

APPROXIMATION OF FRACTIONAL DIFFUSION-WAVE EQUATION

Wojciech MITKOWSKI*

*Department of Automatics, Faculty of Electrical Engineering, Automatics, Computer Science and Electronics, AGH University of Science and Technology, Al. Mickiewicza 30/B-1, 30-059 Kraków, Poland

wojciech.mitkowski@agh.edu.pl

Abstract: In this paper we consider the solution of the fractional differential equations. In particular, we consider the numerical solution of the fractional one dimensional diffusion-wave equation. Some improvements of computational algorithms are suggested. The considerations have been illustrated by examples.

1. INTRODUCTION

Development of fractional calculations have been done probably by Leibniz and Newton (in the years 1695-1822). Further mathematical fractional calculus and its applications have been formulated in the nineteenth century. Due to the development of IT tools in recent years, many authors come back to the problems of fractional dynamic systems (e.g. Lubich, 1986; Weilbeer 2005; Kilbas et. al., 2006; Sabatier et. al., 2007; Ostalczyk, 2008; Kaczorek, 2009, 2011a,b,c; Busłowicz, 2010). For instance, interesting results (important for applications) were obtained in Białystok University of Technology (Busłowicz, 2008; Nartowicz, 2011; Ruszewski, 2009; Sobolewski and Ruszewski, 2011; also Kaczorek, 2011; Trzasko, 2011). Numerical methods for fractional systems were developed (e.g. Lubich, 1986; Podlubny et al., 1995; Podlubny, 2000; Agrawal, 2002; Diethelm and Walz, 1997; Diethelm et al., 2002; Weilbeer, 2005; Ciesielski and Leszczynski, 2006; Murillo and Yuste, 2009, 2011).

The paper is organized as follows. In the sections 2 and 3 we considered Caputo fractional differential equation and its approximation. Fractional diffusion-wave equation and its approximation is considered in the sections 4 and 5 respectively.

2. FRACTIONAL DIFFERENTIAL EQUATION

Fractional order differential equations are associated with the following operators (Weilbeer, 2005; Kaczorek, 2011): D^α – Riemann-Liouville operator (~1837), D_{GL}^α – Grünwald-Letnikov (~1867) operator and D_*^α – Caputo operator (1967).

Dynamical systems are generated by differential equations. For example consider the initial value problem (fractional differential equation of Caputo type):

$$D_*^\alpha u(t) = \lambda u(t), u(0) = 1, u^{(k)}(0) = 0, \quad (1)$$

$$k = 1, 2, \dots, n - 1$$

where $\alpha > 0$, $\lambda \in R$, $n = [\alpha] = \min \{ \xi \in N : \xi \geq \alpha \}$, N is a set of natural numbers and D_*^α is the Caputo fractional differential operator.

The solution of the initial value problem (1) is given by (Weilbeer, 2005)

$$u(t) = E_\alpha(\lambda t^\alpha), t \geq 0 \quad (2)$$

where

$$E_\alpha(z) = 1 + \frac{z}{\Gamma(1+\alpha)} + \frac{z^2}{\Gamma(1+2\alpha)} \dots = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)} \quad (3)$$

is the Mittag-Leffler function with one parameter α . In equation (3) $\Gamma(\alpha)$ denote Euler's continuous gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt = \int_0^1 (\ln \frac{1}{t})^{\alpha-1} dt. \quad (4)$$

General Euler's gamma function is defined in the whole complex plane except zero and negative integers (Weilbeer, 2005). Formula (4) is true for $\text{Re} \alpha > 0$ and the following limit holds

$$\Gamma(\alpha) = \lim_{n \rightarrow \infty} \frac{n! n^\alpha}{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)}. \quad (5)$$

