POSITIVE REALIZATION OF SISO 2D DIFFERENT ORDERS FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

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Abstract: The realization problem for single-input single-output 2D positive fractional systems with different orders is formulated and a method based on the state variable diagram for finding a positive realization of a given proper transfer function is proposed. Sufficient conditions for the existence of a positive realization of this class of 2D linear systems are established. A procedure for computation of a positive realization is proposed and illustrated by a numerical example.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of art in positive systems theory is given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2002). The realization problem for positive discrete-time and continuous-time systems without and with delays was considered in Benvenuti and Farina (2004), Farina and Rinaldi (2000) and Kaczorek (2006a, 2006b, 2004, 2005). A new class of positive 2D hybrid linear system has been introduced in Kaczorek (2007), and the realization problem for this class of systems has been considered in Kaczorek (2008c).

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century (Nishimoto, 1984; Oldham and Spanier, 1974). This idea has been used by engineers for modeling different process (Engheta, 1997; Ferreira and Machado, 2003; Klamka, 2005; Ostalczyk, 2000; Oustaloup, 1993). Mathematical fundamentals of fractional calculus are given in the monographs (Miller and Ross, 1993; Nishimoto, 1984; Oldham and Spanier, 1974; Ortigueira, 1997; Podlubny, 1999). The fractional order controllers have been developed in (Ostalczyk, 2000; Podlubny et al., 1997). A generalization of the Kalman filter for fractional order systems has been proposed in Zaborowsky and Meylaov (2001). A new class of positive fractional 2D hybrid linear system has been introduced in Kaczorek (2008e) and positive fractional 2D linear systems described by the Roesser model in Rogowski and Kaczorek (2010). The realization problem for positive fractional systems was considered in Kaczorek (2008b, 2008d, 2011) and Sajewski (2010).

The main purpose of this paper is to present a method for computation of a positive realization of SISO 2D different orders fractional systems with given proper transfer function using the state variable diagram method. Sufficient conditions for the existence of a positive realization of this class of systems will be established and a procedure for computation of a positive realization will be proposed.

The paper is organized as follows. In section 2 basic definition and theorem concerning positive 2D different orders fractional systems are recalled. Also in this section using the zet transform the transfer matrix (function) of the different orders fractional systems is derived and the positive realization problem is formulated. Main result is given in section 3 where solution to the realization problem for given transfer function of the 2D different orders fractional discrete-time linear systems is given. In the same section the sufficient conditions for the positive realization are derived and the procedure for computation of the positive realization is proposed. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix, $z[f(k)]$ – zet transform of the discrete-time function $f(k)$.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider a 2D system with different fractional orders described by the equations

$$\Delta^{\alpha}x_1(k+1)=A_{11}x_1(k)+A_{12}x_2(k)+B_1u(k) \tag{2.1a}$$

$$\Delta^{\beta}x_2(k+1)=A_{21}x_1(k)+A_{22}x_2(k)+B_2u(k) \tag{2.1b}$$

$$y(k)=C_1x_1(k)+C_2x_2(k)+Du(k),\ k\in Z_+ \tag{2.1c}$$

where $x_1(k)\in\Re^{n_1}$, $x_2(k)\in\Re^{n_2}$ are state vectors and $u(k)\in\Re^m$ is input vector $y(k)\in\Re^p$ is output vector and $A_{ij}\in\Re^{n_i\times n_j}$, $B_i\in\Re^{n_i\times m}$, $C_i\in\Re^{p\times n_i}$, $i,j=1,2$; $D\in\Re^{p\times m}$.

