &= \beta K_w + e_2K_D + e_3K_B \end{aligned} \tag{58}$$

These field intensity factors satisfy the constitutive equations:

$$\left[K_w, K_\phi, K_\psi\right]^T = \mathbf{C}^{-1}\left[K_\tau, K_D, K_B\right]^T \tag{59}$$

The energy release rate is derived in the following in similar manner to proposed by Pak (1990).

The energy release rate of the crack-tip is obtained from the following integral:

$$\begin{aligned} G = \frac{1}{2}\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta \{ &\tau_{yz}(r+a,0)[|w|](r+a-\delta)+ \\ &D_y(r+a,0)[|\phi|](r+a-\delta)+ \\ &B_y(r+a,0)[|\psi|](r+a-\delta)\}dr \end{aligned} \tag{60}$$

where $[|w|]$, $[|\phi|]$ and $[|\psi|]$ are the jumps of displacement, electric potential and magnetic potential field intensity factors given by Eqs. (52).

The energy release rate is defined as:

$$G = \frac{1}{2}\left(K_\tau K_w + K_D K_\phi + K_B K_\psi\right) \tag{61}$$

or :

$$\begin{aligned} G = \frac{2}{\pi^2} f^2(a)a\frac{1}{\tilde{c}_{44}\left(\varepsilon_{11}\mu_{11}-d_{11}^2\right)}\Big[&\left(\varepsilon_{11}\mu_{11}-d_{11}^2\right)\tau_0^2 + \\ &-\left(c_{44}\mu_{11}+q_{15}^2\right)(D-d_0)^2-\left(c_{44}\varepsilon_{11}+e_{15}^2\right)(B-b_0)^2+ \\ &+2\left(e_{15}\mu_{11}-q_{15}d_{11}\right)\tau_0(D-d_0)+ \\ &+2\left(q_{15}\varepsilon_{11}-e_{15}d_{11}\right)\tau_0(B-b_0)+ \\ &+2\left(c_{44}d_{11}+e_{15}q_{15}\right)(D-d_0)(B-b_0)\Big] \end{aligned} \tag{62}$$

8. SOLUTIONS BASED ON IDEAL CRACK-FACE BOUNDARY CONDITIONS

When the crack is one of four ideal crack models: magneto-electrically permeable, magneto-electrically impermeable, magnetically permeable and electrically impermeable, magnetically impermeable and electrically permeable are the limiting cases of the magneto-electrically dielectric crack model.

– fully impermeable case: $\varepsilon_0 \to 0$ and $\mu_0 \to 0$, $D-d_0 \to D$, $B-b_0 \to B$ and the intensity factors are given by:

$$\begin{aligned} K_w^{imp.imp.} &= \frac{2}{\pi}\frac{f(a)\sqrt{a}}{\tilde{c}_{44}}\left(\tau_0+D\alpha+B\beta\right) \\ K_\tau^{imp.imp.} &= \frac{2}{\pi}f(a)\sqrt{a}\tau_0 \\ K_D^{imp.imp.} &= \frac{2}{\pi}f(a)\sqrt{a}D \\ K_B^{imp.imp.} &= \frac{2}{\pi}f(a)\sqrt{a}B \\ K_\phi^{imp.imp.} &= \alpha K_w^{imp.imp.}+e_1K_D^{imp.imp.}+e_2K_B^{imp.imp.} \\ K_\psi^{imp.imp.} &= \beta K_w^{imp.imp.}+e_2K_D^{imp.imp.}+e_3K_B^{imp.imp.} \end{aligned} \tag{63}$$

Equations (63) indicate that K_τ, K_D and K_B are independent on the material constants, while K_w, K_ϕ and K_ψ, depend. Since $f(a)$ depend on the parameter of location of the crack(the thickness h), strictly speaking on h/a, these quantities depend on this location.

