# STABILIZATION OF INERTIAL PLANT WITH TIME DELAY USING FRACTIONAL ORDER CONTROLLER

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**Abstract:** The paper presents the problem of designing of a fractional order controller satisfying the conditions of gain and phase margins of the closed-loop system with time-delay inertial plant. The transfer function of the controller follows directly from the use of Bode's ideal transfer function as a reference transfer function for the open loop system. Using the classical D-partition method and the gain-phase margin tester, a simple computational method for determining stability regions in the controller parameters plane is given. An efficient analytical procedure to obtain controller parameter values for specified gain and phase margin requirements is also given. The considerations are illustrated by numerical examples computed in MATLAB/Simulink.

## 1. INTRODUCTION

In recent years considerable attention has been paid to fractional calculus and its application in many areas in science and engineering (see, e.g. (Das, 2008; Kaczorek, 2011; Kilbas et al., 2006; Ostalczyk, 2008)).

In control system fractional order controllers are used to improve the performance of the feedback control loop. One of the most developed approaches to design robust and fractional order controllers is CRONE control methodology, French acronym of "Commande Robuste d'Ordre Non Entier" (non-integer order robust control) (Oustaloup 1991, 1995, 1999).

The fractional PID controllers, namely $PI^{\lambda}D^{\mu}$ controllers, including an integrator of $\lambda$ order and a differentiator of $\mu$ order were proposed in (Podlubny, 1994, 1999). Several design methods of tuning the $PI^{\lambda}D^{\mu}$ controllers were presented in (Monje et al., 2004; Valerio, 2005; Valerio and Costa, 2006). These methods are based on the mathematical description of the process. The first order-plant with time delay is the most frequently used model for tuning fractional and integral controllers (O'Dwyer, 2003).

The asymptotic stability is the basic requirement of a closed-loop system. Some methods for determining the asymptotic stability regions in the controller parameter space were proposed in (Hamamci, 2007; Ruszewski, 2008). Gain and phase margins are measures of relative stability for a feedback system, therefore the synthesis of control systems is very often based on them. In typical control systems the phase margin is from 30° to 60° whereas the gain margin is from 5dB to 10dB. In paper (Ruszewski, 2010) a simple method of determining the stability region (satisfying the conditions of gain and phase margins) in the parameter space of a fractional-order inertial plant with time delay and a fractional-order PI controller was given.

In this paper the methods for tuning a fractional order controller satisfying the conditions of gain and phase margins are given. The transfer function of the controller follows from the use of Bode's ideal transfer function as a reference transfer function for the open loop system (Barbosa et al., 2004; Boudjehem et al., 2008; Busłowicz and Nartowicz, 2009; Skogestad, 2001; Nartowicz, 2010). Using the D-partition method a simple and efficient computational method for determining stability regions in the controller parameters space is given. Moreover analytical forms directly expressing the controller parameters for specified gain and phase margin requirements are determined.

## 2. PROBLEM FORMULATION

Consider the feedback control system shown in Fig. 1. The main path of the control system includes the gain-phase margin tester $A\exp(-j\phi)$, where $A$ and $\phi$ are gain margin and phase margin respectively. This tester does not exist in the real control system, it is only used for tuning the controller.

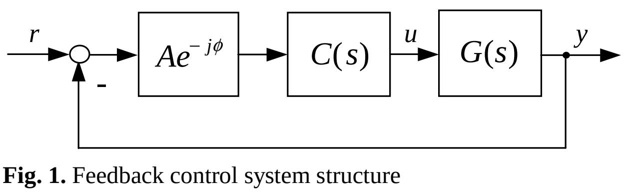


**Fig. 1.** Feedback control system structure

The process to be controlled is described by an inertial plant with time delay

$$G(s) = \frac{k}{1+sT}e^{-sh}, \qquad (1)$$

where $k$, $T$, $h$ are positive real numbers.

The transfer function of controller $C(s)$ directly follows from the use of Bode's ideal transfer function

$$K(s) = \left(\frac{\omega_c}{s}\right)^{\beta}, \tag{2}$$

as a reference transfer function of the open loop system, where $\omega_c$ is the gain crossover frequency ($|K(j\omega_c)| = 1$) and $\beta$ is the fractional order. Transfer function (2) describes the fractional derivative plant for $\beta < 0$ and the fractional integral plant for $\beta > 0$. The open loop system with transfer function (2) has a constant phase margin of the value $\phi_m = (1 - 0.5\beta)\pi$. Hence, such a system is insensitive to gain variation in the open loop system. Detailed analysis (including time domain) of the system considered is presented, for instance, in (Barbosa et al., 2004).