The fractional difference of $\alpha \in \Re$ order is defined by

$$\Delta^{\alpha} x(k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j) \quad (2.2a)$$

and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for} \quad j = 1,2,... \end{cases} \quad (2.2b)$$

Using (2.2a) we can write the equation (2.1a) and (2.1b) in the following form

$$\begin{aligned} x_1(k+1) &= A_{1\alpha} x_1(k) + A_{12} x_2(k) \\ &\quad - \sum_{j=2}^{k+1} (-1)^j \binom{\alpha}{j} x_1(k-j+1) + B_1 u(k) \\ x_2(k+1) &= A_{21} x_1(k) + A_{2\beta} x_2(k) \\ &\quad - \sum_{j=2}^{k+1} (-1)^j \binom{\beta}{j} x_2(k-j+1) + B_2 u(k) \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} A_{1\alpha} &= A_{11} + \alpha I_{n_1} \\ A_{1\beta} &= A_{22} + \beta I_{n_2} \end{aligned} \quad (2.4)$$

Definition 2.1. The fractional system (2.1) is called positive if and only if $x_1(k) \in \Re^{n_1}$, $x_2(k) \in \Re^{n_2}$ and $y(k) \in \Re_+^p$, $k \in Z_+$ for any initial conditions $x_1(0) = x_{10} \in \Re_+^{n_1}$, $x_2(0) = x_{20} \in \Re_+^{n_2}$, and all input sequences $u(k) \in \Re^m$, $k \in Z_+ = \{0, 1, ...\}$.

Theorem 2.1. (Kaczorek, 2011) The fractional discrete-time linear system (2.1) with $0 < \alpha < 1$, $0 < \beta < 1$ is positive if and only if

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \Re_+^{n \times n}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{n \times m},$$

$$[C_1 \quad C_2] \in \Re_+^{p \times n}, \ D \in \Re_+^{p \times m}. \quad (2.5)$$

Proof is given in Kaczorek (2011).

Substituting (2.2a) into (2.1a) and (2.1b) we obtain

$$\begin{aligned} & x_1(k+1) + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x(k-j+1) = A_{11} x_1(k) + A_{12} x_2(k) + B_1 u(k) \\ & x_2(k+1) + \sum_{j=1}^{k+1} (-1)^j \binom{\beta}{j} x(k-j+1) = A_{21} x_1(k) + A_{22} x_2(k) + B_2 u(k) \\ & y(k) = C_1 x_1(k) + C_2 x_2(k) + Du(k) \end{aligned} \quad (2.6a)$$

Performing the zet transform with zero initial conditions we have

$$\begin{aligned} & zX_1(z) + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} z^{1-j} X_1(z) = A_{11} X_1(z) + A_{12} X_2(z) + B_1 U(z) \\ & zX_2(z) + \sum_{j=1}^{k+1} (-1)^j \binom{\beta}{j} z^{1-j} X_2(z) = A_{21} X_1(z) + A_{22} X_2(z) + B_2 U(z) \\ & Y(z) = C_1 X_1(z) + C_2 X_2(z) + DU(z) \end{aligned} \quad (2.7)$$

where $X(z) = Z[x(k)]$, $U(z) = z[u(k)]$, $Y(z) = z[y(k)]$.

The equations (2.7) can be written in the matrix form

$$\begin{aligned} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} &= \begin{bmatrix} I_{n_1}(z - c_\alpha) - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}(z - c_\beta) - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z) \\ Y(z) &= [C_1 \quad C_2] \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} + DU(z) \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} c_\alpha &= c_\alpha(k, z) = \sum_{j=1}^{k+1} (-1)^{j-1} \binom{\alpha}{j} z^{1-j} \\ c_\beta &= c_\beta(k, z) = \sum_{j=1}^{k+1} (-1)^{j-1} \binom{\beta}{j} z^{1-j} \end{aligned} \quad (2.9)$$

The transfer matrix of the system (2.1) is given by

$$T(z) = [C_1 \quad C_2] \begin{bmatrix} I_{n_1}(z - c_\alpha) - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}(z - c_\beta) - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \quad (2.10)$$

In this case the transfer matrix is the function of the operators $w_\alpha = z - c_\alpha$, $w_\beta = z - c_\beta$ and for single-input single-output (shortly SISO) systems it has the following form

$$T(w_\alpha, w_\beta) = \frac{\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} b_{i,j} w_\alpha{}^i w_\beta{}^j}{w_\alpha{}^{n_1} w_\beta{}^{n_2} - \sum_{\substack{i=0 \\ i+j \neq n_1+n_2}}^{n_1} \sum_{j=0}^{n_2} b_{i,j} w_\alpha{}^i w_\beta{}^j} \quad (2.11)$$

for known α, β.