For natural arguments and for half-integer arguments Euler's gamma function has the special form

$$\Gamma(n) = (n-1)!, \quad \Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!! \sqrt{\pi}}{2^{(n-1)/2}}, \lambda \in R \quad (6)$$

$$n \in N = \{1, 2, 3, \dots\}$$

where $n!!$ is the double factorial

$$n!! = \begin{cases} n \cdot (n-2) \dots 5 \cdot 3 \cdot 1 & n > 0 \text{ odd} \\ n \cdot (n-2) \dots 6 \cdot 4 \cdot 2 & n > 0 \text{ even} \\ 1 & n = 0, -1 \end{cases} \quad (7)$$

Example 1. Consider differential equation (1). From (1), (2) and (3), (6) for $\alpha = 1$ and $\alpha = 2$ we have respectively

$$E_1(\lambda t) = e^{\lambda t}, E_2(\lambda t^2) = \cos(\sqrt{|\lambda|}t), \lambda < 0. \quad (8)$$

3. CAPUTO FRACTIONAL DIFFERENTIAL EQUATION AND ITS APPROXIMATION

The Caputo fractional differential operator of order $\alpha > 1$ is defined by (e.g. Kaczorek 2011a, Weilbeer, 2005)

$$D_*^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, \quad (9)$$

where $n = [\alpha] = \min \{ \xi \in N : \xi \geq \alpha \}$.

Now we consider fractional differential equation of order $\alpha > 1$ of Caputo type

$$D_*^\alpha u(t) = f(u(t)), D^k u(0) = b_k, \quad (10)$$

$$k = 0, 1, \dots, n - 1$$

where $b_k \in R$ are given. We are interested in a numerical solution $u(t)$ of equation (10) on a closed interval $[0, T]$ for some $T > 1$. Therefore, we assume that

$$t_m = m\tau, \tau > 0, m = 0, 1, 2, \dots, N$$

$$\text{and } t_0 = 0, t_n = T, N = \frac{T}{\tau}. \quad (11)$$

Furthermore we denote by $u_m = u(t_m)$ and $f_m = f(u_m)$. Precisely u_m is the approximation of $u(t_m)$. From equality (10) for $t = t_m$ we obtain a discrete problem (Weilbeer, 2005; Murillo and Yuste, 2009, 2011)

$$\frac{1}{\tau^2} \left[u_m - \sum_{k=1}^m \omega_k u(t_m - k\tau) + \left(\frac{m^{-\alpha}}{\Gamma(m-\alpha)} \right) u_0 - \sum_{k=0}^{n-1} \frac{b_k t_m^{k-\alpha}}{\Gamma(k+1-\alpha)} \right] = f_m, \quad (12)$$

$$m = 1, \dots, N$$

Note that $t_m - k\tau = t_{m-k}$. Consistently with (12) we obtain

$$u_m = \sum_{k=1}^m \omega_k u_{m-k} - \left(\frac{m^{-\alpha}}{\Gamma(m-\alpha)} - \sum_{j=0}^m \omega_j \right) u_0 + \sum_{k=0}^{n-1} \frac{b_k t_m^{k-\alpha}}{\Gamma(k+1-\alpha)} + \tau^2 f_m, \quad (13)$$

$$m = 1, 2, \dots, N$$

where

$$\omega_k = (-1)^k \binom{\alpha}{k}, \quad (14)$$

$$\binom{\alpha}{k} = \frac{(-1)^{k-1} \Gamma(k-\alpha)}{\Gamma(1-\alpha) \Gamma(k+1)} = \frac{\alpha(\alpha-1)(\alpha-2) \dots (\alpha-k+1)}{k!}$$

$$\alpha \in R, \alpha \in N_0 = \{0, 1, 2, \dots\}.$$

Example 2. We consider equation (1) with $u(0) = u_0$. In this case $f_m = \lambda u_m$. Therefore from (13) we obtain the following discrete equation (numerical solution of equation (1))

$$u_m = \frac{1}{1-\lambda\tau^2} \left\{ \sum_{k=1}^m \omega_k u_{m-k} - \left(\frac{m^{-\alpha}}{\Gamma(m-\alpha)} - \sum_{j=0}^m \omega_j \right) u_0 + \frac{(m\tau)^{-\alpha}}{\Gamma(1-\alpha)} u_0 \right\} \quad (15)$$

$$m = 1, 2, \dots, N$$

In this case we can use the following formula for the $\Gamma(\alpha)$, $\alpha \in C \setminus \{0, -1, -2, \dots\}$, C is a set of complex numbers,

$$\frac{1}{\Gamma(\alpha)} = \alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{n} \right) e^{-\alpha/n} \quad (16)$$

where γ is the Euler's constant (Weilbeer, 2005)

