LINEAR q-DIFFERENCE FRACTIONAL ORDER CONTROL SYSTEMS WITH FINITE MEMORY

Dorota MOZYRSKA*, Ewa PAWŁUSZEWICZ**

*Faculty of Computer Science, Białystok University of Technology, 15-351 Białystok, Poland
**Faculty of Mechanical Engineering, Białystok University of Technology, 15-351 Białystok, Poland

d.mozyrska@pb.edu.pl, e.pawluszewicz@pb.edu.pl

Abstract: The formula for the solution to linear q-difference fractional-order control systems with finite memory is derived. New definitions of observability and controllability are proposed by using the concept of extended initial conditions. The rank condition for observability is established and the control law is stated.

1. INTRODUCTION

Recently the concept of fractional derivatives and differences is under strong consideration as a tool in descriptions of behaviors of real systems. In modeling the real phenomena authors emphatically use generalizations of n-th order differences to their fractional forms and consider the state-space equations of control systems in discrete-time, (e.g. Guermah, Djennoune and Bettayeb, 2008; Sierociuk and Dzieliński, 2006). Some problems and special attempt to the fractional q-calculus was provided and presented in Atici and Eloe (2007). The possible application of fractional q-difference was proposed by Ortigueira (2008).

In the generalization of classical discrete-case differences to fractional-order differences it is convenient to take finite summation (see: Kaczorek, 2007; Kaczorek, 2008; Guermah, Djennoune and Bettayeb, 2008; Sierociuk and Dzieliński, 2006). On the other hand there is no good reason for that. The way we use the fractional difference does not introduce any doubt on the initial condition problems for fractional linear systems in discrete-case. Moreover, what seems to be one of the greatest phenomena in using fractional derivatives and differences in systems modeling real behaviors is the initialization of systems. In fact the initial value problem is an important task in daily applications. Recently we can find papers dealing with the problem how to impose initial conditions for fractional-order dynamics, (e. g. Ortigueira and Coito, 2007; Lorenzo and Hartley, 2009; Atici and Eloe, 2009).

In this paper we deal with q-fractional difference control systems with the initialization by an additional function φ that vanishes on a time interval with infinitely many points. In that way we get only finite number of values of initializing function φ that can be nonzero. We call such set, stated as the extended vector, by l-memory. Hence a control system is defined together with initializing point of time and length of the memory.

We present the construction of the solution to l-memory initial value problem and discuss the observability and controllability in s-steps conditions for such system. Some results concerning the autonomous linear q-difference fractional-order system with l-memory were presented in Mozyrska and Pawłuszewicz (2010). Although we take as initial states the extended vectors for the initial memory, we restrict definition of indistinguishability relation and observability to those defined for s-steps, similarly as it is proposed in Mozyrska and Bartosiewicz (2010). We state the problem in the classical way, using the rank of observability matrix. For controllability we formulate the control law using recursively defined Gramian.

The paper is organized as follows. In Section 2 the foundation of fractional q-derivative is presented and it is showed that forward trajectory of linear q-difference fractional order control system with l-memory is uniquely defined. In Section 3 observability problem in finite memory domain is stated. Proposition 3.3 gives another, then in Mozyrska and Pawłuszewicz (2010), observability rank condition. Section 4 presents solution of controllability problem in finite memory domain.

2. FRACTIONAL q-DERIVATIVE AND q-DIFFERENCE SYSTEMS

Firstly we recall some basic facts connected with q-difference systems. Let $q \in (0,1)$. By q-difference of a function $f\colon \mathrm{R} \to \mathrm{R}$ we mean (see e.g. Jackson, 1910)

$$\Delta_q f(t) = \frac{f(qt) - f(t)}{qt - t},$$

where t is any nonzero real number.