Definition 2.2. The matrices (2.5) are called the positive realization of the transfer matrix $T(z)$ if they satisfy the equality (2.10).

The realization problem can be stated as follows.

Given a proper rational matrix $T(w_\alpha, w_\beta) \in \Re^{p \times m}(w_{\alpha,} w_\beta)$ and fractional orders α, β, find its positive realization (2.5), where $\Re^{p \times m}(w_{\alpha,} w_\beta)$ is the set of $p \times m$ rational matrices in w_α and w_β.

3. PROBLEM SOLUTION FOR SISO SYSTEMS

The essence of proposed method for solving of the realization problem for positive linear systems with different fractional orders will be presented on single-input single-output system. It will be shown that state variable diagram method previously used for standard discrete-time systems and 2D hybrid systems (Kaczorek, 2002, 2008c) is also valid for fractional order discrete-time systems.

In standard (nonfractional) discrete-time systems it is well-known that

$$\mathcal{Z}[x(k+1)] = z \cdot \mathcal{Z}[x(k)] = zX(z) \quad (3.1a)$$

and

$$\mathcal{Z}[x(k)] = \frac{1}{z} \cdot \mathcal{Z}[x(k+1)] \quad (3.1b)$$

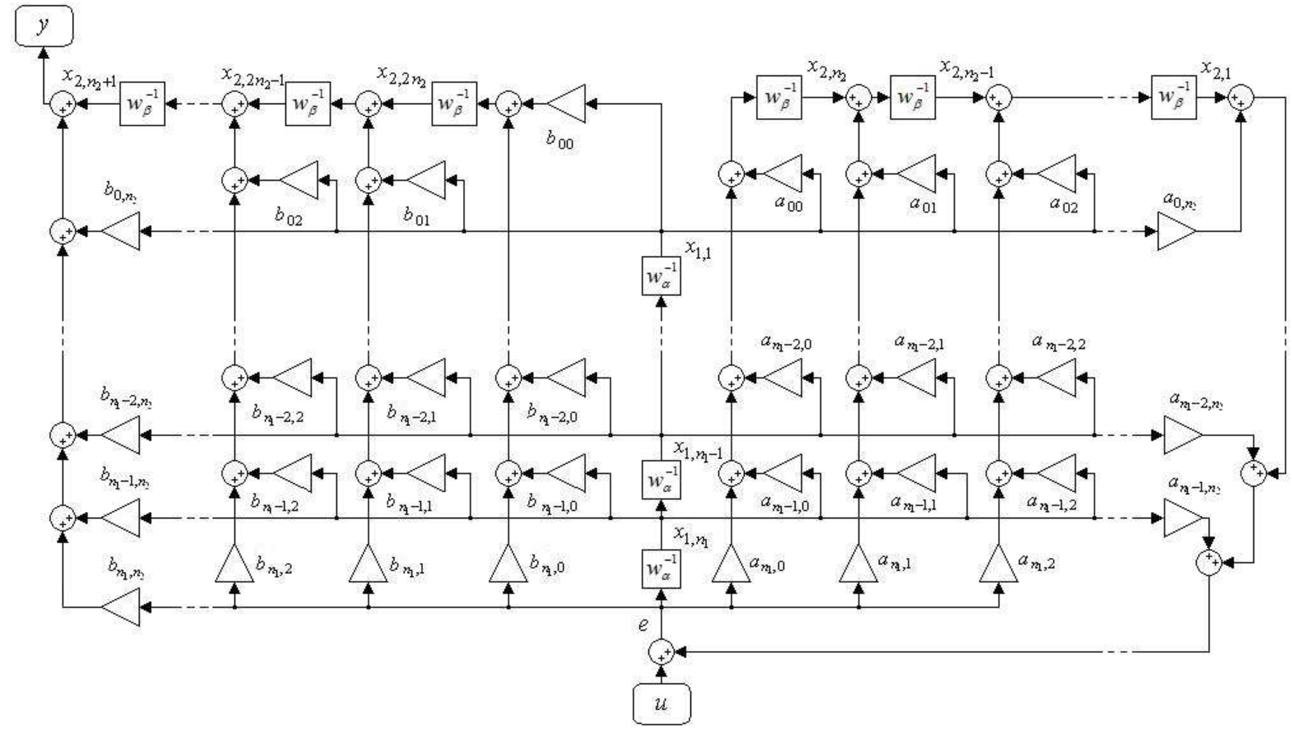

Fig. 3.1. State variable diagram for 2D fractional different orders system

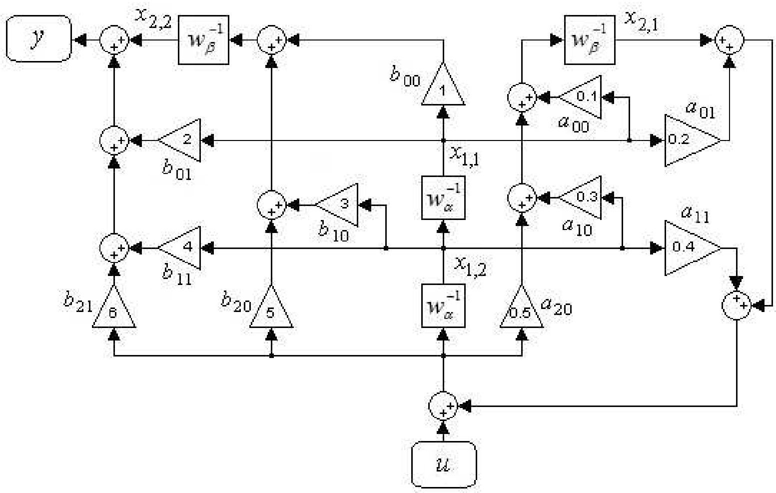

