STABILITY OF STATE-SPACE MODELS OF LINEAR CONTINUOUS-TIME FRACTIONAL ORDER SYSTEMS

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Abstract: The paper considers the stability problem of linear time-invariant continuous-time systems of fractional order, standard and positive, described by the state space model. Review of previous results is given and some new methods for stability checking are presented. Considerations are illustrated by numerical examples and results of computer simulations.

1. INTRODUCTION

In the last decades, the problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations was considered in many papers and books. For review of the previous results see, for example, the monographs (Caponetto et al., 2010; Das, 2008; Diethelm (2010); Kaczorek, 2009, 2011a; Kilbas et al., 2006; Monje et al., 2010; Ostalczyk, 2008; Podlubny, 1994, 1999; Sabatier et al., 2007).

The problems of stability and robust stability of linear fractional order continuous-time systems were studied among others in Matignon (1996, 1998), Busłowicz (2008a, 2008b, 2009), Petras (2008, 2009), Radwan et al. (2009), Sabatier et al. (2008, 2010), Tavazoei and Heri (2009) and in Ahn et al. (2006), Ahn and Chen (2008), Busłowicz (2008c), Lu and Chen (2009), Tan et al. (2009), Zhuang and Yisheng (2010), respectively.

The new class of the linear fractional order systems, namely the positive systems of fractional order was considered by Kaczorek (2008a, 2008b, 2009, 2011a, 2011b).

The aim of the paper is to give the review of the methods for stability analysis of fractional continuous-time linear systems described by the state-space model and presentation of some new results. The standard and positive fractional order systems will be considered.

2. PROBLEM FORMULATION

Consider a linear continuous-time system of fractional order described by the state equation

$$ {}_0D_t^{\alpha} x(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$ and

$$ {}_0D_t^{\alpha} x(t) = \frac{1}{\Gamma(p-\alpha)} \int_0^t \frac{x^{(p)}(\tau)d\tau}{(t-\tau)^{\alpha+1-p}}, \quad p-1 \le \alpha \le p, \tag{2}$$

is the Caputo definition for fractional α-order derivative, where $x^{(p)}(t) = d^P x(t)/dt^P$, p is a positive integer and

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt \tag{3}$$

is the Euler gamma function.

Definition (3) can be written in the equivalent form

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n! n^{\alpha}}{\alpha(\alpha+1)\cdots(\alpha+n)}. \tag{3a}$$

From (2) for $p = 1$ and $p = 2$ we have, respectively

$$ {}_0D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x^{(1)}(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad 0<\alpha<1, \tag{4}$$

$$ {}_0D_t^{\alpha} x(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{x^{(2)}(\tau)d\tau}{(t-\tau)^{\alpha-1}}, \quad 1<\alpha<2. \tag{5}$$

The Laplace transform of the Caputo fractional derivative has the form

$$L\{{}_0D_t^{\alpha} x(t)\} = s^{\alpha}F(s) - \sum_{k=1}^{p} s^{\alpha-k} x^{(k-1)}(0^+). \tag{6}$$

For zero initial conditions, the Laplace transform (6) reduces to

$$L\{{}_0D_t^{\alpha} x(t)\} = s^{\alpha}F(s). \tag{6a}$$

Definition 1. The fractional system (1) will be called positive (internally) if $x(t) \in \Re_+^n$ for any initial condition $x(0) \in \Re_+^n$ and for all inputs $u(t) \in \Re_+^n$, $t \ge 0$.

Positivity condition of the system (1) is known only in the case of fractional order $\alpha \in (0,1]$. In Kaczorek (2008a, 2008b), see also Kaczorek (2009, 2011a), the following theorem has been proved.

Theorem 1. The fractional system (1) with $0 < \alpha \le 1$ is positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \tag{7}$$

where M_n – the set of $n \times n$ real Metzler matrices (matrices with non-negative off-diagonal entries), $\Re_+^{n\times m}$ – the set of $n \times m$ real matrices with non-negative entries.

Characteristic function of the fractional system (1) is the fractional degree polynomial of the form

$$w(s) = \det(s^\alpha I - A) = a_n s^{n\alpha} + a_{n-1} s^{(n-1)\alpha} + \dots + a_0. \tag{8}$$

The associated natural degree polynomial has the form

$$\tilde{w}(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0, \quad \lambda = s^\alpha. \tag{9}$$

The polynomial (8) is a multivalued function whose domain is a Riemann surface. In general, this surface has an infinite number of sheets and the fractional polynomial (8) has an infinite number of zeros. Only a finite number of which will be in the main sheet of the Riemann surface. For stability reasons only the main sheet defined by $-\pi < \arg s < \pi$ can be considered (Petras, 2008, 2009).

