LOCAL CONTROLLABILITY OF FRACTIONAL DISCRETE-TIME SEMILINEAR SYSTEMS

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Abstract: In the paper unconstrained local controllability problem of finite-dimensional fractional discrete-time semilinear systems with constant coefficients is addressed. Using general formula of solution of difference state equation sufficient condition for local unconstrained controllability in a given number of steps is formulated and proved. Simple illustrative example is also presented.

1. INTRODUCTION

Controllability is one of the fundamental concepts in modern mathematical control theory. This is qualitative property of control systems and is of particular importance in control theory. The basic concepts of controllability, reachability and the weaker notion of stabilizability play an essential, fundamental role in dynamical systems analysis and in the solutions of many different important optimal control problems.

Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

During last few years many results concerning theory of fractional control systems both discrete-time and continuous-time have been published in the literature (see e.g. (Kaczorek, 2007a, 2007b, 2009; Klamka, 2002, 2008)). However, it should be pointed out, that the most controllability results are known only for linear fractional control systems both without delays or with delays in control or state variables.

Controllability problems studied in this paper concern semilinear fractional discrete-time control systems. More precisely, in the present paper unconstrained local controllability problem of finite-dimensional fractional discrete-time semilinear systems is addressed. Using general formula of solution of difference state equation, sufficient condition for local controllability in a given number of steps is formulated and proved. The present paper extends for semilinear discrete-time fractional control systems with constant coefficients controllability results given in Kaczorek (2007a, 2007b, 2009) and Klamka (2002, 2008) for linear fractional systems.

The paper is organized as follows. In section 2 using results presented in (Kaczorek, 2007b), general solution of the difference state equation for finite-dimensional fractional linear systems is recalled. Sufficient condition for local unconstrained controllability of the semilinear fractional discrete-time control system with constant parameters is established in section 3. Section 4 contains simple numerical example, which illustrates theoretical considerations. Finally, concluding remarks and propositions for future works are given in section 5.

2. FRACTIONAL SYSTEMS

The set of nonnegative integers will be denoted by Z_+. Let $x_k \in R^n$, $u_k \in R^m$, $k \in Z_+$. In this paper well known extended definition of the fractional difference of the form (Kaczorek, 2007a, 2007b, 2009; Klamka, 2002, 2008)

$$\Delta^\alpha x_k = \sum_{j=0}^{j=k} (-1)^j \binom{\alpha}{j} x_{k-j} \quad \text{for } n-1<\alpha<n \in N=\{1,2,\ldots\}, k \in Z_+ \tag{1}$$

will be used, where $\alpha \in R$ is the order of the fractional difference and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j=0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j=1,2,\ldots \end{cases} \tag{2}$$

where $\binom{\alpha}{j}$ is so called generalized Newton symbol. Let us observe, that in the case when $\alpha = n$ we have well known standard Newton symbol

$$\binom{\alpha}{j} = \frac{n!}{j!(n-j)!}$$

Let us consider the fractional discrete-time linear system, described by the semilinear difference state-space equation

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k + f(x_k, u_k) \tag{3}$$

where $x_k \in R^n$, $u_k \in R^m$ are the state and input and A and B are $n\times n$ and $n\times m$ constant matrices, $f: R^n\times R^m \to R^n$ is nonlinear function differentiable near zero in the space $R^n\times R^m$ and such that $f(0,0) = 0$.

Let us observe, that semilinear discrete-time control system is described by the difference state equation, which contains both pure linear and pure nonlinear parts in the right hand side of the state equation.

Using definition of fractional difference (1) we may write semilinear difference equation (3) in the equivalent form

$$x_{k+1} + \sum_{j=1}^{j=k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k + f(x_k, u_k)$$

Next, using standard linearization method (Klamka, 1995) it is possible to find the associated linear difference state equation

$$x_{k+1} + \sum_{j=1}^{j=k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k + Fx_k + Gu_k$$

where $n\times n$ dimensional matrix

$$F = \frac{d}{dx} f(x,u)\Big|_{\substack{x=0,\\ u=0}}$$

and $n\times m$ dimensional matrix

$$G = \frac{d}{du} f(x,u)\Big|_{\substack{x=0,\\ u=0}}$$

Moreover, for simplicity of notation let us denote $A + F = C$ and $D = B + G$.