Fig. 3.2. State variable diagram for 2D fractional different orders transfer function (3.14)

Therefore, to draw the state variable diagram for standard discrete-time linear systems (Kaczorek, 2002) we use the of delay element $1/z$.

By similarity, for the fractional discrete-time linear systems we have

$$
\begin{aligned}
z[\Delta^{\alpha} x_1(k+1)] &= z\left[x_1(k+1)+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_1(k-j+1)\right] \\
&= \left(z-\sum_{j=1}^{k+1}(-1)^{j-1}\binom{\alpha}{j}z^{1-j}\right)X_1(z)=(z-c_\alpha)X_1(z)=w_\alpha X_1(z), \\
z[\Delta^{\beta} x_2(k+1)] &= (z-c_\beta)X_2(z)=w_\beta X_1(z)
\end{aligned}
\tag{3.2}
$$

and to draw the state variable diagram we have to use the fractional of delay elements $\frac{1}{w_\alpha}=w_\alpha^{-1}$ and $\frac{1}{w_\beta}=w_\beta^{-1}$.

Consider a 2D different orders fractional discrete-time linear system described by the transfer function (2.11). Multiplying the numerator and denominator of transfer function (2.11) by $w_\alpha^{-n_1}w_\beta^{-n_2}$ we obtain

$$
\begin{aligned}
T(w_\alpha,w_\beta) &= \frac{Y}{U} \\
&= \frac{b_{n_1,n_2}+b_{n_1,n_2-1}w_\beta^{-1}+b_{n_1-1,n_2}w_\alpha^{-1}+\ldots+b_{00}w_\alpha^{-n_1}w_\beta^{-n_2}}{1-a_{n_1,n_2-1}w_\beta^{-1}-a_{n_1-1,n_2}w_\alpha^{-1}-\ldots-a_{00}w_\alpha^{-n_1}w_\beta^{-n_2}}
\end{aligned}
\tag{3.3}
$$