In order to obtain the transfer function of the open loop system in the form of transfer function (2), with expected time delay, we simplify the plant transfer function

$$G(s) = \frac{k}{1+sT}e^{-sh} \approx \frac{k}{sT}e^{-sh}. \tag{3}$$

Then the transfer function of the controller must have the form

$$C(s) = k_c s^{1-\alpha}, \tag{4}$$

where $\alpha$ is a positive real number. We will assume $\alpha > 1$.

The characteristic function of the closed-loop system with simplified transfer function (3), transfer function of controller (4) and gain-phase tester is given by

$$w(s) = Akk_c s^{1-\alpha} e^{-j\phi} e^{-sh} + sT. \tag{5}$$

The closed-loop system in Fig. 1 is said to be bounded-input bounded-output stable if and only if all the zeros o characteristic function (5) have negative real parts. It is noted that (5) is the fractional order quasi-polynomial which has an infinite number of zeros. This makes the problem of analysing the stability of the closed-loop system difficult. There is no general algebraic methods available in the literature for the stability test of fractional order quasi-polynomials. The next problem of closed-loop system synthesis is how to choose such a fractional order $\alpha$ of the controller that the closed-loop system will be stable and characterized by specified gain and phase margins.

The aim of the paper is to propose tuning methods based on gain and phase margin specifications. The first one is to give the method for determining the stability region in the parameter plane ($\alpha$, $k_c$). The second is to give a simple analytical formula to obtain the controller parameter values for specified gain and phase margin requirements.

## 3. MAIN RESULT

By using the D-partition method (Gryazina, 2004) the stability region in the parameter plane ($\alpha$, $k_c$) can be determined and the parameters can be specified. The plane ($\alpha$, $k_c$) is decomposed by the boundaries of the D-partition into finite number regions $D(k)$. Any point in $D(k)$ corresponds to such values of $k_c$ and $\alpha$ that quasi-polynomial (5) has exactly $k$ zeros with positive real parts. The region $D(0)$, if it exists, is the stability region of quasi-polynomial (5). The D-partition boundaries are curves on which each point corresponds to quasi-polynomial (5) having zeros on the imaginary axis. It may be the real zero boundary or the complex zero boundary. It is easy to see that quasi-polynomial (5) has zero $s = 0$ if $k_c = 0$ (the real zero boundary). The complex zero boundary corresponds to the pure imaginary zeros of (5). We obtain this boundary by solving the equation

$$w(j\omega) = Akk_c (j\omega)^{1-\alpha} e^{-j\phi} e^{-j\omega h} + j\omega T = 0, \tag{6}$$

which we obtain by substituting $s = j\omega$ in quasi-polynomial (5) and equating to 0. The term of $j^{\alpha}$ which is required for equation (6) can be expressed by

$$j^{\alpha} = \cos\left(\alpha\frac{\pi}{2}\right) + j\sin\left(\alpha\frac{\pi}{2}\right). \tag{7}$$

Using (7) equation (6) takes the form

$$\begin{aligned} &Akk_c\omega^{1-\alpha}\cos\left(\frac{\pi}{2}(\alpha-1)\right) - \omega T\sin(\omega h + \phi) + \\ &- jAkk_c\omega^{1-\alpha}\sin\left(\frac{\pi}{2}(\alpha-1)\right) + j\omega T\cos(\omega h + \phi) = 0. \end{aligned} \tag{8}$$

Complex equation (8) can be rewritten as a set of real equations in the form

$$Akk_c\omega^{1-\alpha}\cos\left(\frac{\pi}{2}(\alpha-1)\right) - \omega T\sin(\omega h + \phi) = 0, \tag{9}$$

$$-Akk_c\omega^{1-\alpha}\sin\left(\frac{\pi}{2}(\alpha-1)\right) + \omega T\cos(\omega h + \phi) = 0. \tag{10}$$

Finally, by solving equations (9) and (10) we get

$$\alpha = \frac{2(\pi - \omega h - \phi)}{\pi}, \tag{11}$$

$$k_c = \frac{T}{Ak}\omega^{\frac{2(\pi-\omega h-\phi)}{\pi}}. \tag{12}$$

Equations (11) and (12) determine the complex zero boundary as a function of $\omega$. The real zero boundary and the complex zero boundary for $\omega \geq 0$ decompose plane ($\alpha$, $k_c$) into regions $D(k)$. The stability region $D(0)$ is chosen by testing an arbitrary point from each region and checking the stability of quasi-polynomial (5) using the methods proposed in (Busłowicz, 2008). In this paper only the stability region $D(0)$ in the parameter plane of quasi-polynomial (5) is presented.