From the theory of stability of linear fractional order systems given by Matignon (1996, 1998) and Petras (2008, 2009), we have the following theorem.

Theorem 2. The fractional order system (1) is stable if and only if the fractional degree characteristic polynomial (8) has no zeros in the closed right-half of the Riemann complex surface, i.e.

$$w(s) = \det(s^\alpha I - A) \neq 0 \text{ for } \operatorname{Re} s \geq 0, \tag{10}$$

or equivalently, the following condition is satisfied

$$|\arg \lambda_i(A)| > \alpha\frac{\pi}{2}, \quad i = 1,2,\dots,n, \tag{11}$$

where $\lambda_i(A)$ is the i-th eigenvalues of matrix A.

From Radwan et al. (2009) it follows that the fractional system with the characteristic polynomial (8) is unstable for all $\alpha > 2$. Therefore, in this paper we consider the fractional system (1) of fractional order $\alpha \in (0,2)$.

The stability regions of the system (1), described by (11) are shown in Fig. 1 and 2 for $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$, respectively. Parametric description of the boundary of the stability regions has the form

$$(j\omega)^\alpha = |\omega|^\alpha e^{j\pi\alpha/2}, \quad \omega \in (-\infty, \infty). \tag{12}$$

The polynomial (8) with $\alpha = 1$ is a natural degree polynomial and from (12) for $\alpha = 1$ we have that the imaginary axis of the complex plane is the boundary of the stability region.

The aim of this paper is to give the review of the methods for stability analysis of the fractional system (1) and presentation of some new results. We consider the stability problem of standard and positive fractional order systems.

3. STABILITY OF FRACTIONAL SYSTEMS

The following lemma can be used to checking the condition (11) of Theorem 2.

Lemma 1. The fractional order system (1) is stable if and only if

$$\gamma > \alpha\frac{\pi}{2}, \tag{13}$$

where

$$\gamma = \min_i |\arg \lambda_i(A)| \tag{14}$$

and $\lambda_i(A)$ is the i-th eigenvalue of A.

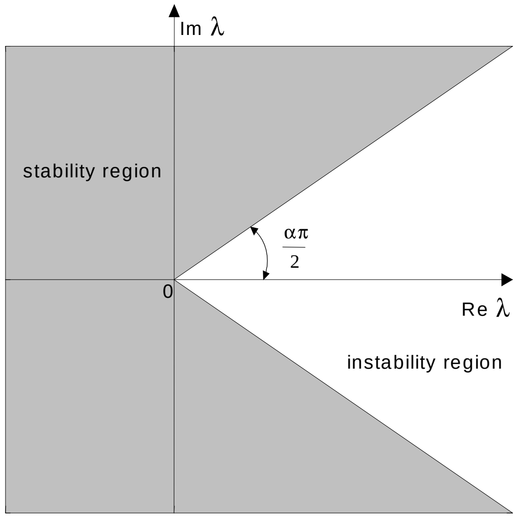

Fig. 1. Stability region for $0 < \alpha \leq 1$

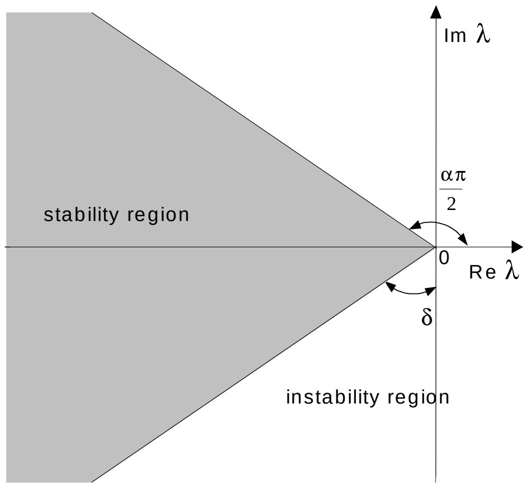

Fig. 2. Stability region for $1 \leq \alpha < 2$

From Theorem 2, Lemma 1 and Fig. 1 and 2 we have the following important lemmas and remark.

Lemma 2. The fractional system (1) is unstable for all $\alpha \in (0,2)$ if the matrix A has at least one non-negative real eigenvalue. In particular, this holds if $\det A = 0$.

Lemma 3. Assume that the state matrix A has no real non-negative eigenvalues. Then the fractional system (1) is stable if and only if $\alpha \in (0, \alpha_0)$, where $\alpha_0 = 2\gamma/\pi$ and γ is computed from (14).

