# SOLUTIONS TO TIME-FRACTIONAL DIFFUSION-WAVE EQUATION IN SPHERICAL COORDINATES

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**Abstract:** Solutions to time-fractional diffusion-wave equation with a source term in spherical coordinates are obtained for an infinite medium. The solutions are found using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate $\varphi$, the Legendre transform with respect to the spatial coordinate $\mu$, and the Hankel transform of the order n+1/2 with respect to the radial coordinate $r$. In the central symmetric case with one spatial coordinate $r$ the obtained results coincide with those studied earlier.

## 1. INTRODUCTION

The time-fractional diffusion-wave equation

$$\frac{\partial^{\alpha} c}{\partial t^{\alpha}} = a \Delta c \tag{1}$$

is a mathematical model of important physical phenomena ranging from amorphous, colloid, glassy and porous materials through fractals, percolation clusters, random and disordered media to comb structures, dielectrics and semiconductors, polymers and biological systems (see Metzler and Klafter, 2000, 2004; Povstenko, 2005; Magin, 2006; Uchaikin, 2008, among many others, and references therein).

The fundamental solution for the time-fractional diffusion-wave equation in one Cartesian space-dimension was obtained by Mainardi (1996). Wyss (1986) obtained the solutions to the Cauchy problem in terms of *H*-functions using the Mellin transform. Schneider and Wyss (1989) converted the diffusion-wave equation with appropriate initial conditions into the integrodifferential equation and found the corresponding Green functions in terms of Fox functions. Fujita (1990) treated integrodifferential equation which interpolates the heat conduction equation and the wave equation.

Previously, in studies concerning this equation in spherical coordinates only central symmetric case has been investigated (Povstenko, 2008a, 2008b, 2008c; Lenci et al., 2009, Qi and Liu, 2010). In this paper we investigate solutions to time-fractional diffusion-wave equation in an infinite medium in spherical coordinate system in the case of three spatial coordinates $r$, $\theta$, and $\varphi$.

Consider the time-fractional diffusion-wave equation with a source term

$$\frac{\partial^{\alpha} c}{\partial t^{\alpha}} = a\left[\frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial c}{\partial \theta}\right)\right.$$

$$\left. + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 c}{\partial \varphi^2}\right] + Q(r,\theta,\varphi,t), \tag{2}$$

$$0 \le r < \infty \,,\; 0 \le \theta \le \pi \,,\; 0 \le \varphi \le 2\pi \,,\; 0 < t < \infty \,,\; 0 < \alpha \le 2 \,.$$

Here we use the Caputo fractional derivative (see Gorenflo and Mainardi, 1997; Kilbas et al., 2006; Klimek, 2009)

$$\frac{d^{\alpha}c(t)}{dt^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1}\dfrac{d^n c(\tau)}{dt^n}d\tau, \\ \qquad\qquad\qquad\qquad n-1 < \alpha < n, \\ \dfrac{d^n c(t)}{dt^n}, \qquad\qquad\qquad \alpha = n, \end{cases} \tag{3}$$

where $\Gamma(\alpha)$ is the gamma function.

For its Laplace transform rule the Caputo fractional derivative requires the knowledge of the initial values of the function $c(t)$ and its integer derivatives of the order $k = 1,2,...,n-1$:

$$L\left\{\frac{d^{\alpha}c(t)}{dt^{\alpha}}\right\} = s^{\alpha}L\{c(t)\} - \sum_{k=0}^{n-1} c^{(k)}(0^+)s^{\alpha-1-k}\,, \tag{4}$$

$$n-1 < \alpha < n,$$

where $s$ is the transform variable.

Change of variable $\mu = cos\theta$ in (2) leads to the following equation

$$\frac{\partial^{\alpha} c}{\partial t^{\alpha}} = a\left[\frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left(\left(1-\mu^2\right)\frac{\partial c}{\partial \mu}\right)\right.$$

$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 c}{\partial \varphi^2}\right] + Q(r,\mu,\varphi,t), \tag{5}$$

$$0 \le r < \infty \,,\; -1 \le \mu \le 1 \,,\; 0 \le \varphi \le 2\pi \,,\; 0 < t < \infty \,,\; 0 < \alpha \le 2 \,.$$

For simplicity, we have not introduced different letters for $Q(r,\theta,\varphi,t)$ and $Q(r,\mu,\varphi,t)$. For equation (5) the initial conditions are prescribed:

$$t = 0: \quad c = f(r, \mu, \varphi), \; 0 < \alpha \le 2, \tag{6}$$

$$t = 0: \quad \frac{\partial c}{\partial t} = F(r, \mu, \varphi), \; 1 < \alpha \le 2. \tag{7}$$