For $A = 1$ and $\phi = 0$ in (11) and (12) the stability boundaries are calculated. To determine the complex zero boundary for a given value of gain margin $A$ of the control system we should set $\phi = 0$ in (11) and (12). On the other hand by setting $A = 1$ in (12), we can obtain the boundary for a given phase margin $\phi$.

The complex zero boundary (11) and (12) is determined for parameter $\omega \geq 0$. The complex zero boundary for a given value of gain margin $A$ begins at the point $\alpha = 2$, $k_c = 0$ which we obtain by substituting $\omega = 0$ in (11) and (12). However, the complex zero boundary for the given phase margin $\phi$ starts at the point $\alpha = 2(\pi - \phi)/\pi$, $k_c = 0$. If $\omega \to \infty$ plot of the complex zero boundary tends towards $k_c$ -axis.

**Example 1.** Consider the feedback control system shown in Fig. 1. in which the process to be controlled is described by transfer function

$$G(s) = \frac{0.55}{1 + 62s} e^{-10s}. \tag{13}$$

On computing by the proposed method complex zero boundaries (11) and (12) we obtain the stability regions in controller parameter plane ($\alpha$, $k_c$).

Fig. 2 shows boundaries in controller parameter plane ($\alpha$, $k_c$) for gain margin $A = 1$ and a few values of phase margin $\phi$. The stability regions lie between line $k_c = 0$ (the real zero boundary) and the curve assigned to specified phase margin $\phi$ (the complex zero boundary).

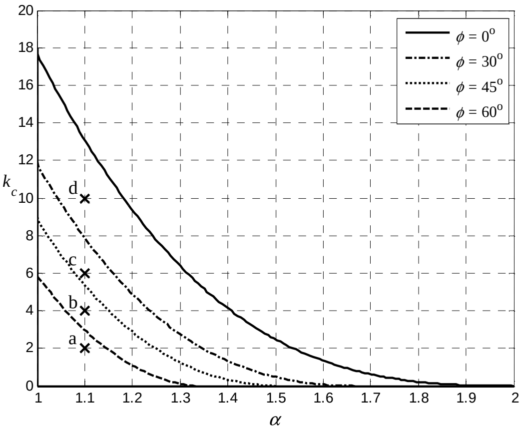


**Fig. 2.** Stability regions of quasi-polynomial (5) for $A$ = 1 and different values of $\phi$

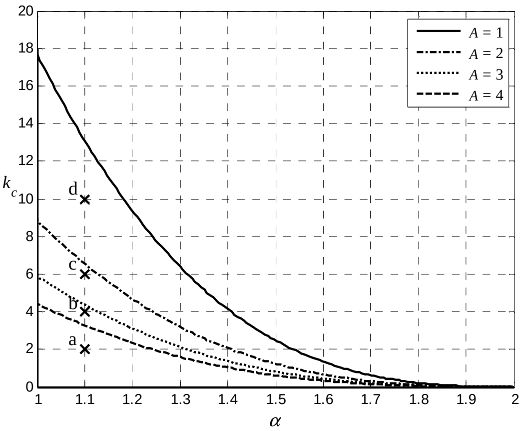


**Fig. 3.** Stability regions of quasi-polynomial (5) for $\phi = 0$ and different values of $A$

On choosing any point from the stability region we obtain the controller parameter values provided the phase margin of this system not less than specified for drawing the complex boundary. For example, any point from the region limited by the line $k_c = 0$ and the curve corresponding to $\phi = 60°$ provides a phase margin of this system not less than 60°. From Fig.2 we see that the increasing value of $\phi$ results in the disappearance of the stability region.

The stability regions of quasi-polynomial (5) for phase margin $\phi = 0$ and a few values of gain margin $A$ are shown in Fig.3. We see that increasing value of $A$ results in the disappearance of the stability region. On choosing any point from the stability region we obtain the controller parameter values provided that the gain margin of this system is not less than specified for drawing the complex boundary. For example a choosing point between $k_c = 0$ and the complex boundary for $A = 4$ we obtain the controller parameters satisfying a gain margin of not less than 4.

The controller parameters and stability margins of the control system for all points marked in Fig. 2 and Fig. 3 are listed in Tab. 1. It is shown that the stability margin values are larger than specified for drawing the complex boundaries of the stability regions. Gain and phase margins of the control system are calculated for transfer function (1).