Thus we have

$$x_{k+1} + \sum_{j=1}^{j=k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Cx_k + Du_k \tag{4}$$

Lemma 1. (Kaczorek, 2007b) The solution of linear difference equation (4) with initial condition $x_0 \in R^n$ is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{i=k-1} \left(\Phi_{k-i-1} D u_i\right) \tag{5}$$

where $n\times n$ dimensional state transition matrices Φ_k, $k = 0,1,2,\ldots$ are determined by the recurrent formula

$$\Phi_{k+1} = (C + I_n\alpha)\Phi_k + \sum_{i=2}^{i=k+1} (-1)^{j+1} \binom{\alpha}{i} \Phi_{k-i+1} \tag{6}$$

with $\Phi_0 = I_n$, where I_n is $n\times n$ dimensional identity matrix and by assumption matrices $\Phi_k = 0$ for $k < 0$.

Moreover, it should be pointed out, that the matrices Φ_k, $k = 0,1,2,\ldots$ defined above are extensions for fractional linear discrete-time control systems, the well known state transition matrices (see e.g. (Klamka, 1991)) for standard linear discrete-time control systems.

3. CONTROLLABILITY

First of all, in order to define global and local controllability concepts for semilinear and linear finite-dimensional discrete-time control systems let us introduce the notion of reachable set or in other words attainable set in q steps (Kaczorek, 2007a, 2007b, 2009; Klamka, 1991, 1995, 2002, 2008).

Definition 1. For fractional semilinear system (3) or linear system (4) reachable set in q steps from initial condition $x_0 = 0$ is defined as follows:

$K_q = \{x(q) \in R^n$: $x(q)$ is a solution of semilinear system (3) or linear system (4) in step q for sequence of admissible controls $u_0, u_1, \ldots u_k, \ldots, u_{q-1}\}$ (7)

Definition 2. The fractional semilinear discrete-time control system (3) is locally controllable in q-steps if there exists a neighborhood of zero $N \subset R^n$, such that

$$K_q = N \tag{8}$$

Definition 3. The fractional linear discrete-time linear control system (4) is globally controllable in q-steps if

$$K_q = N \tag{9}$$

For linear control system (4) let us introduce the $n\times qm$ dimensional controllability matrix

$$H_q = [D, (\Phi_1 D), (\Phi_2 D), \ldots, (\Phi_i D), \ldots, (\Phi_{q-1} D)] \tag{10}$$

In order to prove sufficient condition for local controllability of semilinear discrete-time fractional control systems (3), we shall use certain result taken directly from nonlinear functional analysis. This result concerns so called nonlinear covering operators.

Lemma 2. (Robinson, 1986) Let W: Z$\to$Y be a nonlinear operator from a Banach space Z into a Banach space Y and W(0) = 0. Moreover, it is assumed, that operator W has the Frechet derivative dW(0): Z$\to$Y, whose image coincides with the whole space Y. Then the image of the operator W will contain a neighborhood of the point W(0)$\in$Y.

Now, we are in the position to formulate and prove the main result on the local unconstrained controllability in the interval $[0, q]$ for the nonlinear discrete-time system (1). This result is known for semilinear or nonlinear continuous-time control system and is given in Klamka (1995), as a sufficient condition for local controllability.

Theorem 1. Semilinear discrete-time control system (3) is locally controllable in q steps if the associated linear discrete-time control system (4) is globally controllable in q-steps.

Proof. Proof of the Theorem 1 is based on Lemmas 1 and 2. Let the nonlinear operator W transform the space of admissible control sequence $\{u(i): 0 \le i \le q\}$ into the space of solutions at the step q for the semilinear discrete-time fractional control system (3).

More precisely, the nonlinear operator

W: $R^m\times R^m\times\ldots\times R^m \to R^n$

asssociated with semilinear control system (3) is defined as follows (Klamka, 1995):

$$W\{u(0), u(1), u(2),..., u(i),..., u(q-1)\} = x_{sem}(q)$$

where $x_{sem}(q)$ is the solution at the step q of the semilinear discrete-time fractional control system (3) corresponding to an admissible controls sequence $u_q = \{u(i): 0 \le i < q\}$.

Therefore, for zero initial condition Frechet derivative at point zero of the nonlinear operator W denoted as dW(0) is a linear bounded operator defined by the following formula

$$dW(0)\{u(0), u(1), u(2),..., u(i),..., u(q-1)\} = x_{lin}(q)$$

where $x_{lin}(q)$ is the solution at the step q of the linear system (4) corresponding to an admissible controls sequence $u_q = \{u(i): 0 \le i < q\}$ for zero initial condition.