Remark 1. If the fractional system (1) is stable for a fixed $\alpha \in [1,2)$ then it is also stable for all fractional orders $\alpha \in (0,1]$.

3.1. Stability of system of fractional order $\alpha \in [1, 2)$

The system (1) of fractional order $\alpha \in [1,2)$ is stable if and only if all eigenvalues of A lie in the stability region shown in Fig. 2. Hence, this system may be unstable in the case of negative real parts of all eigenvalues of matrix A if $\alpha \arg\lambda_i(A) < \alpha\pi/2$, $i = 1, 2, \dots, n$.

The following lemma can be used to stability checking of the fractional system (1) of order $\alpha \in [1,2)$.

Lemma 4 (Anderson et al., 1974; Davison and Ramesh, 1970). The eigenvalues of an $n \times n$ matrix A lie in the sector shown in Fig. 2 if and only if the eigenvalues of $2n \times 2n$ matrix

$$\tilde{A} = \begin{bmatrix} A\cos\delta & -A\sin\delta \\ A\sin\delta & A\cos\delta \end{bmatrix} \tag{15}$$

have negative real parts, where $\delta = (\alpha - 1)\pi/2$.

From the above and the result given in (Hostetter, 1975), see also (Tavazoei and Haeri, 2009) it follows that if $p(s) = \det(sI - A)$ then

$$\det(sI - \tilde{A}) = p(se^{j\delta})\,p(se^{-j\delta}), \quad \delta = (\alpha - 1)\pi/2.$$

Based on Lemma 4, the following theorem has been proved in Tavazoei and Haeri (2009).

Theorem 3. The fractional system (1) with $1 \le \alpha < 2$ is stable if and only if the eigenvalues of the matrix $\tilde{A}$ have negative real parts, where

$$\tilde{A} = \begin{bmatrix} A\sin(\alpha\pi/2) & A\cos(\alpha\pi/2) \\ -A\cos(\alpha\pi/2) & A\sin(\alpha\pi/2) \end{bmatrix}. \tag{16}$$

Proof. Substitution $\delta = (\alpha - 1)\pi/2$ in (15) gives (16). The proof follows directly from Theorem 2 and Lemma 4.

In Molinary (1975) it has been proved that if there exist positive definite Hermitian matrices $P > 0$ and $Q > 0$ such that

$$\beta PA + \beta^* A^{\mathrm{T}} P = -Q, \tag{17}$$

where $\beta = \eta + j\xi$ with $\tan(\pi - \alpha\pi/2) = \eta/\xi$ (equivalently, $\tan(\pi/2 - \delta) = \eta/\xi$), then all eigenvalues of A are within the stable area shown in Fig. 2. From the above and Theorem 2 one obtains the following theorem (see also Ahn et. al. (2006), Sabatier et al. (2008, 2010)).

Theorem 4. The fractional system (1) with $1 \le \alpha < 2$ is stable if and only if there exist positive definite Hermitian matrices $P > 0$ and $Q > 0$ such that (17) holds.

The stability region shown in Fig. 2 is convex. Therefore, to the stability analysis of the system (1) with $1 \le \alpha < 2$ the LMI based conditions can be applied.

In Chilali et al. (1999) it has been shown that the eigenvalues of matrix A lie in the sector shown in Fig. 2 if and only if there exists a matrix $P = P^{\mathrm{T}} > 0$ such that

$$\begin{bmatrix} (AP + PA^{\mathrm{T}})\sin(\theta) & (AP - PA^{\mathrm{T}})\cos(\theta) \\ (PA^{\mathrm{T}} - AP)\cos(\theta) & (AP + PA^{\mathrm{T}})\sin(\theta) \end{bmatrix} < 0, \tag{18}$$

where $\theta = \pi - \alpha\pi/2$.

Substitution $\theta = \pi - \alpha\pi/2$ in (18) gives

$$\begin{bmatrix} (AP + PA^{\mathrm{T}})\sin(\alpha\pi/2) & (AP - PA^{\mathrm{T}})\cos(\alpha\pi/2) \\ (PA^{\mathrm{T}} - AP)\cos(\alpha\pi/2) & (AP + PA^{\mathrm{T}})\sin(\alpha\pi/2) \end{bmatrix} < 0. \tag{19}$$

Hence, we prove the following theorem.

Theorem 5. The fractional system (1) with $1 \le \alpha < 2$ is stable if and only if there exists a matrix $P = P^{\mathrm{T}} > 0$ such that the condition (19) holds.