Then $\Delta_q t^k = \frac{q^k - 1}{q-1} t^{k-1}$ and, if $p(t) = \sum_{k=0}^{n} a_k t^k$, then $\Delta_q p(t) = \sum_{k=0}^{n-1} a_{k+1} \frac{q^{k+1}-1}{q-1} t^k$. In the natural way this leads to the problem of solving q-difference equation in x with known function f: $\Delta_q x(t) = f(t)$. Detailing with this, last equation gives $x(t) = (1-q)t\sum_{i=0}^{\infty} q^i f(q^i t)$ under the assumption of the convergency of the series on the right side.

Let $q \in (0,1)$ and let α be any nonzero rational number. We need the following q-analogue of $n!$, introduced in Kac and Cheung (2001):

$$[n]! = \begin{cases} 1, & if \ n = 0, \\ [n]\cdot[n-1]\cdot\dots\cdot[1], & if \ n = 1,2,\dots \end{cases}$$

Hence $[n+1]! = [n]!\,[n+1]$ for each $n \in \mathrm{N}$. Also, following the notations in Kac and Cheung (2001), we write $[\alpha] = \frac{1-q^\alpha}{1-q}$ and for generalization of the q-binomial coefficients

$$\begin{bmatrix}\alpha\\0\end{bmatrix} = 1, \begin{bmatrix}\alpha\\j\end{bmatrix} = \frac{[\alpha][\alpha-1]\cdots[\alpha-j+1]}{[j]!},\ j \in N.$$

Note that:

1. $[1] = 1$ but $[n+1] = 1 + q + \dots + q^n$ and $\lim_{n\to+\infty}[n] = \frac{1}{1-q}$;
2. For $n \in \mathrm{N}$: $\lim_{q\to 1}[n]! = n!$;
3. $\begin{bmatrix}\alpha\\1\end{bmatrix} = [\alpha]$, $\begin{bmatrix}\alpha\\2\end{bmatrix} = \frac{(1-q^{\alpha-1})(1-q^\alpha)}{(1-q^2)(1-q)}$

Example 2.1. Let $q = \alpha = 0.5$. Then the sequence $\left(\begin{bmatrix}\alpha\\j\end{bmatrix}, j = 1..4\right) \approx (0.586, -0.324, 0.676, -3.358)$, according to computations in Maple package.

In Ortigueira (2008), the q-difference of fractional order is defined by

$$\Delta_q^\alpha x(t) := t^{-\alpha}\ \frac{\sum_{j=0}^{\infty}\begin{bmatrix}\alpha\\j\end{bmatrix}(-1)^j q^{\frac{j(j+1)}{2}} q^{-j\alpha}}{(1-q)^\alpha} x(q^j t).$$

Let us denote $b_j = \begin{bmatrix}\alpha\\j\end{bmatrix}(-1)^j q^{\frac{j(j+1)}{2}} q^{-j\alpha}$. Then

$$(1-q)^\alpha \Delta_q^\alpha x(t) = t^{-\alpha}\sum_{j=0}^{\infty} b_j x(q^j t). \tag{1}$$

It is easy to check that $b_0 = 1$. The series on the right side of (1) needs the infinite values of the function x(·). But if $x(\cdot)$ is such that it vanishes besides finite number of points, then summation is finite.

If s is a natural number or $s = 0$, and $t \in R_+$ then let $\mathbf{\Omega}_s(t_0) := \{q^k t_0 : k \in \mathrm{Z}, k \le s\}$.

Let $\alpha \in R_+$. By $u_\alpha : \mathrm{R} \to \{0,1\}$ we denote the Heaviside step function such that $u_\alpha(t) = 0$ for $t < \alpha$ and $u_\alpha(t) = 1$ for $t \ge \alpha$. Then we can easily deduce the following:

Proposition 2.2. Let $\alpha > 0$, $s \in \mathrm{Z}$. Let $\varphi : \mathrm{R} \to \mathrm{R}^n$ be any function and $x(t) = \varphi(t) u_\alpha(t)$. Then,

$$\Delta_q^\alpha x(t) = \begin{cases} 0 & for \ t < a, \\ t^{-\alpha}\sum_{j=0}^{N(t,a)} \frac{b_j}{(1-q)^\alpha} x(q^j t) & for \ t \ge a, \end{cases} \tag{2}$$

where $N(t,\alpha) = E[\frac{\ln\alpha - \ln t}{\ln q}]$ and $E[x]$ denotes the integer value of x.