Following Kaczorek (2002, 2008c) we define

$$E=\frac{U}{1-a_{n_1,n_2-1}w_\beta^{-1}-a_{n_1-1,n_2}w_\alpha^{-1}-\ldots-a_{00}w_\alpha^{-n_1}w_\beta^{-n_2}} \tag{3.4}$$

and from (3.3) we have

$$E=U+(a_{n_1,n_2-1}w_\beta^{-1}+a_{n_1-1,n_2}w_\alpha^{-1}+\ldots+a_{00}w_\alpha^{-n_1}w_\beta^{-n_2})E$$
$$Y=(b_{n_1,n_2}+b_{n_1,n_2-1}w_\beta^{-1}+b_{n_1-1,n_2}w_\alpha^{-1}+\ldots+b_{00}w_\alpha^{-n_1}w_\beta^{-n_2})E \tag{3.5}$$

Using (3.5) we may draw the state variable diagram shown in Fig. 3.1.

As a state variable we choose the outputs of fractional (order α) of delay elements $(x_{1,1}(k),x_{1,2}(k),\ldots,x_{1,n_1}(k))$ and fractional (order β) of delay elements $(x_{2,1}(k),x_{2,2}(k),\ldots,x_{2,2n_2}(k))$. Using state variable diagram (Fig. 3.1) we can write the following discrete-time different orders fractional equations

$$\begin{aligned}
&\Delta^\alpha x_{1,1}(k+1)=x_{1,2}(k)\\
&\Delta^\alpha x_{1,2}(k+1)=x_{1,3}(k)\\
&\vdots\\
&\Delta^\alpha x_{1,n_1-1}(k+1)=x_{1,n_1}(k)\\
&\Delta^\alpha x_{1,n_1}(k+1)=e(k)
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\Delta^\beta x_{2,1}(k+1)&=a_{0,n_2-1}x_{1,1}(k)+a_{1,n_2-1}x_{1,2}(k)\\
&\quad+\ldots+a_{n-1,n_2-1}x_{1,n_1}(k)+x_{2,2}(k)+a_{n_1,n_2-1}e(k)\\
\Delta^\beta x_{2,2}(k+1)&=a_{0,n_2-2}x_{1,1}(k)+a_{1,n_2-2}x_{1,2}(k)\\
&\quad+\ldots+a_{n-1,n_2-2}x_{1,n_1}(k)+x_{2,3}(k)+a_{n_1,n_2-2}e(k)\\
&\vdots\\
\Delta^\beta x_{2,n_2-1}(k+1)&=a_{0,1}x_{1,1}(k)+a_{1,1}x_{1,2}(k)\\
&\quad+\ldots+a_{n_1-1,1}x_{1,n_1}(k)+x_{2,n_2}(k)+a_{n_1,1}e(k)\\
\Delta^\beta x_{2,n_2}(k+1)&=a_{00}x_{1,1}(k)+a_{10}x_{1,2}(k)\\
&\quad+\ldots+a_{n_1-1,0}x_{1,n_1}(k)+a_{n_1,0}e(k)\\
\Delta^\beta x_{2,n_2+1}(k+1)&=b_{0,n_2-1}x_{1,1}(k)+b_{1,n_2-1}x_{1,2}(k)\\
&\quad+\ldots+b_{n_1-1,n_2-1}x_{1,n_1}(k)+x_{2,n_2+2}(k)+b_{n_1,n_2-1}e(k)\\
\Delta^\beta x_{2,n_2+2}(k+1)&=b_{0,n_2-2}x_{1,1}(k)+b_{1,n_2-2}x_{1,2}(k)\\
&\quad+\ldots+b_{n_1-1,n_2-2}x_{1,n_1}(k)+x_{2,n_2+3}(k)+b_{n_1,n_2-2}e(k)\\
&\vdots\\
\Delta^\beta x_{2,2n_2-1}(k+1)&=b_{0,1}x_{1,1}(k)+b_{1,1}x_{1,2}(k)\\
&\quad+\ldots+b_{n_1-1,1}x_{1,n_1}(k)+x_{2,2n_2}(k)+b_{n_1,1}e(k)\\
\Delta^\beta x_{2,2n_2}(k+1)&=b_{00}x_{1,1}(k)+b_{10}x_{1,2}(k)\\
&\quad+\ldots+b_{n_1-1,0}x_{1,n_1}(k)+b_{n_1,0}e(k)\\
y(k)&=b_{0,n_2}x_{1,1}(k)+b_{1,n_2}x_{1,2}(k)+\ldots+b_{n_1-1,n_2}x_{1,n_1}(k)\\
&\quad+x_{2,n_2+1}(k)+b_{n_1,n_2}e(k)
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
e(k)&=a_{0,n_2}x_{1,1}(k)+a_{1,n_2}x_{1,2}(k)\\
&\quad+\ldots+a_{n_1-1,n_2}x_{1,n_1}(k)+x_{2,1}(k)+u(k)
\end{aligned} \tag{3.7}$$