The same criterion has been obtained by Sabatier et al. (2008, 2010). In this criterion, the condition (19) is written in the equivalent form

$$\begin{bmatrix} (A^{\mathrm{T}}P + PA)\sin(\alpha\pi/2) & (A^{\mathrm{T}}P - PA)\cos(\alpha\pi/2) \\ (PA - A^{\mathrm{T}}P)\cos(\alpha\pi/2) & (A^{\mathrm{T}}P + PA)\sin(\alpha\pi/2) \end{bmatrix} < 0. \tag{19a}$$

To checking the condition (19) (or (19a)), a LMI solver can be used.

3.2. Stability of system of fractional order $\alpha \in (0, 1]$

The system (1) of fractional order $\alpha \in (0,1]$ is stable if and only if all eigenvalues of A lie in the stability region shown in Fig. 1. Hence, this system may be stable in the case when not all eigenvalues of A lie in open left half-plane. Moreover, this system may be stable when all eigenvalues of the matrix A are complex with positive real parts.

From the above we have the following simple sufficient condition for the stability.

Lemma 5. The fractional system (1) with $0 < \alpha \le 1$ is stable if all eigenvalues of A lie in open left half-plane of the complex plane.

Using Lemma 4 and taking into account that the system (1) with $0 < \alpha \le 1$ is unstable if all eigenvalues of A lie in the instability region shown in Fig. 1, we obtain the following theorem.

Theorem 6 (Tavazoei and Haeri, 2009). The fractional system (1) with $0 < \alpha \le 1$ is unstable and all eigenvalues of A lie in the instability region shown in Fig. 1 if and only if the eigenvalues of $\bar{A}$ have negative real parts, where

$$\bar{A} = \begin{bmatrix} -A\sin(\alpha\pi/2) & A\cos(\alpha\pi/2) \\ -A\cos(\alpha\pi/2) & -A\sin(\alpha\pi/2) \end{bmatrix}. \tag{20}$$

Proof. If all eigenvalues of A lie in the instability sector shown in Fig. 1, then all eigenvalues of $-A$ satisfy the inequality

$$|\arg\lambda_i(-A)| > \pi - \alpha\frac{\pi}{2}, \quad i = 1, 2, \dots, n, \tag{21}$$

i.e. lie in sector shown in Fig. 2 if we consider angle $\pi - \alpha\pi/2$ with $\alpha \in (0,1]$ instead of angle $\alpha\pi/2$. Then $\delta = (1 - \alpha)\pi/2$. The proof follows directly from Lemma 4 for $\delta = (1 - \alpha)\pi/2$ and substitution $-A$ instead of A.

Based on instability analysis, the following condition has been given in Sabatier et al. (2008, 2010).

Theorem 7. The fractional system (1) with $0 < \alpha < 1$ is stable if and only if there does not exist any non-negative rank one complex matrix Q such that

$$rAQ + QA^T\bar{r} \ge 0, \tag{22}$$

where $r = \sin(\alpha\pi/2) + j\cos(\alpha\pi/2)$ and $\bar{r}$ denotes the complex conjugate of r.

The stability region shown in Fig. 1 is not convex. Therefore, to the stability analysis of the fractional system (1) with $0 < \alpha < 1$ the LMI conditions can not be applied.

In Sabatier et al. (2008, 2010) the following sufficient and necessary and sufficient conditions have been proved.

Theorem 8. The fractional system (1) with $0 < \alpha < 1$ is asymptotically stable if there exists a matrix $P > 0$ such that

$$(A^{1/\alpha})^{\mathrm{T}} P + P(A^{1/\alpha}) < 0. \tag{23}$$

Theorem 9. The fractional system (1) with $0 < \alpha < 1$ is stable if and only if there exists a symmetric matrix $P > 0$ such that

$$\left(-(-A)^{1/(2-\alpha)}\right)^{\mathrm{T}} P + P\left(-(-A)^{1/(2-\alpha)}\right) < 0. \tag{24}$$

Based on the Generalized LMI (GLMI), in Sabatier et al. (2008, 2010) the following criterion has been given.

Theorem 10. The fractional system (1) with $0 < \alpha < 1$ is stable if there exist positive definite complex matrices $X_1 = X_1^*$ and $X_2 = X_2^*$ such that

$$\bar{r}X_1A^{\mathrm{T}} + rAX_1 + rX_2A^{\mathrm{T}} + rAX_2 < 0, \tag{25}$$

where $r = \exp\,(j(1-\alpha)\pi/2)$.