Let $l \in \mathrm{N}\cup\{0\}$, $t_0 = q^{j_0}$, $\alpha = q^l t_0$, $\varphi : \mathrm{R} \to \mathrm{R}^n$. The vector

$$M(l, t_0, \varphi) := \begin{bmatrix}\varphi(t_0)\\ \varphi(qt_0)\\ \vdots \\ \varphi(q^l t_0)\end{bmatrix}$$ of ordered values of function φ on $\Omega_l(t_0)$, is called a finite l-memory at t_0. Observe that if $l \in \mathrm{N}\cup\{0\}$ and $s \in \mathrm{N}\cup\{0\}$, $\varphi : R \to R^n$, then

- $M(l, t_0, \varphi) \in R^{n+nl}$
- if $l_1, l_2 \in \mathrm{N}\cup\{0\}$, $l_2 \ge l_1$ and $t_0 > 0$, then $\Omega_{l_1}(t_0) \subset \Omega_{l_2}(t_0)$ and if I_{nl_1} is a matrix of the form
$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0\\ 0 & 1 & \dots & 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$
with the first block of the dimension $l_1 \times l_1$, then also $[I_{nl_1}, 0_{nl_1 \times n(l_2 - l_1)}]\cdot M(l_2, t_0, \varphi) = M(l_1, t_0, \varphi)$.

Definition 2.3. Let $l \in \mathrm{N}\cup\{0\}$ and $t_0 > 0$, $a = q^l t_0 \in \Omega_l(t_0)$, $\varphi : R \to R^n$. A linear q-difference fractional-order time-varying control system with l-memory is a system given by the following set of equations, denoted by $\Sigma_{(\varphi,l)}$:

$$\Delta_q^\alpha x(t) = A(qt)x(qt) + B(qt)u(qt), \quad t > t_0 \tag{3}$$

$$x(t) = (\varphi u_a)(t), \qquad t \le t_0 \tag{4}$$

$$y(t) = C(t)x(t), \tag{5}$$

where $A(\cdot) \in R^{n\times n}$, $B(\cdot) \in R^{m\times n}$, $C(\cdot) \in R^{p\times n}$ are matrices with elements depending on time, and $u : q^k \mapsto u(q^k) \in R^m$, $k \in \mathrm{Z}$, is any measurable function.

Remark 2.4. If $l \to +\infty$ then $q^l t_0 \to 0$ for any $t_0 > 0$ and the vector $M(l, t_0, \varphi)$ becomes infinite.

From equation (1) and (3) we have

$$x\left(\frac{t_0}{q}\right) = G\left(\frac{t_0}{q}\right)x(t_0) - \sum_{j=1}^{l} b_{j+1} x(q^j t_0) + f\left(\frac{t_0}{q}\right),$$

where $G(t) = (t(1-q))^\alpha A(qt) - b_1 I_n$, $f(t) = (t(1-q))^\alpha B(qt)u(qt)$.

Then,

$$G\left(\frac{t_0}{q^{k+1}}\right) = \left(\frac{t_0(1-q)}{q^{k+1}}\right)^\alpha A\left(\frac{t_0}{q^k}\right) - b_1 I_n$$

and $A_0 = \mathbf{0}_n$, while for $j > 0$: $A_j = -b_{j+1} I_n$, where I_n is the $n \times n$ – identity matrix. Moreover,

$$x\left(\frac{t_0}{q^{k+1}}\right) = G\left(\frac{t_0}{q^{k+1}}\right) x\left(\frac{t_0}{q^k}\right) + \sum_{j=1}^{k+l} A_j x\left(q^{k-j} t_0\right) + f\left(\frac{t_0}{q^{k+1}}\right).$$