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right) \approx 0,5772156649. \quad (17)$$

4. FRACTIONAL DIFFUSION-WAVE EQUATION

Let $u(x, t)$ be a function, $x \in [0, L]$ and $t \in [0, T]$. Denote by $D_{t,*}^\alpha$ the Caputo fractional differential operator at the variable t (see (9)). Consider the continuous-time fractional diffusion-wave system described by equation

$$D_*^\alpha u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) \quad (18)$$

with initial and boundary conditions

$$u(x, 0) = \varphi(x), u_t(x, 0) = 0 \quad (19)$$

$$u(0, t) = 0, u(L, t) = 0. \quad (20)$$

The solution of the homogeneous boundary problem (18), (19), (20) is given by (Weilbeer, 2005)

$$u(x, t) = \frac{2}{L} \sum_{k=1}^{\infty} c_k E_\alpha \left(-\frac{k^2 \pi^2}{L^2} t^\alpha \right) \sin \left(\frac{k\pi}{L} x \right), \quad (21)$$

$$c_k = \int_0^L \varphi(x) \sin \left(\frac{k\pi}{L} x \right) dx$$

Equation (18) for $\alpha = 1$ is the classical diffusion equation and for $\alpha = 2$ is the classical wave equation. Thus (18) for $\alpha \in (0, 2]$ is the diffusion-wave equation. The fractional diffusion-wave equation plays an intermediate role between classical wave and diffusion equations (Weilbeer, 2005; Jafari and Momani, 2007; Povstenko, 2011; Murillo and Yuste, 2009, 2011).

Example 3. For $\alpha = 1$ and $\alpha = 2$ we obtain respectively

$$E_1 \left(-\frac{k^2 \pi^2}{L^2} t \right) = \exp \left(-\frac{k^2 \pi^2}{L^2} t \right) \quad (22)$$

$$E_2 \left(-\frac{k^2 \pi^2}{L^2} t \right) = \cos \left(\frac{k\pi}{L} t \right)$$

Therefore using (22) from (21) for $\alpha = 1$ and $\alpha = 2$ we obtain the solution of classical diffusion equation and the solution of classical wave equation respectively.

5. APPROXIMATION OF FRACTIONAL DIFFUSION-WAVE EQUATION

Let $h = L/M$ and $\tau = T/N$ (see (19)) denote the step size of the discretization in the space and time axis respectively. Next let

$$x_i = ih, i = 0, 1, \dots, M \quad (23)$$

and using the discretization on the space axis, the second derivative can be approximated by the central difference of second order

$$\frac{\partial^2}{\partial x^2} u(x_i, t) \approx \frac{1}{h^2} [u(x_{i-1}, t) - 2u(x_i, t) + u(x_{i+1}, t)] \quad (24)$$

$$i = 1, 2, \dots, M - 1,$$

where from (20) $x_0 = x_M = 0$. From (18) for $x = x_i$ and (24) we have

$$D_{t,*}^\alpha u(x_i, t) = \frac{1}{h^2} [u(x_{i-1}, t) - 2u(x_i, t) + u(x_{i+1}, t)] \quad (25)$$

$$i = 1, 2, \dots, M - 1$$

Let $t = t_m = m\tau$, $\tau > 0$, $m = 0, 1, 2, \dots, N$, where $N = T/\tau$. Thus from (25) and (13) we obtain