3.3. Generalization of frequency domain methods

The frequency domain methods for stability analysis of fractional systems described by the transfer function have been proposed in Busłowicz (2008a, 2009), see also Kaczorek (2011a, Chapter 9). These methods can be applied to the system (1) of any fractional order $\alpha \in (0,2)$.

By generalization of the results of Busłowicz (2008a, 2009) to the case of fractional system (1) we obtain the following methods for stability checking.

Theorem 11. The fractional system (1) with characteristic polynomial (8) is stable if and only if

$$\underset{0\le\omega<\infty}{\Delta \arg}\; w(j\omega) = n\pi/2, \tag{26}$$

where $w(j\omega) = w(s)$ for $s = j\omega$, i.e. plot of the function $w(j\omega)$ starts for $\omega = 0$ in the point $w(0) = \det(-A)$ and with ω increasing from 0 to ∞ turns strictly counter-clockwise and goes through n quadrants of the complex plane.

Plot of the function $w(j\omega)$ is called the generalised (to the class of fractional degree polynomials) Mikhailov plot.

Checking the condition (26) is difficult in general (for large values of n), because $w(j\omega)$ quickly tends to infinity as ω grows to ∞.

To remove this difficulty, we consider the rational function

$$\psi(s) = \frac{\det(s^{\alpha}I - A)}{w_r(s)} \tag{27}$$

instead of the polynomial (8), where $w_r(s)$ is stable the reference fractional polynomial of degree αn, i.e.

$$w_r(s) \neq 0 \text{ for } \operatorname{Re} s \geq 0. \tag{28}$$

The reference fractional polynomial can be chosen in the form

$$w_r(s) = (s+c)^{\alpha n}, \quad c > 0. \tag{29}$$

Theorem 12. The fractional system (1) with $0 < \alpha < 2$ is stable if and only if

$$\underset{\omega\in(-\infty,\infty)}{\Delta \arg}\; \psi(j\omega) = 0, \tag{30}$$

where $\psi(j\omega) = \psi(s)$ for $s = j\omega$ and $\psi(s)$ is defined by (27), i.e. plot of the function $\psi(j\omega)$ does not encircle or cross the origin of the complex plane as ω runs from $-\infty$ to ∞.

Plot of the function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$, is called the generalised modified Mikhailov plot.

From (8), (27) and (29) we have

$$\psi(\infty) = \lim_{\omega\to\pm\infty} \psi(j\omega) = 1 \tag{31}$$

and

$$\psi(0) = \frac{\det(-A)}{c^{\alpha n}}. \tag{32}$$

From (32) it follows that $\psi(0) \leq 0$ if $\det(-A) \leq 0$. Hence, from Theorem 12 we have the following important lemma.

Lemma 6. If $\det(-A) \leq 0$ then the fractional system (1) is unstable for all $\alpha \in (0,2)$.

Lemma 6 also follows from the Hurwitz stability test because if $\det(-A) \leq 0$ then not all coefficients of the characteristic polynomial of *A* are non-zero and positive.

3.4. Stability of positive systems

Now we consider the stability problem of the positive system (1) of fractional order $\alpha \in (0,1]$. In this case, according to Theorem 1, the condition (7) holds, i.e. the matrix *A* has non-negative off-diagonal entries.

Positive linear systems are sub-class of linear systems. Therefore, the stability conditions given in this paper can also be applied to the stability analysis of the positive system (1).

Stability conditions of positive natural number systems, continuous-time and discrete-time, are very simple in comparison with the stability conditions of standard systems (Farina and Rinaldi, 2000; Kaczorek, 2000, 2002). Therefore, we consider the possibilities of simplification of the stability conditions of standard fractional system (1) with $\alpha \in (0,1]$.

From Theorems 1 and 2 it follows that the positive system (1) with $\alpha \in (0,1)$ is stable if and only if all eigenvalues of the Metzler matrix *A* lie in the stability region shown in Fig. 1.

From (Farina and Rinaldi, 2000; Kaczorek, 2011b) we have that the dominant eigenvalue (eigenvalue with the largest real part) of the Metzler matrix is real. Therefore, the positive system (1) with $\alpha \in (0,1)$ is stable if and only if all eigenvalues of the Metzler matrix A have negative real parts.

Hence, using the well-known stability conditions of positive systems given in Kaczorek (2000, 2002), we obtain the following simple necessary and sufficient condition for the asymptotic stability.

Lemma 7. The positive system (1) is asymptotically stable for all $\alpha \in (0,1)$ if and only if one of the following equivalent conditions holds:

1. eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix A have negative real parts,
2. all the leading principal minors $\Delta_1, \Delta_2, \dots, \Delta_n$ of the matrix $-A$ are positive,
3. all the coefficients of the characteristic polynomial of the matrix A are positive.