The solution to the initial value problem (5)-(7) can be written in the following form

$$c = \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} f(\rho, \zeta, \phi) G_f(r, \mu, \varphi, \rho, \zeta, \phi, t) \rho^2 d\rho \, d\zeta \, d\phi$$

$$+ \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} F(\rho, \zeta, \phi) G_F(r, \mu, \varphi, \rho, \zeta, \phi, t) \rho^2 d\rho \, d\zeta \, d\phi$$

$$+ \int_0^{t} \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} Q(\rho, \zeta, \phi, \tau)$$

$$\times G_Q(r, \mu, \varphi, \rho, \zeta, \phi, t - \tau) \rho^2 d\rho \, d\zeta \, d\phi \, d\tau. \tag{8}$$

In the subsequent text, we investigate the fundamental solutions for the first Cauchy problem $G_f(r, \mu, \varphi, \rho, \xi, \phi, t)$, to the second Cauchy problem $G_F(r, \mu, \varphi, \rho, \xi, \phi, t)$, and for the source problem $G_Q(r, \mu, \varphi, \rho, \xi, \phi, t)$.

## 2. FUNDAMENTAL SOLUTION TO THE FIRST CAUCHY PROBLEM

Let us examine the time-fractional diffusion-wave equation

$$\frac{\partial^\alpha G_f}{\partial t^\alpha} = a \left[ \frac{\partial^2 G_f}{\partial r^2} + \frac{2}{r} \frac{\partial G_f}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial G_f}{\partial \mu} \right) \right.$$

$$\left. + \frac{1}{r^2(1 - \mu^2)} \frac{\partial^2 G_f}{\partial \varphi^2} \right], \tag{9}$$

$$0 \le r < \infty, \; -1 \le \mu \le 1, \; 0 \le \varphi \le 2\pi, \; 0 < t < \infty, \; 0 < \alpha \le 2,$$

with the prescribed initial value of a function

$$t = 0: \quad G_f = \frac{1}{r^2} \delta(r - \rho) \delta(\mu - \zeta) \delta(\varphi - \phi), \tag{10}$$

$$0 < \alpha \le 2,$$

$$t = 0: \quad \frac{\partial G_f}{\partial t} = 0, \; 1 < \alpha \le 2. \tag{11}$$

The three-dimensional Dirac delta function $\delta(x)\delta(y)\delta(z)$ after passing to the spherical coordinates takes the form $\frac{1}{\pi r^2} \delta_+(r)$, but for the sake of simplicity we have omitted the factor $4\pi$ in the solution (8) as well as the factor $(4\pi)^{-1}$ in the initial condition (10).

Now we introduce the new looked-for function $v = \sqrt{r}c$ and use the Laplace transform with respect to time $t$, the finite Fourier transform with respect to the angular coordinate $\varphi$, the Legendre transform with respect to the coordinate $\mu$, and the Hankel transform of the order $n + 1/2$ with respect to the radial coordinate $r$. The details of application the integral transform technique to the Laplace operator in spherical coordinates can be found in the book of Özişik (1980). In the transforms domain we obtain

$$v^*(\xi, m, n, \varphi, \rho, \zeta, \phi, s) = \frac{1}{\sqrt{\rho}} J_{n+1/2}(\rho\xi) P_n^m(\zeta)$$

$$\times \cos[m(\varphi - \phi)] \frac{s^{\alpha-1}}{s^\alpha + a\xi^2}, \tag{12}$$

where the asterisk indicates the transforms, $J_{n+1/2}(r)$ is the Bessel function of the first kind of order $n + 1/2$, $P_n^m(\xi)$ are the associated Legendre polynomials of degree $n$ and order $m$, $s$ is the Laplace transform variable, $\xi$ is the Hankel transform variable, the integer $m$ is the Fourier transform variable, and the integers $n$ and $m$ are the Legendre transform variables.

To invert the Laplace transform we use the following result (Gorenflo and Mainardi, 1997; Kilbas et al., 2006)

$$L^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + a\xi^2} \right\} = E_\alpha(-a\xi^2 t^\alpha), \tag{13}$$

where $E_\alpha(z)$ is the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \; \alpha > 0, \; z \in C. \tag{14}$$

For large values of argument the Mittag-Leffler function is represented as

$$E_\alpha(-a\xi^2 t^\alpha) \approx \frac{1}{\Gamma(1 - \alpha)} \frac{1}{a\xi^2 t^\alpha}. \tag{15}$$

Inversion of all the integral transforms gives:

$$G_f(r, \mu, \varphi, \rho, \zeta, \phi, t) = \frac{1}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} {}' \frac{2n+1}{2}$$

$$\times \frac{(n-m)!}{(n+m)!} P_n^m(\mu) P_n^m(\zeta) \cos[m(\varphi - \phi)] \tag{16}$$

$$\times \int_0^{\infty} E_\alpha(-a\xi^2 t^\alpha) J_{n+1/2}(r\xi) J_{n+1/2}(\rho\xi) \xi \, d\xi,$$

where the prime near the summation symbol denotes that the term corresponding to $m = 0$ in the sum should be multiplied by the factor $1/2$.