Since from the assumption nonlinear function $f(0,0) = 0$, then for zero initial condition the nonlinear operator W transforms zero in the space of admissible controls into zero in the state space i.e., W(0) = 0.

Moreover, let us observe, that if the associated linear discrete-time fractional control system (4) is globally controllable in the interval $[0, q]$, then by Definition 1 the image of the Frechet derivative dW(0) covers whole state space R^n.

Therefore, by the result stated at the beginning of the proof, the nonlinear operator W covers some neighborhood of zero in the state space R^n. Hence, by Definition 2 semilinear discrete-time fractional control system (3) is locally controllable in the interval $[0, q]$. This completes the proof.

Now, for the convenience, let us recall some well known (see e.g. (Kaczorek, 2007a, 2007b, 2009; Klamka, 1991, 2002, 2008)) facts from the controllability theory of linear finite-dimensional discrete-time fractional control systems.

Theorem 2. (Klamka, 2008) The fractional discrete-time linear system (4) is globally controllable in q steps if and only if

$$rank\, H_q = n \tag{11}$$

Taking into account the form of controllability matrix, from Theorem 2 immediately follows the simple Corollary.

Corollary 1. (Klamka, 2008) The fractional linear control system (4) is controllable in q steps if and only if $n \times n$ dimensional constant matrix $H_q H_q^T$ is invertible, i.e. there exists the inverse matrix $(H_q H_q^T)^{-1}$.

Corollary 2. The fractional semilinear control system (3) is controllable in q steps if equality (11) holds or equivalently if $n \times n$ dimensional constant matrix $H_q H_q^T$ is invertible, i.e. there exists the inverse matrix $(H_q H_q^T)^{-1}$.

4. EXAMPLE

Let us consider the semilinear fractional discrete-time control system with constant coefficients of the form (3) for $0 \le \alpha \le 1$ with the following matrices and vectors in the difference state equation

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad f(x,u) = f(x_1, x_2, u) = \begin{bmatrix} e^u - 1 \\ 2\sin x_1 \end{bmatrix} \tag{12}$$

Hence we have

$$f(0,0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F = \frac{d}{dx} f(x_1, x_2, u)\Big|_{\substack{x=0 \\ u=0}} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

$$G = \frac{d}{du} f(x_1, x_2, u)\Big|_{\substack{x=0 \\ u=0}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence we have

$$C = A + F = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad D = B + G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Using formula (6) for $k = 0$ we obtain

$$\Phi_1 = (C + I\alpha)\Phi_0 = \begin{bmatrix} 1+\alpha & 0 \\ 2 & 1+\alpha \end{bmatrix}$$

Controllability matrix (10) for $q = 2$ has the form

$$H_2 = [D, (\Phi_1 D)] = \begin{bmatrix} 1 & 1+\alpha \\ 1 & 3+\alpha \end{bmatrix}$$

Therefore, since $rank\, H_2 = 2 = n$ then taking into account Theorem 2 the fractional associated linear discrete-time system with constant coefficients is globally controllable in two steps, hence by Theorem 1 the semilinear fractional discrete-time system (12) is locally controllable in two steps.

For comparison let us consider linear fractional discrete system (4) with the matrices A and B given equalities (12). In this case using formula (6) for $k = 0$ we have

$$\Phi_1 = (A + I\alpha)\Phi_0 = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 1+\alpha \end{bmatrix}$$

Controllability matrix (10) for $q = 2$ has the form

$$H_2 = [B, (\Phi_1 B)] = \begin{bmatrix} 0 & 0 \\ 1 & 1+\alpha \end{bmatrix}$$

Therefore, since $rank\, H_2 = 1 < n$ then taking into account Corollary 1 the fractional linear discrete-time system with constant coefficients is not globally controllable in two steps and consequently in any number of steps.

5. CONCLUDING REMARKS

In the present paper unconstrained local controllability problem of finite-dimensional fractional discrete-time semilinear systems has been addressed. Using linearization method and solution formula for linear difference equation sufficient condition for unconstrained local controllability in q steps of the discrete-time fractional control system has been established as rank condition of suitably defined controllability matrix. In the proof of the main result certain theorem taken directly from nonlinear functional analysis has been used. Moreover, simple illustrative numerical example has been also presented.

There are many possible extensions of the results given in the paper. First of all it is possible to consider semilinear infinite-dimensional fractional control systems. Moreover, it should be mentioned, that controllability considerations presented in the paper can be extended for fractional discrete-time linear systems with multiple delays both in the controls and in the state variables.

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