It is easy to see that if $A \in M_n$ then the matrix (20) is not a Metzler matrix. This means that is not possible simplification of the condition given in Theorem 6 for the positive system (1).

4. ILLUSTRATIVE EXAMPLES

Example 1. Check stability of the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}, \quad a, b \in \Re. \tag{33}$$

Eigenvalues of A are as follows

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \tag{34}$$

If $a^2 = 4b$ then $\lambda_{1,2} = -a/2$ Hence, from Lemmas 2 and 3 we have the following:

- if $a < 0$ then eigenvalues of A are positive and the system is unstable for all fractional orders α
- if $a > 0$ then eigenvalues of A are negative and the system is stable for all fractional orders $\alpha \in (0,2)$.

If

$$a^2 > 4b \text{ and } -a + \sqrt{a^2 - 4b} \geq 0 \text{ or } -a - \sqrt{a^2 - 4b} \geq 0, \tag{35}$$

then from Lemma 2 it follows that the system is unstable for all values $\alpha \in (0,2)$.

If

$$a^2 > 4b \text{ and } -a \pm \sqrt{a^2 - 4b} < 0, \tag{36}$$

then from Lemma 5 it follows that the system is stable for all $\alpha \in (0,1)$.

If $a^2 < 4b$ then the matrix (33) has two complex eigenvalues

$$\lambda_{1,2} = \frac{-a \pm j\sqrt{4b - a^2}}{2}. \tag{37}$$

If $a < 0$ then from (14) and (37) we have

$$\gamma = \arctan\sqrt{4\tau - 1}, \quad \tau = b/a^2, \tag{38}$$

and

$$\alpha_0 = \frac{2}{\pi}\gamma = \frac{2}{\pi}\arctan\sqrt{4\tau - 1}. \tag{39}$$

From Lemma 3 it follows that the system with $a^2 < 4b$ and $a < 0$ is stable for any $\alpha \in (0, \alpha_0)$ where α_0 is computed from (39).

Similarly, we can show that if $a > 0$ and $a^2 < 4b$ then the system is stable for any $\alpha \in (0, \alpha_{01})$ where

$$\alpha_{01} = \frac{2}{\pi}\gamma = \frac{2}{\pi}(\pi - \arctan\sqrt{4\tau - 1}), \quad \tau = b/a^2. \tag{40}$$

Plots of $\alpha_0(\tau)$ and $\alpha_{01}(\tau)$ for $\tau \in [1,10]$ are shown in Fig. 3. It is easy to check that $\alpha_0 \to 1$ and $\alpha_{01} \to 1$ if $\tau \to \infty$.

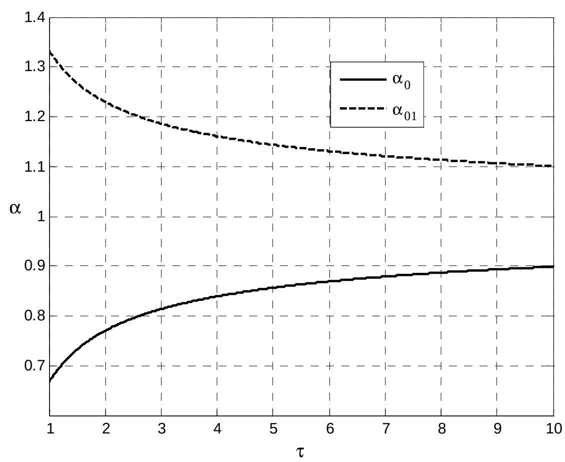

Fig. 3. Plot of the functions (39) and (40) vs. $\tau \in [1,10]$

From Fig. 3 and (39), (40) it follows that $\alpha_0 < \alpha_{01}$ for all fixed τ.

If $\tau = 4$ (i.e. $b = 4a^2$), for example, then the system

- with $a < 0$ is stable if and only if $\alpha \in (0, \alpha_0)$, $\alpha_0 = 0.8391$
- with $a > 0$ is stable if and only if $\alpha \in (0, \alpha_{01})$, where $\alpha_{01} = 1.1609$.

Assume that the output equation and the input matrix of the system (1), (33) are as follows

$$y(t) = Cx(t), \quad C = [1 \quad 0], \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then, the transfer function has the form

$$G(s) = C(s^\alpha I - A)^{-1}B = \frac{1}{\det(s^\alpha I - A)} = \frac{1}{s^{2\alpha} + as^\alpha + b}.$$

Step responses of the system for $b = 4$, $a = 1$ and $b = 4$, $a = -1$ are shown in Figs 4 and 5, respectively, for few values of fractional order α.