In the central symmetric case ($m = 0, n = 0$), taking into account that the Bessel functions of the first kind of the order one half can be represented as

$$J_{1/2}(r) = \sqrt{\frac{2r}{\pi}} \frac{\sin r}{r}, \tag{17}$$

from (16) we get

$$G_f(r, \rho, t) = \frac{1}{2\pi^2 r\rho} \int_0^{\infty} E_\alpha(-a\xi^2 t^\alpha)$$

$$\times \sin(r\xi) \sin(\rho\xi) d\xi. \tag{18}$$

Solution (18) was obtained earlier by Povstenko (2008c) using sin-Fourier transform with respect to the radial coordinate $r$. The limiting case of (18) under $\rho \to 0$,

$$G_f(r,\rho,t) = \frac{1}{2\pi^2 r} \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) \sin(r\xi)\xi\, d\xi\,, \qquad (19)$$

was also investigated earlier (Povstenko, 2008b).

Asymptotic behavior of Mettag-Leffler function $E_\alpha(-\alpha\xi^2 t^\alpha)$ (15) is responsible for appearance of singularity of the solution (16) at the point of applying the delta pulse: $r = \rho$, $\mu = \xi$, $\varphi = \phi$ also for $t > 0$. The sign of the singularity depends on $\alpha$: plus for $0 < \alpha < 1$ and minus for $1 < \alpha < 2$. Only the solution to the classical diffusion equation ($\alpha = 1$ and $E_1(-\alpha\xi^2 t^\alpha) = \exp(-\alpha\xi^2 t^\alpha)$) has no singularity.

## 3. FUNDAMENTAL SOLUTION TO THE SECOND CAUCHY PROBLEM

In the case of the second Cauchy problem, which is considered for the order of time derivative $1 < \alpha \le 2$, the initial value of the time derivative of the sought-for function is prescribed, and for the corresponding fundamental solution we have

$$\frac{\partial^\alpha G_F}{\partial t^\alpha} = a\left[\frac{\partial^2 G_F}{\partial r^2} + \frac{2}{r}\frac{\partial G_F}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left(\left(1-\mu^2\right)\frac{\partial G_F}{\partial \mu}\right)\right.$$

$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 G_F}{\partial \varphi^2}\right], \qquad (20)$$

$0 \le r < \infty$, $-1 \le \mu \le 1$, $0 \le \varphi \le 2\pi$, $0 < t < \infty$, $1 < \alpha \le 2$,

with the following initial conditions:

$$t = 0: \quad G_F = 0,\ 1 < \alpha \le 2\,, \qquad (21)$$

$$t = 0: \quad \frac{\partial G_F}{\partial t} = \frac{1}{r^2}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\delta(\varphi-\phi), $$

$$1 < \alpha \le 2\,. \qquad (22)$$

The integral transform technique allows us to remove the partial derivatives and to get the expression for the auxiliary function $v$ in the transforms domain

$$v*(\xi,m,n,\varphi,\rho,\zeta,\phi,s) = \frac{1}{\sqrt{\rho}}J_{n+1/2}(\rho\xi)P_n^m(\zeta)$$

$$\times \cos[m(\varphi-\phi)]\frac{s^{\alpha-2}}{s^\alpha + a\xi^2}\,. \qquad (23)$$

After inversion of integral transforms we gain

$$G_F(r,\mu,\varphi,\rho,\zeta,\phi,t) = \frac{1}{\pi\sqrt{r\rho}}\sum_{n=0}^\infty \sum_{m=0}^n {}' \frac{2n+1}{2}$$

$$\times \frac{(n-m)!}{(n+m)!}P_n^m(\mu)P_n^m(\zeta)\cos[m(\varphi-\phi)] \qquad (24)$$

$$\times \int_0^\infty t\, E_{\alpha,2}\left(-a\xi^2 t^\alpha\right)J_{n+1/2}(r\xi)J_{n+1/2}(\rho\xi)\xi\, d\xi,$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function in two parameters $\alpha$ and $\beta$ (Gorenflo and Mainardi, 1997; Kilbas et al., 2006)

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}\,, \qquad (25)$$

$\alpha > 0$ , $\beta > 0$, $z \in C$.