The idea of the construction given in the next lines follows from Guermah, Djennoune and Bettayeb (2008). Here we extend the construction to q-difference with finite l-memory. Let us define the following sequence of matrices from $R^{n\times(nl+n)}$:

$$\tilde{\Phi}(t_0) = [I_n, \mathbf{0}_n, \dots \mathbf{0}_n], \quad \tilde{\Phi}\left(\frac{t_0}{q}\right) = \left[G\left(\frac{t_0}{q}\right), A_1, \dots A_l\right]$$

$$\tilde{\Phi}\left(\frac{t_0}{q^2}\right) = G\left(\frac{t_0}{q^2}\right)\tilde{\Phi}\left(\frac{t_0}{q}\right) + [A_1, \ldots, A_{l+1}] \tag{6}$$

and for $k \geq 2$:

$$\tilde{\Phi}\left(\frac{t_0}{q^{k+1}}\right) = G\left(\frac{t_0}{q^{k+1}}\right)\tilde{\Phi}\left(\frac{t_0}{q^k}\right) + \sum_{j=1}^{k-1} A_j \tilde{\Phi}\left(\frac{t_0}{q^{k-j}}\right) + [A_k, A_{k+1}, \ldots, A_{k+l}].$$

With the sequence $\left\{\tilde{\Phi}(\frac{t_0}{q^k})\right\}_{k\in N\cup\{0\}}$ we connect the sequence $\left\{\Phi(\frac{t_0}{q^k})\right\}_{k\in N\cup\{0\}}$ of their sub-matrices in $R^{n\times n}$ that we subtract from $\left\{\tilde{\Phi}(\frac{t_0}{q^k})\right\}_{k\in N\cup\{0\}}$ by the following operation

$$\Phi\left(\frac{t_0}{q^k}\right) = \tilde{\Phi}\left(\frac{t_0}{q^k}\right) \cdot \begin{bmatrix} I_n \\ \mathbf{0}_{n\times(nl)} \end{bmatrix}. \tag{7}$$

Theorem 2.5. Let $l \in N \cup \{0\}$ and $t_0 > 0$; $a = q^l t_0 \in \Omega_l(t_0)$, $\varphi: R \to R^n$. The solution of the system $\Sigma_{(\varphi,l)}$ stated in Definition 2.3, corresponding to control u and a memory function φ is given by values for $t \geq t_0$:

$$x\left(\frac{t_0}{q^k}\right) = \tilde{\Phi}\left(\frac{t_0}{q^k}\right)\tilde{x}(t_0) + F\left(\frac{t_0}{q^k}\right), \tag{8}$$

where $\tilde{x}(t_0) = M(l, t_0, \varphi)$ and for $k > 2$

$$F\left(\frac{t_0}{q^k}\right) = G\left(\frac{t_0}{q^k}\right)F\left(\frac{t_0}{q^{k-1}}\right) + \sum_{j=1}^{k-2} A_j F\left(\frac{t_0}{q^{k-j-1}}\right) + f\left(\frac{t_0}{q^k}\right),$$

while $F\left(\frac{t_0}{q}\right) = f\left(\frac{t_0}{q}\right)$ and

$$F\left(\frac{t_0}{q^2}\right) = G\left(\frac{t_0}{q^2}\right) f\left(\frac{t_0}{q}\right) + f\left(\frac{t_0}{q^2}\right).$$

Proof. For the proof we use the mathematical induction with respect to $k \in N \cup \{0\}$, where $t = t_0/q^k$. First we check steps for $k \in \{1,2\}$. For $k = 1$:

$$\tilde{\Phi}\left(\frac{t_0}{q}\right)\tilde{x}(t_0) = G\left(\frac{t_0}{q}\right)\varphi(t_0) + [A_1, \ldots, A_l] \cdot \begin{bmatrix} \varphi(qt_0) \\ \vdots \\ \varphi(q^l t_0) \end{bmatrix}$$

and then:

$$x\left(\frac{t_0}{q}\right) = G\left(\frac{t_0}{q}\right)\varphi(t_0) + \sum_{j=1}^{l} A_j \varphi(q^j t_0) + f\left(\frac{t_0}{q}\right)$$

$$= \tilde{\Phi}\left(\frac{t_0}{q}\right)\tilde{x}(t_0) + f\left(\frac{t_0}{q}\right).$$