$$u_m(x_i) - \frac{\tau^2}{h^2} [u_m(x_{i-1}, t) - 2u_m(x_i, t) + u_m(x_{i+1}, t)] = z(u_{m-1}(x_i)u_{m-2}(x_i), \dots, u_0(x_i)),$$

$$z(u_{m-1}(x_i)u_{m-2}(x_i), \dots, u_0(x_i)) = \sum_{k=1}^m \omega_k u_{m-k}(x_i) - \left(\frac{m^{-\alpha}}{\Gamma(m-\alpha)} - \sum_{j=0}^m \omega_j \right) u_0(x_i) + \sum_{k=0}^{n-1} \frac{b_k(x_i) t_m^{k-\alpha}}{\Gamma(k+1-\alpha)} \quad (26)$$

$$m = 1, 2, \dots, N, i = 1, 2, \dots, M - 1$$

where $u_m(x_i) = u(x_i, t_m)$. Let $Z = [z(x_i)]$ be vector $(M - 1) \times 1$. Let $U_m = [u_m(x_i)]$ be vector $(M - 1) \times 1$, $i = 1, 2, \dots, M - 1$. Denote by $A = [a_{ki}]$ tridiagonal matrix

$(M - 1) \times (M - 1)$ with $\alpha_{kk} = -2, \alpha_{k,k-1} = 1, \alpha_{k,k+1} = 1$. From (26) we have

$$U_m = \left[I - \frac{\tau^2}{h^2} A \right]^{-1} Z \quad (27)$$

At the time-step $t_m, m = 1, 2, \dots, N$, the values for $u_m(x_i) = u(x_i, t_k)$, for $i = 0, 1, \dots, M$ and $k = 0, 1, \dots, m - 1$ are known (in (27) Z is known).

6. NUMERICAL EXAMPLES

Example 4. A very simple approximation of the system (25) can be the equation (1) with suitably chosen parameter λ . For fixed $x = x_i$ the nature of the function $u(x_i, t)$ will be similar to the solution $u(t)$ of the system (1). Thus, in this paper we will present only the simulation of solutions of equation (1). In the calculations formula (3) and the Matlab Gamma function were used.

Let us consider the system (1). Let $kk = 20$ (see (3)). In Fig. 1 trajectories $u(t)$ for $\lambda = -1$ and $\alpha = 0,5$ (solid line), $\alpha = 1$ (dotted line) are shown. In Fig. 2 trajectories $u(t)$ for $\lambda = -10$ and $\alpha = 1,5$ (solid line), $\alpha = 2$ (dotted line) are shown.

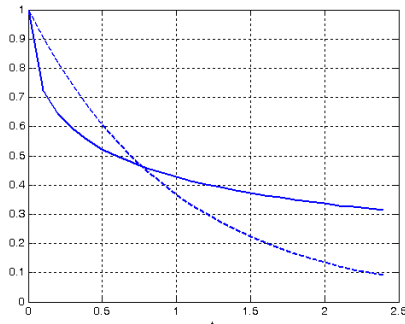


Fig. 1. Trajectories of the system (1) for $\alpha = 0,5$ (solid line), $\alpha = 1$ (dotted line)

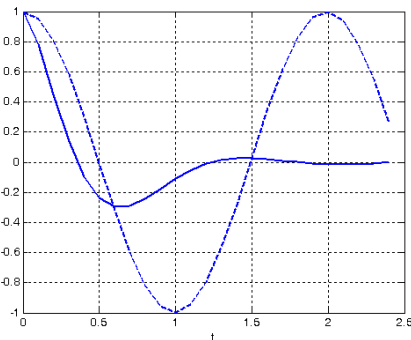


Fig. 2. Trajectories of the system (1) for $\alpha = 1,5$ (solid line), $\alpha = 2$ (dotted line)

Example 5. Consider the continuous-time fractional diffusion-wave system (18) with initial and boundary conditions (19), (20). The solution of the homogeneous boundary problem (18), (19), (20) is given by (21). In the calculations the Matlab Gamma function and the Matlab Mittag-Leffler function (Podlubny and Kacanak 2001) were used.