We have used the following formula for the inverse Laplace transform

$$L^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha + a\xi^2}\right\} = t^{\beta-1}E_{\alpha,\beta}\left(-a\xi^2 t^\alpha\right). \qquad (26)$$

In the central symmetric case we have (Povstenko, 2008c)

$$G_F(r,\rho,t) = \frac{1}{2\pi^2 r\rho}\int_0^\infty t\, E_{\alpha,2}\left(-a\xi^2 t^\alpha\right)$$

$$\times \sin(r\xi)\sin(\rho\xi)d\xi\,. \qquad (27)$$

It should be noted that due to the behavior of the Mittag-Leffler function $E_{\alpha,2}(-\alpha\xi^2 t^\alpha)$ for large values of argument

$$E_{\alpha,2}\left(-a\xi^2 t^\alpha\right) \approx \frac{1}{\Gamma(2-\alpha)}\frac{1}{a\xi^2 t^\alpha}\,,\ 1 < \alpha < 2\,, \qquad (28)$$

the fundamental solution (24) has the singularity with the positive sign at the point of applying the delta pulse for $t > 0$ and all values of $1 < \alpha < 2$.

## 4. FUNDAMENTAL SOLUTION TO THE SOURCE PROBLEM

Consider the time-fractional diffusion equation with a source term being the time and space delta pulse applied at point with the spatial coordinates $\rho, \xi$ and $\phi$.

$$\frac{\partial^\alpha G_Q}{\partial t^\alpha} = a\left[\frac{\partial^2 G_Q}{\partial r^2} + \frac{2}{r}\frac{\partial G_Q}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left(\left(1-\mu^2\right)\frac{\partial G_Q}{\partial \mu}\right)\right.$$

$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 G_Q}{\partial \varphi^2}\right]$$

$$+ \frac{1}{r^2}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\delta(\varphi-\phi)\,\delta_+(t), \qquad (29)$$

$0 \le r < \infty$, $-1 \le \mu \le 1$, $0 \le \varphi \le 2\pi$, $0 < t < \infty$, $0 < \alpha \le 2$,

under zero initial conditions

$$t = 0: \quad G_Q = 0,\ 0 < \alpha \le 2\,, \qquad (30)$$

$$t = 0: \quad \frac{\partial G_Q}{\partial t} = 0,\ 1 < \alpha \le 2\,. \qquad (31)$$

Using integral transform, we arrive at

$$v* = \frac{1}{\sqrt{\rho}}J_{n+1/2}(\rho\xi)P_n^m(\zeta)\cos[m(\varphi-\phi)]\frac{1}{s^\alpha + a\xi^2}\,, \qquad (32)$$

and after inversion of integral transforms

$$G_Q(r,\mu,\varphi,\rho,\zeta,\phi,t) = \frac{1}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} {}' \frac{2n+1}{2}$$

$$\times \frac{(n-m)!}{(n+m)!} P_n^m(\mu) P_n^m(\zeta) \cos[m(\varphi-\phi)] \qquad (33)$$

$$\times \int_0^{\infty} t^{\alpha-1} E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) J_{n+1/2}(r\xi) J_{n+1/2}(\rho\xi)\, \xi\, d\xi\, .$$

In the central symmetric case we have (Povstenko, 2008c)

$$G_Q(r,\rho,t) = \frac{1}{2\pi^2 r\rho} \int_0^{\infty} t^{\alpha-1} E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right)$$

$$\times \sin(r\xi)\sin(\rho\xi)d\xi\, . \qquad (34)$$

Due to the behavior of the Mittag-Leffler function $E_{\alpha,\alpha}(-\alpha\xi^2 t^\alpha)$ for large values of argument

$$E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) \approx -\frac{1}{\Gamma(-\alpha)} \frac{1}{a^2\xi^4 t^{2\alpha}} \qquad (35)$$

the solution (33) has no singularity at the point of applying the delta pulse for $t > 0$.

## 5. CONCLUSIONS

The new solutions to the Cauchy and source problems for time-fractional diffusive-wave equation have been obtained for an infinite medium referred to spherical coordinate system $r, \theta, \varphi$. For the first time, the non-central-symmetric case has been considered. The found solutions satisfy the appropriate initial conditions and reduce to the solutions of classical diffusion equation in the limit $\alpha = 1$ and of the standard wave equation in the case of ballistic diffusion ($\alpha = 2$). Our results provide a new analytical tool for studying anomalous diffusion.

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