Similarly for $k = 1$ holds

$$x\left(\frac{t_0}{q^2}\right) = G\left(\frac{t_0}{q^2}\right)x\left(\frac{t_0}{q}\right) + A_1\varphi(t_0) + \sum_{j=1}^{l} A_{j+1}\varphi(q^j t_0) + f\left(\frac{t_0}{q^2}\right)$$

Using the formula for $x(t_0/q)$ we get

$$x\left(\frac{t_0}{q^2}\right) = G\left(\frac{t_0}{q^2}\right)G\left(\frac{t_0}{q}\right)\varphi(t_0) + A_1\varphi(t_0)$$

$$+ G\left(\frac{t_0}{q^2}\right)\left(\sum_{j=1}^{l} A_j\varphi(q^j t_0) + f\left(\frac{t_0}{q}\right)\right) + \sum_{j=1}^{l} A_{j+1}\varphi(q^j t_0)$$

$$+ f\left(\frac{t_0}{q^2}\right) = \Phi\left(\frac{t_0}{q^2}\right)\varphi(t_0) + \sum_{j=1}^{l}\left(G\left(\frac{t_0}{q^2}\right)A_j + A_{j+1}\right)\varphi(q^j t_0)$$

$$+ F\left(\frac{t_0}{q^2}\right) = \tilde{\Phi}\left(\frac{t_0}{q^2}\right)\tilde{x}(t_0) + F\left(\frac{t_0}{q^2}\right)$$

Now let us assume that the solution formula holds for all $t \in \Omega_k(t_0)$, $k \in Z_-$. Let us take now $t = t_0/q^{k+1}$. Hence

$$x(t) = G(t)x(qt) + \sum_{j=1}^{k+l} A_j x\left(\frac{t_0}{q^{k-j}}\right) + f(t).$$

Using the inductive assumption we get

$$x(t) = G(t)\tilde{\Phi}\left(\frac{t_0}{q^k}\right)\tilde{x}(t_0) + A_1 x\left(\frac{t_0}{q^{k-1}}\right) + \cdots + A_{k-1}x\left(\frac{t_0}{q}\right) + A_k\varphi(t_0)$$

$$+ A_{k+1}\varphi(qt_0) + \cdots + A_{k+l}\varphi(q^l t_0) + G(t)F\left(\frac{t_0}{q^k}\right) + f\left(\frac{t_0}{q^{k+1}}\right).$$

We can also use again inductive assumption for each of $x(t_0/q^j)$, $j = 1, \ldots, k-1$:

$$x\left(\frac{t_0}{q^j}\right) = \tilde{\Phi}\left(\frac{t_0}{q^j}\right)\tilde{x}(t_0) + F\left(\frac{t_0}{q^j}\right)$$

and

$$A_1 x\left(\frac{t_0}{q^{k-1}}\right) + \cdots + A_{k-1}x\left(\frac{t_0}{q}\right) = \sum_{j=1}^{k-1} A_j\tilde{\Phi}\left(\frac{t_0}{q^{k-j}}\right)\tilde{x}(t_0) + \sum_{j=1}^{k-1} A_j F\left(\frac{t_0}{q^{k-j}}\right).$$

In the consequence

$$x(t) = G(t)\tilde{\Phi}\left(\frac{t_0}{q^k}\right) + \sum_{j=1}^{k-1} A_j\tilde{\Phi}\left(\frac{t_0}{q^{k-j}}\right) + [A_k, \ldots, A_{k+l}]\ \tilde{x}(t_0)$$

$$+ G\left(\frac{t_0}{q^{k+1}}\right)F\left(\frac{t_0}{q^k}\right) + \sum_{j=1}^{k-1} A_j F\left(\frac{t_0}{q^{k-j}}\right) + f\left(\frac{t_0}{q^{k+1}}\right)$$

$$= \tilde{\Phi}\left(\frac{t_0}{q^{k+1}}\right)\tilde{x}(t_0) + F\left(\frac{t_0}{q^{k+1}}\right).$$

Hence from the mathematical induction the formula for solution holds for all $k \in N \cup \{0\}$.