Let $L = \pi$ and $kk = 40, \tau = 0,1; h = 0,0314$. Let $\varphi(x) = \sin(2x)$. Solutions of the boundary problem (18), (19), (20) with $\alpha = 0,5; 1,0; 1,5; 1,8; 2,0$ are shown in Fig. 3, 4, 5, 6 and 7 respectively.

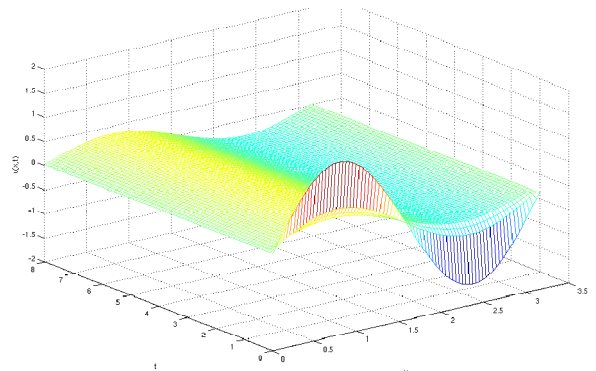


Fig. 3. Solution of the boundary problem (18)-(20) for $\alpha = 0,5$

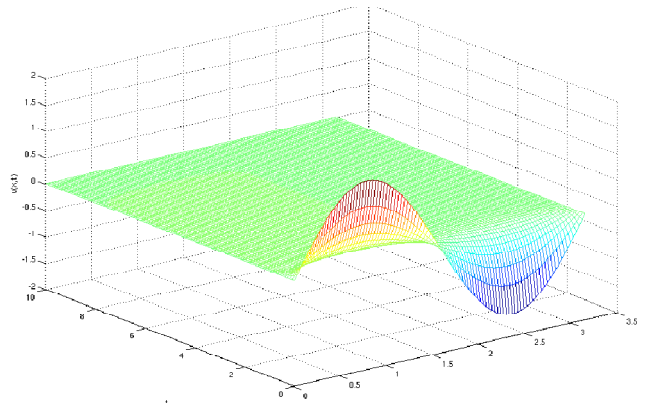


Fig. 4. Solution of the boundary problem (18)-(20) for $\alpha = 1,0$

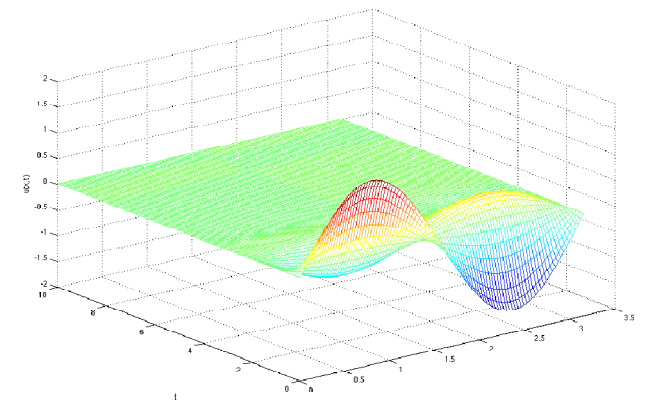


Fig. 5. Solution of the boundary problem (18)-(20) for $\alpha = 1,5$

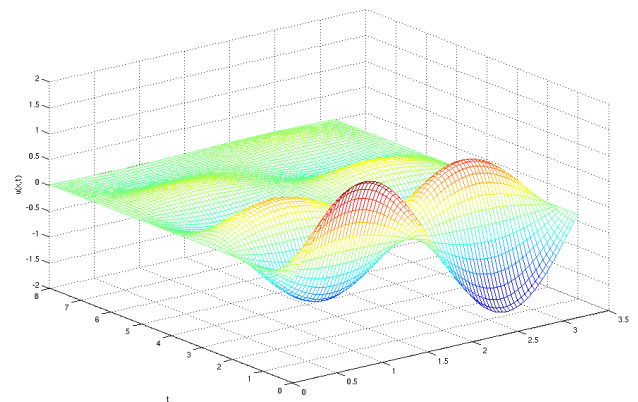


Fig. 6. Solution of the boundary problem (18)-(20) for $\alpha = 1,8$

The calculations were performed on a computer with dual-core processor Intel Core 2 Duo (T7500) 2.2 GHz / core, 3.5 GB memory. It took approximately 1 hour to calculate the data for each Figure.