Defining

$$x_1(k)=\begin{bmatrix} x_{1,1}(k)\\ \vdots\\ x_{1,n_1}(k)\end{bmatrix},\quad x_2(k)=\begin{bmatrix} x_{2,1}(k)\\ \vdots\\ x_{2,2n_2}(k)\end{bmatrix} \tag{3.8}$$

and substituting (3.7) into (3.6) we can write the equations (3.6) in the form

$$\begin{aligned}
\begin{bmatrix}\Delta^\alpha x_1(k+1)\\ \Delta^\beta x_2(k+1)\end{bmatrix}&=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u(k)\\
y(k)&=\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}+Du(k)
\end{aligned} \tag{3.9}$$

where

$$A_{11}=\begin{bmatrix} 0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 1\\ a_{0,n_2} & a_{1,n_2} & a_{2,n_2} & \ldots & a_{n_1-1,n_2}\end{bmatrix}\in R^{n_1\times n_1},$$

$$A_{12}=\begin{bmatrix} 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 0 & 0\\ 1 & 0 & \ldots & 0 & 0\end{bmatrix}\in R^{n_1\times 2n_2},$$

$$A_{21}=\begin{bmatrix} \bar a_{0,n_2-1} & \bar a_{1,n_2-1} & \bar a_{2,n_2-1} & \ldots & \bar a_{n_1-1,n_2-1}\\ \bar a_{0,n_2-2} & \bar a_{1,n_2-2} & \bar a_{2,n_2-2} & \ldots & \bar a_{n_1-1,n_2-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \bar a_{00} & \bar a_{10} & \bar a_{20} & \ldots & \bar a_{n-1,0}\\ \bar b_{0,n_2-1} & \bar b_{1,n_2-1} & \bar b_{2,n_2-1} & \ldots & \bar b_{n_1-1,n_2-1}\\ \bar b_{0,n_2-2} & \bar b_{1,n_2-2} & \bar b_{2,n_2-2} & \ldots & \bar b_{n_1-1,n_2-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \bar b_{00} & \bar b_{10} & \bar b_{20} & \ldots & \bar b_{n_1-1,0}\end{bmatrix}\in R^{2n_2\times n_1},$$

$$A_{22}=\begin{bmatrix}
a_{n_1,n_2-1} & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
a_{n_1,n_2-2} & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
a_{n_1,2} & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
a_{n_1,1} & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
a_{n_1,0} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
b_{n_1,n_2-1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0\\
b_{n_1,n_2-2} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
b_{n_1,2} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0\\
b_{n_1,1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1\\
b_{n_1,0} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}$$
$$\in R^{2n_2\times 2n_2},$$

$$B_1=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}\in R^{n_1\times 1},\quad B_2=\begin{bmatrix}a_{n_1,n_2-1}\\a_{n_1,n_2-2}\\\vdots\\a_{n_1,0}\\b_{n_1,n_2-1}\\b_{n\ ,n_2-2}\\\vdots\\b_{n_1,0}\end{bmatrix}\in R^{2n_2\times 1},$$

$$C_1=\begin{bmatrix}\overline{b}_{0,n_2} & \overline{b}_{1,n_2} & \dots & \overline{b}_{n_1-1,n_2}\end{bmatrix}\in R^{1\times n}, \tag{3.10}$$

$$C_2=\begin{bmatrix}C_{21} & C_{22}\end{bmatrix}\in R^{1\times 2n_2},\quad D=\begin{bmatrix}b_{n_1,n_2}\end{bmatrix}\in R^{1\times 1}$$