– full permeable case: $\varepsilon_0 \to 0$ and $\mu_0 \to 0$
Then:

$$D-d_0=\frac{e_{15}\tau_0}{c_{44}}, \quad B-b_0=\frac{q_{15}\tau_0}{c_{44}} \tag{64}$$

And:

$$\begin{aligned} K_w^{per.per.} &= \frac{2}{\pi}\frac{f(a)\sqrt{a}}{c_{44}}\tau_0 \\ K_\tau^{per.per.} &= \frac{2}{\pi} f(a)\sqrt{a}\tau_0 \\ K_D^{per.per.} &= e_{15}K_w^{per.per.} \\ K_B^{per.per.} &= q_{15}K_w^{per.per.} \\ K_\phi^{per.per.} &= 0 \\ K_\psi^{per.per.} &= 0 \end{aligned} \tag{65}$$

The energy release rate is:

$$G=\frac{2}{\pi^2}\frac{\tau_0^2 af^2(a)}{c_{44}} \tag{66}$$

– electrically impermeable and magnetically permeable: $\varepsilon_0 \to 0$ and $\mu_0 \to 0$, $D-d_0 \to D$, $d_0 \to 0$

$$\begin{aligned} K_D^{imp.per} &= K_D^{imp.imp.} \\ B-b_0 &= \left[D\left(\frac{\alpha\beta}{\tilde{c}_{44}}+e_2\right)f(a)+\frac{\tau_0\beta}{\tilde{c}_{44}}\right]\left[1+\left(\frac{\beta^2}{\tilde{c}_{44}}+e_3\right)f\right]^{-1} \\ K_B^{imp.per.} &= \frac{2}{\pi} f(a)\sqrt{a}(B-b_0) \\ K_w^{imp.per.} &= \frac{2}{\pi} f(a)\sqrt{a}\,\frac{\tau_0+D\alpha}{\tilde{c}_{44}}+\beta K_B^{imp.per.} \end{aligned} \tag{67}$$

The solutions for the electrically impermeable and magnetically permeable crack are independent of the applied magnetic field.

– electrically permeable and magnetically impermeable: $\varepsilon_0 \to 0$ and $\mu_0 \to 0$, $B-b_0 \to B$, $b_0 \to 0$

$$\begin{aligned} K_B^{per.imp.} &= K_B^{imp.imp.} \\ D-d_0 &= \left[\frac{\tau_0\alpha}{\tilde{c}_{44}}+B\left(e_2+\frac{\alpha\beta}{\tilde{c}_{44}}\right)f(a)\right]\left[1+\left(e_1+\frac{\alpha^2}{\tilde{c}_{44}}\right)f(a)\right]^{-1} \\ K_D^{per.imp.} &= \frac{2}{\pi} f(a)\sqrt{a}(D-d_0) \\ K_w^{per.imp} &= \frac{2}{\pi} f(a)\sqrt{a}\,\frac{\tau_0+B\beta}{\tilde{c}_{44}}+\alpha K_D^{per.imp.} \end{aligned} \tag{68}$$

The solution for the electrically permeable and magnetically impermeable crack are independent of the applied electric displacement.

In practical applications the following cases appear:

– Let ε_0 tends to infinity and μ_0 is finite
Then:

$$\begin{aligned} K_D^{perm.\mu_c} &= K_D^{per.imp.}\left(1-f_1(\bar{\mu})\right)+K_D^{per.per.}f_1(\bar{\mu}) \\ K_B^{perm.\mu_c} &= K_B^{imp.imp.}\left(1-f_1(\bar{\mu})\right)+K_B^{per.per.}f_1(\bar{\mu}) \end{aligned} \tag{69}$$

where:

$$f_1(\bar{\mu})=\frac{1}{1+\bar{\mu}}, \quad \bar{\mu}=\frac{\pi}{2}\frac{\mu_{11}}{\mu_c}\frac{\delta_0}{a}\frac{1}{f(a)}\left(1+\frac{q_{15}^2}{\mu_{11}c_{44}}\right) \tag{70}$$