**Tab. 1.** Gain and phase margins

| Point | Controller parameters | Gain margin | | Phase margin |
|---|---|---|---|---|
| a | $\alpha = 1.1,\ k_c = 2$ | 7.13 | 17.06 dB | 107.36° |
| b | $\alpha = 1.1,\ k_c = 4$ | 3.56 | 11.64 dB | 74.38° |
| c | $\alpha = 1.1,\ k_c = 6$ | 2.38 | 7.52 dB | 55.51° |
| d | $\alpha = 1.1,\ k_c = 10$ | 1.43 | 3.08 dB | 26.64° |

Tab. 1 confirms the results received on the basis of the D-partition method showing that the points from the stability regions satisfy the gain and phase margin requirements.

The step responses of the control system are presented in Fig. 4. It can be seen that the increasing value of $\phi$ results in smaller oscillations.

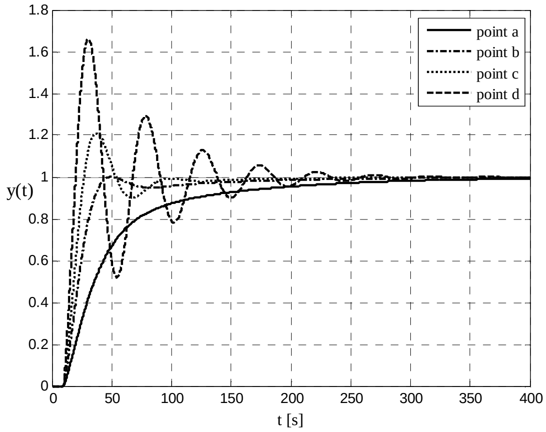


**Fig. 4.** Step responses of control system

By using the stability regions we can obtain the controller parameter values for specified gain and phase margins requirements simultaneously. For this purpose we draw in one plot the complex zero boundary for specified phase margin $\phi$ with $A = 1$ and the complex zero boundary for specified gain margin $A$ with $\phi = 0$. Intersection point of the complex zero boundaries determines the controller parameter values.

**Example 2.** Consider the feedback control system as in Example 1. Calculate the controller parameter values so that the control system has the gain margin $A = 4$ (about 12 dB) and the phase margin $\phi = 55°$.

On computing the complex zero boundaries (11) and (12) for specified gain margin $A = 4$ with $\phi = 0$ and for specified phase margin $\phi = 55°$ with $A = 1$ we obtain the stability regions which are shown in Fig. 5. The intersection point of the complex zero boundaries is marked on Fig. 5 and has coordinates $\alpha = 1.1339$, $k_c = 2.9358$. On calculating the stability margins of control system for simplified transfer function (3) we obtain $A = 4$ and $\phi = 55°$. Whereas stability margins for model plant (1) are $A = 4.4$ and $\phi = 80°$ because of simplification (3). Fig. 6 shows the Bode plot with the gain and phase margins marked for controller parameters $\alpha = 1.1339$, $k_c = 2.9358$.

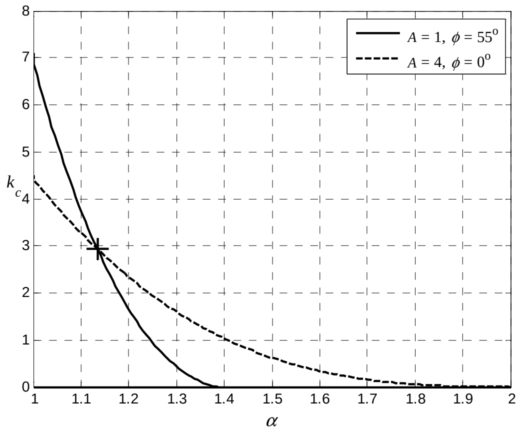


**Fig. 5.** Stability regions of quasi-polynomial (5) for $A = 1$, $\phi = 55°$ and $A = 4$, $\phi = 0$

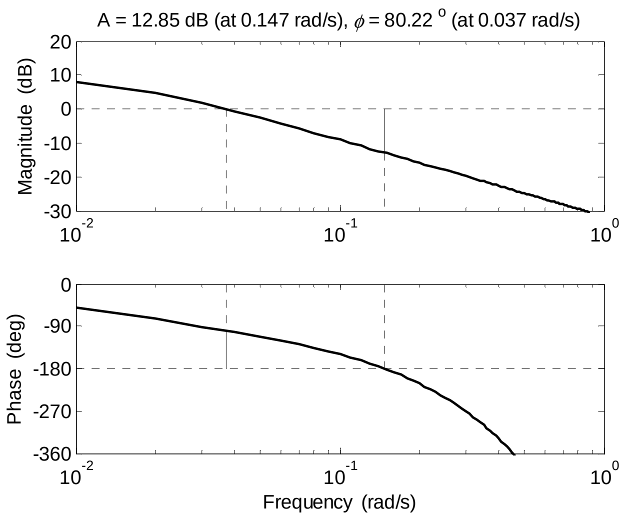


**Fig. 6.** Bode plot with gain and phase margins

By using expressions of the stability boundaries (11) and (12) we can determine analytical description for direct calculations of the controller parameter values for specified gain and phase margins requirements without drawing the stability region.