Numerical simulations are performed using Ninteger v. 2.3 − Fractional Control Toolbox for MatLab, see Valério (2005).

From Figs 4 and 5 it follows that simulations confirm the above theoretical results that the system with $b = 4a^2$ and $a < 0$ is stable for all positive $\alpha < 0.8391$, whereas this system with $a > 0$ is stable for all positive $\alpha < 1.1609$.

Now we consider the stability problem of positive system (1) with (33).

From Theorem 1 it follows that the system (1) with A of the form (33) and $\alpha \in (0,1]$ is positive if and only if $b < 0$. If $b < 0$ then from (34) it follows that A has two real eigenvalues, one negative and one positive. Hence, from the above and Lemma 2 we have that the positive system (1) with the matrix (33) with $b < 0$ is unstable for all fractional orders $\alpha \in (0,1]$. In particular, this system is unstable for $\alpha = 1$ (the natural number positive system).

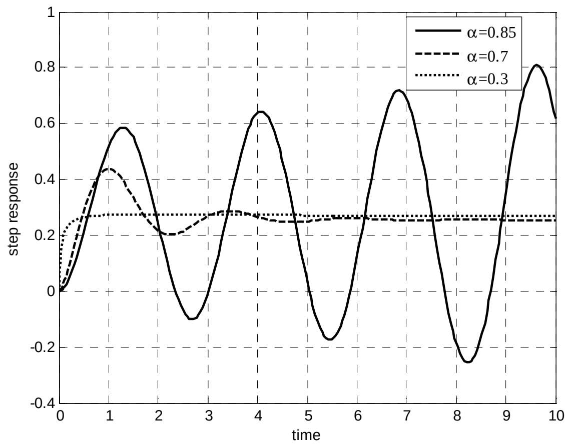

Fig. 4. Step responses of the system with $a = -1$, $b = 4$

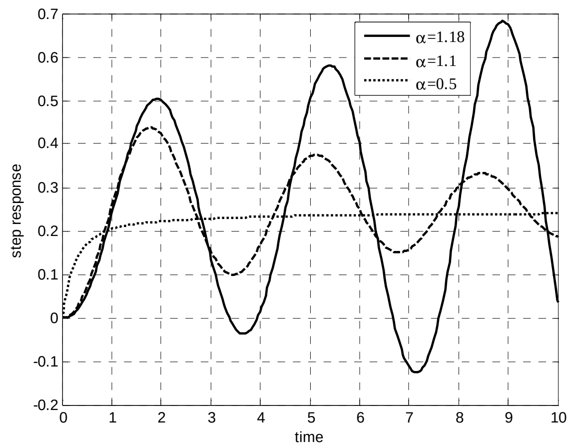

Fig. 5. Step responses of the system with $a = 1$, $b = 4$

Example 2. Consider the fractional system (1) with

$$A = \begin{bmatrix} -1 & 0.8 & 1.1 \\ -0.8 & -2 & 0.9 \\ -0.3 & -1.2 & -1.6 \end{bmatrix}. \tag{41}$$

Check stability of the system for $\alpha = 1.4$ and $\alpha = 1.9$.

Plot of the function

$$\psi(j\omega) = \frac{\det((j\omega)^{\alpha} I - A)}{(j\omega + 1)^{3\alpha}}, \quad \omega \in (-\infty, \infty), \tag{42}$$

with $\alpha = 1.4$ and $\alpha = 1.9$ is shown in Figs 6 and 7, respectively.

According to (31) and (32) we have (independently of the value of α)

$$\psi(\infty) = \lim_{\omega \to \pm\infty} \psi(j\omega) = 1, \quad \psi(0) = \det(-A) = 5.1240.$$

From Figs 6, 7 and Theorem 12 it follows that the system with $\alpha = 1.4$ is stable (plot of (42) does not encircle the origin of the complex plane) and with $\alpha = 1.9$ is unstable (plot of (42) encircles the origin of the complex plane).