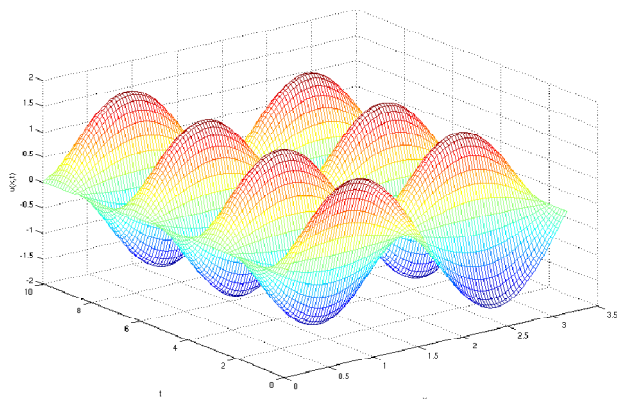


Fig. 7. Solution of the boundary problem (18)-(20) for $\alpha = 2,0$.

7. CONCLUDING REMARKS

In this paper we consider the selection of the fractional differential equations. The considerations have been illustrated by a numerical examples. The effectiveness of computational algorithms is dependent on the possibility of determining the Euler's continuous gamma function and depends on the possibility of calculating of the Mittag-Leffler function $E_\alpha(z)$. The function $E_\alpha(z)$ was first introduced in 1903 by Mittag-Leffler (Pillai, 1990).

Some recent interesting results in fractional systems theory and its applications in automatic control can be found in (Liang et al., 2004; Busłowicz, 2008; Ostalczyk, 2008; Ruszewski, 2009; Nartowicz, 2011; Sobolewski and Ruszewski, 2011).