To determine the complex zero boundary for a given value of gain margin $A$ of the control system we set $\phi = 0$ in (11) and (12). On solving system of equations (11) and (12) for the unknown quantities of $\omega$ and $k_c$ with $\phi = 0$ we obtain

$$\omega = \frac{\pi(2-\alpha)}{2h}, \tag{14}$$

$$k_c = \frac{T}{Ak}\left(\frac{\pi(2-\alpha)}{2h}\right)^{\alpha}. \tag{15}$$

Expression (15) gives the relationship between $k_c$ and $\alpha$ for specified gain margin $A$.

Similarly to determine the complex zero boundary for a given phase margin $\phi$ of the control system we set $A = 1$ in (12). On solving system of equations (11) and (12) for the unknown quantities of $\omega$ and $k_c$ with $A = 1$ we obtain

$$\omega = \frac{\pi(2-\alpha)-2\phi}{2h}, \tag{16}$$

$$k_c = \frac{T}{k}\left(\frac{\pi(2-\alpha)-2\phi}{2h}\right)^{\alpha}. \tag{17}$$

Expression (17) gives the relationship between $k_c$ and $\alpha$ for specified phase margin $\phi$.

Note from Fig.5 that for fixed value of $\alpha$ which guarantees gain and phase margins requirements simultaneously the values of $k_c$ in the two complex boundaries are the same (the intersection point). Therefore the value of $\alpha$ which ensures gain and phase margins requirements can be calculated by solving following nonlinear equation

$$\frac{T}{Ak}\left(\frac{\pi(2-\alpha)}{2h}\right)^{\alpha} = \frac{T}{k}\left(\frac{\pi(2-\alpha)-2\phi}{2h}\right)^{\alpha}. \tag{18}$$

After simplifications equation (18) can be rewritten in the form

$$A = \left(\frac{\pi(2-\alpha)}{\pi(2-\alpha)-2\phi}\right)^{\alpha}. \tag{19}$$

If we get the value of $\alpha$ from (19) we can calculate controller gain $k_c$ from expression (15) or (17).

From the above it can be seen that the procedure for calculating parameters of controller (4) for specified gain and phase margins requirements is as follows:

1. Solve the nonlinear equation (19) and determine $\alpha$.
2. Calculate controller gain $k_c$ from expression (15) or (17).

Note that in the procedure proposed the calculation of the gain crossover frequency or the phase crossover frequency is not necessary in contrast to methods presented in (Boudjehem et al., 2008; Busłowicz and Nartowicz, 2009; Nartowicz, 2010). The advantage of the procedure proposed is that the controller settings are easily calculated.

**Example 3.** Consider the feedback control system as in Example 2. Using the procedure presented calculate the controller parameters values so that the control system has gain margin $A = 4$ (about 12 dB) and phase margin $\phi = 55°$.

On solving nonlinear equation (19) we have $\alpha = 1.1339$. From (15) or (17) we calculate controller gain $k_c = 2.9358$. Note that we obtain the same values of the controller parameter as in Example 2.

Gain and phase margins are measures of relative stability for a feedback system. Although the phase margin is used more frequently than both margins. The phase margin is closely related to transient response i.e. overshoot.

From the above it can be seen that the procedure for calculating parameters of controller (4) for specified phase margin requirement is as follows:

1. Calculate the start point of the complex zero boundary $\alpha = 2(\pi - \phi)/\pi$.
2. Choose any positive value smaller than determined $\alpha$.
3. Calculate controller gain $k_c$ from expression (17).

In the above procedure solving nonlinear equation is not necessary.

## 4. CONCLUSION

In this paper the stability problem of control systems composed of the fractional-order controller and the inertial plant with time delay is examined. On the basis of the D-partition method analytical forms expressing the boundaries of stability regions in the parameter space for specified gain and phase margin requirements were determined. When the the stability regions are known the tuning of the fractional controller can be carried out. Simple analytical formulas for obtaining the controller parameter values for specified gain and phase margins requirements were also given. In the method proposed the controller settings are easily calculated.

The calculations and simulations were made using the Matlab/Simulink programme.

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