and

$$C_{21}=\begin{bmatrix}b_{n_1,n_2} & 0 & \dots & 0\end{bmatrix}\in R^{1\times n_2},\quad C_{22}=\begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}\in R^{1\times n_2} \tag{3.11}$$

where

$$\overline{a}_{i,j}=a_{i,j}+a_{i,n_2}a_{n_1,j},\ \overline{b}_{i,j}=b_{i,j}+a_{i,n_2}b_{n_1,j}\ \text{for}$$

$$i=0,1,\dots,n_1-1;\ j=0,1,\dots,n_2-1. \tag{3.12}$$

Taking under consideration that $A_{1\alpha}=A_{11}+\alpha I_{n_1}$, $A_{1\beta}=A_{22}+\beta I_{n_2}$ the following theorem has been proved.

Theorem 3.1. There exists a positive realization (2.5) of the 2D different orders fractional system (2.1) with $0<\alpha<1$, $0<\beta<1$ if all coefficients of the numerator and denominator of the transfer function $T(w_{\alpha,}\, w_\beta)$ are nonnegative.

If the assumptions of Theorem 3.1 are satisfied then a positive realization (2.5) of (2.11) can be found by the use of the following procedure.

Procedure 3.1.

Step 1. Write the transfer function $T(w_{\alpha,}\, w_\beta)$ in the form (3.3) and the equations (3.5).

Step 2. Using (3.5) draw the state variable diagram shown in Fig. 3.1.

Step 3. Choose the state variables and write equations (3.4).

Step 4. Using (3.10) to (3.12) find the realization (3.10).

Step 5. Knowing fractional orders $\alpha,\ \beta$ and using (2.4) to matrices (3.10) compute the desired positive realization of the transfer function (2.11).

Example 3.1. Find a positive realization (2.5) of the proper transfer function where $\alpha=\beta=0{,}5$.

$$T(w_\alpha,w_\alpha)=\frac{6w_\alpha^{\ 2}w_\beta+5w_\alpha^{\ 2}+4w_\alpha w_\beta+3w_\alpha+2w_\beta+1}{w_\alpha^{\ 2}w_\beta-0.5w_\alpha^{\ 2}-0.4w_\alpha w_\beta-0.3w_\alpha-0.2w_\beta-0.1} \tag{3.13}$$

In this case $n_1=2$ and $n_2=1$.

Using Procedure 3.1 we obtain the following.

Step 1. Multiplying the nominator and denominator of Transfer function (3.13) by $w_\alpha^{-2}w_\beta^{-1}$ we obtain

$$T(s,z)=\frac{Y}{U}=\frac{6+5w_\beta^{\ -1}+4w_\alpha^{\ -1}+3w_\alpha^{\ -1}w_\beta^{\ -1}+2w_\alpha^{\ -2}+w_\alpha^{\ -2}w_\beta^{\ -1}}{1-0.5w_\beta^{\ -1}-0.4w_\alpha^{\ -1}-0.3w_\alpha^{\ -1}w_\beta^{\ -1}-0.2w_\alpha^{\ -2}-0.1s^{-2}w_\beta^{\ -1}} \tag{3.14}$$

and

$$\begin{aligned}E&=U+(0.5w_\beta^{\ -1}+0.4w_\alpha^{\ -1}+0.3w_\alpha^{\ -1}w_\beta^{\ -1}+0.2w_\alpha^{\ -2}+0.1s^{-2}w_\beta^{\ -1})E\\ Y&=(6+5w_\beta^{\ -1}+4w_\alpha^{\ -1}+3w_\alpha^{\ -1}w_\beta^{\ -1}+2w_\alpha^{\ -2}+w_\alpha^{\ -2}w_\beta^{\ -1})E\end{aligned} \tag{3.15}$$