REFERENCES

1. **Agrawal O. P.** (2002), Solution for a fractional diffusion-wave equation defined in a bounded domain, *Nonlinear Dynam.*, 29, No. 1-4, 145-155.
2. **Busłowicz M.** (2008), Stability of linear continuous time fractional order systems with delay of the retarder type, *Bull. Pol. Acad. Sci. Tech.*, vol. 56, no. 4, 319-324.
3. **Busłowicz M.** (2010), Wybrane zagadnienia z zakresu liniowych ciągłych układów niecałkowitego rzędu, *Pomiary Automatyka Robotyka*, nr 2, 93-114.
4. **Ciesielski M., Leszczynski J.** (2006), Numerical Treatment of An Initial-Boundary Value Problem for Fractional Partial Differential Equations, *Signal Process.* 86, 2619.
5. **Diethelm K., Ford N. J., Freed A. D.** (2002), A predictor-corrector approach for the numerical solution of fractional differential equations, *Nonlinear Dynamics*, 29, 3- 22.
6. **Diethelm K., Walz G.** (1997), Numerical solution of fractional order differential equations by extrapolation, *Numer. Algorithms*, 16, 231-253.
7. **Jafari H., Momani S.** (2007), Solving fractional diffusion and wave equations by modified homotopy perturbation method, *Physics Letters A*, Elsevier, 370, 388-396.
8. **Kaczorek T.** (2009), *Wybrane zagadnienia teorii układów niecałkowitego rzędu*, Oficyna Wydawnicza Politechniki Białostockiej, Białystok.
9. **Kaczorek T.** (2011a), Positive fractional linear systems. *Pomiary Automatyka Robotyka*, No. 2, 91-112.
10. **Kaczorek T.** (2011b), Decomposition of the positive fractional discrete-time linear system, *Pomiary Automatyka Robotyka*, No. 2, 504-511.
11. **Kaczorek T.** (2011c), *Selected Problems of Fractional Systems Theory*, Springer, Berlin (in print).
12. **Kilbas A. A., Srivastava H. M. Trujillo J. J.** (2006), *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam.
13. **Liang J., Chen Y. Q., Meng Max Q.-H., Fullmer R.** (2004), Fractional-order Boundary Control of Fractional Wave Equation with Delayed Boundary Measurement Using Smith Predictor, *43rd IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, 5088-5093.
14. **Lubich C.** (1986), Discretized fractional calculus, *SIAM J. Math. Anal.*, 17, No. 3, 704-719.
15. **Murillo J. Q., Yuste S. B.** (2009), On three explicit difference schemes for fractional diffusion and diffusion-wave equations, *Physica Scripta*, T136, 014025 (6pp).
16. **Murillo J. Q., Yuste S. B.**, (2011), An Explicit Difference Method for Solving Fractional Diffusion and Diffusion-Wave Equations in the Caputo Form, *Journal of Computational and Nonlinear Dynamics*, April 2011, Vol. 6, 021014-1-6.
17. **Nartowicz T.** (2011), Obszary stabilności układu regulacji z regulatorem ułamkowym dla niestabilnego obiektu pierwszego rzędu z opóźnieniem (Design of fractional order controller for a first order unstable plant with delay), *Pomiary Automatyka Robotyka*, No. 2, 595-602.
18. **Ostalczyk P.** (2008), *Zarys rachunku różniczkowo-całkowego ułamkowych rzędów. Teoria i zastosowania w automatyce (Epitome of the Fractional Calculus, Theory and its Applications in Automatics)*, Publishing Department of Technical University of Łódź, Łódź (in Polish).
19. **Pillai R. N.** (1990), On Mittag-Leffler functions and related distributions, *Ann. Inst. Statist. Math.* Vol. 42, No. 1, 157-161.
20. **Podlubny I.** (2000), Matrix approach to discrete fractional calculus, *Fract. Calc. Appl. Anal.*, 3, No. 4, 359-386.
21. **Podlubny I., Dorcak L., Misaneck J.** (1995), Application of fractional-order derivatives to calculation of heat load intensity change in blast furnace walls, *Transactions of the Technical University of Kosice*, 5, 137-144.
22. **Podlubny I., Kacena M.** (2001), <http://www.mathworks.com/matlabcentral/fileexchange/8738>
23. **Povstenko Y. Z.** (2011), Solutions to Time-Fractional Diffusion-Wave Equation in Cylindrical Coordinates, *Advances in Difference Equations*, Hindawi Publishing Corporation, Article ID 930297, Vol. 2011, Article ID 930297, 14 pages.
24. **Ruszewski A.** (2009), Stabilizacja układów inercyjnych ułamkowego rzędu z opóźnieniem za pomocą ułamkowego regulatora PID, *Pomiary Automatyka Robotyka*, No. 2, 406-414.
25. **Sabatier J., Agrawal O. P., Machado J. A. T.** (Eds) (2007), *Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering*, Springer, London.
26. **Sobolewski A., Ruszewski A.** (2011), Realizacja praktyczna regulatora niecałkowitego rzędu (Practical realization of fractional-order controller), *Pomiary Automatyka Robotyka*, No. 2, 589-594
27. **Trzasko W.** (2011), Względna punktowa zupełność dodatnich układów ciągle-dyskretnych niecałkowitego rzędu (Relative pointwise completeness of positive continuous-discrete time fractional order systems), *Pomiary Automatyka Robotyka*, No. 2, 528-537.
28. **Weilbeer M.** (2005), Efficient Numerical Methods for Fractional Differential Equations and their Analytical, *Technischen Universität Braunschweig, Doktors Dissertation*, 1-224.

I thank Mrs. Anna Obraczka very much for performing the calculations in Example 5.