Example 2.6. Let $t_0 = 1$, $l = 1$, $q = \alpha = 0{,}5$ and $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Let us take also the control $u(t) \equiv 1$. Then using Maple and formula given in Theorem 2.5 we can do calculations recursively. In this case we get:

$$\tilde{\Phi}(t_0/q) \approx \begin{bmatrix} 0.414 & -1 & 0.081 & 0 \\ 1 & 0.414 & 0 & 0.081 \end{bmatrix},$$

$$\tilde{\Phi}(t_0/q^2) \approx \begin{bmatrix} -1.162 & -1 & 0.062 & -0.114 \\ 1 & -1.162 & 0.114 & 0.063 \end{bmatrix}.$$

Moreover, as we take $l = 1$ we need to start memory in four dimensional space for $\tilde{x}(t_0)$. Let us take $\tilde{x}(t_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Hence the initial state is in the origin, while from the memory we have (1,1). Then $x(2) = \begin{bmatrix} 1{,}081 \\ 0{,}081 \end{bmatrix}$, $x(4) = \begin{bmatrix} 1{,}777 \\ 1{,}592 \end{bmatrix}$, $x(8) = \begin{bmatrix} -0{,}347 \\ 4{,}234 \end{bmatrix}$, $x(16) = \begin{bmatrix} -9{,}109 \\ 0{,}0909 \end{bmatrix}$, $x(32) = \begin{bmatrix} -3{,}367 \\ -35{,}612 \end{bmatrix}$, $x(64) = \begin{bmatrix} 205{,}288 \\ -33{,}612 \end{bmatrix}$.

3. OBSERVABILITY IN FINITE MEMORY DOMAIN

In this section we recall some facts related to the concept of the observability of linear q-difference fractional system with l-memory given by Definition 2.3. The standard definition of observability says that a system is observable on time-interval if from the knowledge of the output of a given system we can reconstruct uniquely the initial condition. As we consider here systems together with the extended initial conditions, called l-memory, we want to determine the extended initial condition $\tilde{x}(t_0)$ from the knowledge of $\mathbf{Y} \coloneqq \{y(t_0/q^k), k = 0, \dots, s\}$. Hence we need to distinguish in our definitions the starting point t_0, it is the similar situation as for time-varying systems (discrete or continuous). For that we use the definition of an l-event as a pair $(t, \tilde{x}) \in \{q^k : k \in Z\} \times R^{n+nl}$, as the idea comes from Sontag (1990).

Let us consider the linear q-difference fractional-order system $\Sigma_{(\varphi,l)}$.

Definition 3.1. Let l, s be any natural number, $t_0 = q^{j_0} \in \{q^k : k \in Z\}$ and let $\varphi_{1,2}$ be maps from the set $\{q^k : k \in Z\} \cup \{0\}$ into R^n. We say that two l-events $(t_0, \tilde{x}_1)$, $(t_0, \tilde{x}_2)$, where $\tilde{x}_1 = M(l, t_0, \varphi_1)$, $\tilde{x}_2 = M(l, t_0, \varphi_2)$, are indistinguishable with respect to $\Sigma_{(\varphi,l)}$ in s-steps if and only if there is a control u such that for all $t \in \Omega_s(t_0)$, $s \in Z_-$,

$$C(t)x_1(t) = C(t)x_2(t), \tag{9}$$

where functions $x_1(\cdot)$, $x_2(\cdot)$ are given by (8) and correspond respectively to φ_1, φ_2. Otherwise, the l-events $(t_0, \tilde{x}_1)$, $(t_0, \tilde{x}_2)$ are distinguishable with respect to $\Sigma_{(\varphi,l)}$ in s-steps.