– Let μ_0 tends to infinity and ε_0 is finite
Then:

$$\begin{aligned} K_D^{\varepsilon_c perm.} &= K_D^{imp.imp.}\left(1-f_2(\bar{\varepsilon})\right)+K_D^{per.per.}f_2(\bar{\varepsilon}) \\ K_B^{\varepsilon_c perm.} &= K_B^{imp.per.}\left(1-f_2(\bar{\varepsilon})\right)+K_B^{per.per.}f_2(\bar{\varepsilon}) \end{aligned} \tag{71}$$

where:

$$f_2(\bar{\varepsilon})=\frac{1}{1+\bar{\varepsilon}}, \quad \bar{\varepsilon}=\frac{\pi}{2}\frac{\varepsilon_{11}}{\varepsilon_c}\frac{\delta_0}{a}\frac{1}{f(a)}\left(1+\frac{e_{15}^2}{\varepsilon_{11}c_{44}}\right) \tag{72}$$

In above equations the notation K^{perimp} denotes the intensity factors (51) and (58) for electrically permeable and magnetically impermeable crack boundary conditions i.e. for the values (68). Similarly $K^{imp.per.}$ are defined by Eqs (51) or (58) and (67).

The functions of permittivity ε_c and permeability μ_c approaches zero as ε_c and μ_c tends to zero and are unity as ε_c and μ_c tends to infinity.

The solution perfectly matches the exact solution in both limiting cases, namely permeable and/or impermeable electric and/or magnetic boundary conditions.

9. RESULT AND DISCUSSION

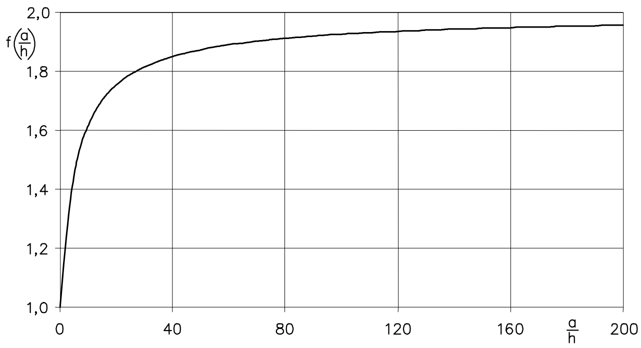

Fig. 2 Variation of $f(a/h)$ versus ratio of a/h; stress, electric displacement and magnetic induction intensity factors are proportional to $f(a/h)$ since: $f(a/h)=K_\tau/\tau_0\sqrt{a}(2/\pi)=K_B/B\sqrt{a}(2/\pi)$ in fully impermeable case

The electric and magnetic response, in fully impermeable case, is proportional to the applied electric and magnetic load, respectively, and is independent on the mechanical loads, as Eq. (63) implies. Similarly is for stress intensity factor. The intensity factors of stress, electric displacement and magnetic induction, therefore, are just a function of the geometry of the cracked PEMO-elastic half-space as shown in Fig. 2.

From the Fig. 2 we can see that the SIF, EDIF and MIIF increase with a/h. For small values of a/h these quantities grow at an approximately constant rate with increasing a/h. For very large a/h (the crack near the boundary of a half-space) $f(a/h)$ increases slowly tending to 2.

10. CONCLUSIONS

From analytical and numerical results, several conclusions can be formulated:

– The electric displacement intensity factor is independent of the applied magnetic field in the special case of electrically impermeable and magnetically permeable crack;
– The magnetic induction intensity factor is independent of the applied electric displacement in the special case of electrically permeable and magnetically impermeable crack;
– Applications of electric and magnetic fields do not alter the stress intensity factors;
– The analytical solution (43) is new to the author' best knowledge. Accordingly, the behaviour of a crack which lies near of the boundary of the medium may be investigated exactly;
– Note that the plane $x = 0$ is a plane of symmetry ($\tau_{xz} = 0, D_x = 0$ and $B_x = 0$ on this plane). In consequence the solutions are valid for quarter-plane with edge crack of length a.

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