Step 2. State variable diagram has the form shown in Fig. 3.2

Step 3. Using state variable diagram we can write the following different orders fractional equations

$$\begin{aligned}&\Delta^\alpha x_{1,1}(k+1)=x_{1,2}(k)\\ &\Delta^\alpha x_{1,2}(k+1)=0.2x_{1,1}(k)+0.4x_{1,2}(k)+x_{2,1}(k)+u(k)\\ &\Delta^\beta x_{2,1}(k+1)=0.2x_{1,1}(k)+0.5x_{1,2}(k)+0.5x_{2,1}(k)+0.5u(k)\\ &\Delta^\beta x_{2,2}(k+1)=2x_{1,1}(k)+5x_{1,2}(k)+5x_{2,1}(k)+5u(k)\\ &y(k)=3.2x_{1,1}(k)+6.4x_{1,2}(k)+6x_{2,1}(k)+x_{2,2}(k)+6u(k)\end{aligned} \tag{3.16}$$

Step 4. Defining state vectors

$$x_1(k)=\begin{bmatrix}x_{1,1}(k)\\x_{1,2}(k)\end{bmatrix},\quad x_2(k)=\begin{bmatrix}x_{2,1}(k)\\x_{2,2}(k)\end{bmatrix} \tag{3.17}$$

we can write the equations (3.16) in the form

$$\begin{aligned}\begin{bmatrix}\Delta^\alpha x_1(k+1)\\\Delta^\beta x_2(k+1)\end{bmatrix}&=\begin{bmatrix}0&1&0&0\\0.2&0.4&1&0\\0.2&0.5&0.5&0\\2&5&5&0\end{bmatrix}\begin{bmatrix}x_{1,1}(k)\\x_{1,2}(k)\\x_{2,1}(k)\\x_{2,2}(k)\end{bmatrix}+\begin{bmatrix}0\\1\\0.5\\5\end{bmatrix}u(k)\\ &=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+\begin{bmatrix}B_1\\B_2\end{bmatrix}u(k)\\ y(k)&=\begin{bmatrix}3.2&6.4&6&1\end{bmatrix}\begin{bmatrix}x_{1,1}(k)\\x_{1,2}(k)\\x_{2,1}(k)\\x_{2,2}(k)\end{bmatrix}+[6]u(k)\\ &=\begin{bmatrix}C_1&C_2\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+Du(k)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&A_{11}=\begin{bmatrix}0&1\\0.2&0.4\end{bmatrix},\quad A_{12}=\begin{bmatrix}0&0\\1&0\end{bmatrix},\\ &A_{21}=\begin{bmatrix}0.2&0.5\\2&5\end{bmatrix},\quad A_{22}=\begin{bmatrix}0.5&0\\5&0\end{bmatrix},\\ &B_1=\begin{bmatrix}0\\1\end{bmatrix},\quad B_2=\begin{bmatrix}0.5\\5\end{bmatrix},\quad C_1=\begin{bmatrix}3.2&6.4\end{bmatrix}\\ &C_2=\begin{bmatrix}6&1\end{bmatrix},\quad D=[6].\end{aligned} \tag{3.19}$$

Step 5. Knowing that $\alpha = \beta = 0.5$ and using (2.4) we have

$$A_{1\alpha} = A_{11} + \alpha I_{n_1} = \begin{bmatrix} 0 & 1 \\ 0.2 & 0.4 \end{bmatrix} + 0.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.2 & 0.9 \end{bmatrix},$$
$$A_{1\beta} = A_{22} + \beta I_{n_2} = \begin{bmatrix} 0.5 & 0 \\ 5 & 0 \end{bmatrix} + 0.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}. \quad (3.20)$$

The conditions of Theorem 2.1 are satisfied and obtained realization (3.19) with (3.20) is positive.

4. CONCLUDING REMARKS

A method for computation of a positive realization of a given proper transfer matrix of 2D different orders fractional discrete-time linear systems has been proposed. Sufficient conditions for the existence of a positive realization of this class of systems have been established. A procedure for computation of a positive realization has been proposed. The effectiveness of the procedure has been illustrated by a numerical example. In general case the proposed procedure does not provide a minimal realization of a given transfer matrix. An open problem is formulation of the necessary and sufficient conditions for the existence of positive minimal realizations for 2D fractional systems in the general case as well as connection between minimal realization and controllability (observability) of this class of systems.

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