Definition 3.2. Let $l, s \in N \cup \{0\}$, $\varphi_{1,2} : R \to R^n$. We say that the system $\Sigma_{(\varphi,l)}$ is observable at t_0 in s-steps if any two l-events $(t_0, \tilde{x}_1)$, $(t_0, \tilde{x}_2)$, $\tilde{x}_1 = M(l, t_0, \varphi_1)$, $\tilde{x}_2 = M(l, t_0, \varphi_2)$, are distinguishable with respect to $\Sigma_{(\varphi,l)}$ in s-steps.

Directly from Definition 3.2 follows that the system $\Sigma_{(\varphi,l)}$ is observable at t_0 in l-memory domain in s-steps if and only if the initial extended state $\tilde{x}(t_0) = M(l, t_0, \varphi)$ can be uniquely determined from the knowledge of $\mathbf{Y} \coloneqq \{y(t_0/q^k), k = 0, \dots, s\}$.

Proposition 3.3. Let $l, s \in N \cup \{0\}$. The system $\Sigma_{(\varphi,l)}$ is observable at t_0 in s-steps if and only if one of the following conditions holds

1. the $n \times n$ real matrix:
$$W(s, t_0) = \sum_{k=0}^{s} \Phi^T\left(\frac{t_0}{q^k}\right) C^T C \Phi\left(\frac{t_0}{q^k}\right) \text{ is nonsingular;}$$
2. the matrix $\Phi(t_0/q^k)$ has linearly independent columns for all $k \in \{0, \dots, s\}$;
3. $$\operatorname{rank} \mathrm{O}(s) = \operatorname{rank} \begin{bmatrix} C\Phi(t_0) \\ C\Phi\left(\frac{t_0}{q}\right) \\ \vdots \\ C\Phi\left(\frac{t_0}{q^s}\right) \end{bmatrix} = n.$$

Proof. Proof goes in the same manner as in the classical linear control theory, see for example Kaczorek (2007).

Example 3.4. For the system in Example 2.6 we have $W(1, t_0) = \begin{bmatrix} 0 & 0{,}414 \\ 0{,}414 & 1{,}172 \end{bmatrix}$. Hence system $\Sigma_{(\varphi,l=1)}$ is observable in $s = 1$ steps, because rank $W(1, t_0) = 2$.

4. CONTROLLABILITY LAW

In the literature one can find many various concepts of controllability. In our case is that we start our system at $t_0 \in R_+$, not exactly at a point from the set $\{q^k : k \in Z\}$.

Definition 4.1. The system $\Sigma_{(\varphi,l)}$ is said to be completely l-memory controllable from $t_0 \in R_+$ in s-steps, if for any $\varphi = \varphi(t)$, $t \in \Omega_l(t_0)$, and any final value $x_f \in R^n$ there is a control $u = u(t)$, $t \in \Omega_{-s}(t_0)$, such that $x(t_0/q^s) = x_f$.

Definition 4.2. Let $t_0 \in R_+$ and $s \in N$. The (l, φ) – memory controllability Gramian for the system $\Sigma_{(\varphi,l)}$ on $\Omega_{-s}(t_0)$ we define recursively in the sequel

$$W\left(\frac{t_0}{q}\right) = \left(\frac{t_0(1-q)}{q}\right)^{-\alpha} B^T(t_0) B(t_0),$$

$$W\left(\frac{t_0}{q^2}\right) = \left(\frac{t_0(1-q)}{q}\right)^{-\alpha} G\left(\frac{t_0}{q^2}\right) B^T(t_0) B(t_0) + \left(\frac{t_0(1-q)}{q^2}\right)^{-\alpha} B^T\left(\frac{t_0}{q}\right) B\left(\frac{t_0}{q}\right)$$

and for $k \geq 3$:

$$W\left(\frac{t_0}{q^k}\right) = G\left(\frac{t_0}{q^k}\right) W\left(\frac{t_0}{q^{k-1}}\right) + \sum_{j=1}^{k-2} A_j W\left(\frac{t_0}{q^{k-j-1}}\right) + \left(\frac{t_0(1-q)}{q^k}\right)^{-\alpha} B^T\left(\frac{t_0}{q^{k-1}}\right) B\left(\frac{t_0}{q^{k-1}}\right).$$

Theorem 4.3. Let $t_0 \in R_+$ and $s \in N$. If the matrix $W(t_0/q^s)$ is nonsingular, then the control function given for $k \in \{1, \dots, s\}$

$$\bar{u}\left(\frac{t_0}{q^k}\right) = -\left(\frac{t_0(1-q)}{q^{k+1}}\right)^{-\alpha} B^T\left(\frac{t_0}{q^k}\right) W^{-1}\left(\frac{t_0}{q^s}\right)\left(x_f - \tilde{\Phi}\left(\frac{t_0}{q^s}\right) \tilde{x}(t_0)\right)$$

transfers $x(t_0) = \varphi(t_0)$ to $x_f = x\left(\frac{t_0}{q^s}\right)$.

Proof. If $W(t_0/q^s)$ is nonsingular, then the proof is by direct substitution the form of control $\bar{u}(t_0/q^k)$ for $k \in \{1, \dots, s\}$ to the formula of solution $x(t_0/q^s)$.

REFERENCES

1. A**tici F. M., Eloe P. W.** (2007), Fractional q-calculus on a time scale, *Journal of Nonlinear Mathematical Physics*, Vol.14, No 3, 333-344.
2. **Atici F. M., Eloe P.W.** (2009), Initial value problems in discrete fractional calculus, *Proc. AMS*, No. 13, 7981-989.
3. **Guermah S., Djennoune S.**, **Bettayeb M.** (2008), Controllability and the observability of linear discrete-time fractional-order systems, *Int. J. Appl. Math. Comput. Sci.*, Vol. 18, No 2, 213-222.
4. **Jackson H. F.** (1910), q-Difference equations, *Am. J. Math.*, No 32, 305-314.
5. **Kac V., Cheung P.** (2001), *Quantum calculus*, New York Berlin Heidelberg.
6. **Kaczorek T.** (2007), Reachability and controllability to zero of positive fractional discrete--time systems, *Proceedings of European Control Conference ECC'07*, Kos, Greece.
7. **Kaczorek T**. (2008), Reachability of fractional positive continuous--time linear systems and their reachability, *Int. J. Appl. Math. Comput. Sci*, Vol. 18, No 2, 223-228.
8. **Lorenzo C. F., Hartley T. T.** (2009), On self-consistent operators with application to operators of fractional order, *Proceedings of the ASME 2009 International Design Engineering technical Conferences & Computers and Information in Engineering Conference IDETC/CIE 2009*, San Diego, California, USA.
9. **Mozyrska D., Bartosiewicz Z.** (2010), On observability concepts for nonlinear discrete--time fractional order control systems, *New Trends in Nanotechnology and Fractional Calculus Applications*, 305-312.
10. **Mozyrska D., Pawłuszewicz E**. (2010), Observability of linear q-difference fractional-order systems with finite initial memory, *Bull. Pol. Acad. Sci. Tech. Sci.*, Vol. 58, No 4, 601-605.
11. **Ortigueira M. D.** (2008), The fractional quantum derivative and the generalised Euler-Cauchy equation, *J. Inequal. Appl*. 956-962.
12. **Ortigueira M. D., Coito F. J.** (2007), Revisiting the Initial Conditions Problem in Fractional Linear Systems, *Symposium on Applied Fractional Calculus SAFC07*, University of Extremadura, Badajoz, Spain.
13. **Sierociuk D., Dzieliński A.** (2006), Fractional Kalman filter algorithn for the states, parameters and order of fractonal system estimation, *Int. J. Appl. Math. Comput. Sci.*, Vol. 16, No 1, 129-140.
14. **Sontag E.** (1990), *Mathematical Control Theory*, Springer – Verlag.

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