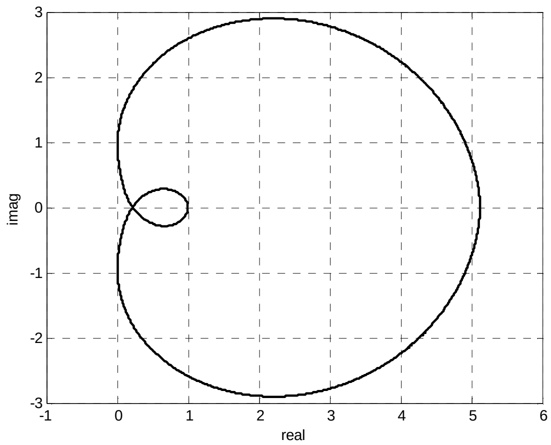

Fig. 6. Plot of the function (42) with $\alpha = 1.4$

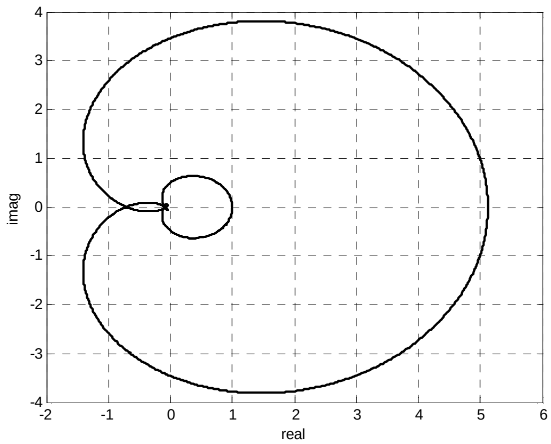

Fig. 7. Plot of the function (42) with $\alpha = 1.9$

Now we apply Theorem 5. Using the LMI toolbox of Matlab, we obtain the following feasible solution of (19):

– for $\alpha = 1.4$

$$P = \begin{bmatrix} 0.7751 & -0.0939 & 0.0750 \\ -0.0939 & 0.4212 & -0.0232 \\ 0.0750 & -0.0232 & 0.4510 \end{bmatrix} \tag{43}$$

– for $\alpha = 1.9$

$$P = \begin{bmatrix} 1.4859 & -0.6659 & 0.5467 \\ -0.6659 & 0.2984 & -0.2450 \\ 0.5467 & -0.2450 & 0.2012 \end{bmatrix}. \tag{44}$$

Computing the leading principal minors of the matrices (43) and (44) we obtain, respectively,

$$\Delta_1 = 0.7751,\ \Delta_2 = 0.3277,\ \Delta_3 = 0.1408,$$

$$\Delta_1 = 1.4850,\ \Delta_2 = -1.676 \cdot 10^{-9},\ \Delta_3 = -3.025 \cdot 10^{-5}.$$

From the above it follows that the matrix (43) is positive definite (all the leading principal minors are positive) and the matrix (44) is not positive definite. This means, according to Theorem 5, that the system with $\alpha = 1.4$ is stable and with $\alpha = 1.9$ is unstable.

Now we apply Lemma 3 to stability checking of the system.

The matrix (41) has the following eigenvalues:

$$\lambda_1 = -0.9538;\ \lambda_{2,3} = -1.8231 \pm j1.4313.$$

From (14) we have $\gamma = 2.4760$ and from Lemma 3 it follows that the system is stable for all $\alpha \in (0, \alpha_0)$ where $\alpha_0 = 2\gamma/\pi = 1.4305$. Hence, the system is stable for $\alpha = 1.4 < \alpha_0$ and unstable for $\alpha = 1.9 > \alpha_0$.

Now we assume $\alpha = 0.5$ and check stability using Theorems 8 and 9.

Computing the feasible solutions of (23) and (24) with $\alpha = 0.5$ we obtain respectively

$$P = \begin{bmatrix} 0.3866 & -0.0039 & 0.1038 \\ -0.0039 & 0.2308 & -0.0216 \\ 0.1038 & -0.0216 & 0.3173 \end{bmatrix}, \tag{45}$$

$$P = \begin{bmatrix} 0.6392 & -0.0125 & 0.1085 \\ -0.0125 & 0.4703 & -0.0301 \\ 0.1085 & -0.0301 & 0.5521 \end{bmatrix}. \tag{46}$$

It is easy to check that the matrices (45) and (46) are positive definite. From Theorems 8 and 9 it follows the system with $\alpha = 0.5$ is stable.

Example 3. Check stability of the system (1) with

$$A = \begin{bmatrix} -1.4 & 0 & 0.1 & 1.8 \\ 0.1 & -1.5 & 1.7 & 0.5 \\ 0.1 & 0.08 & -1.4 & 1.1 \\ 0 & 0.4 & 0.5 & -1.4 \end{bmatrix}. \tag{47}$$

The matrix (47) is a Metzler matrix. Therefore, the system (1), (47) with $\alpha \in (0,1]$ is a positive system. To stability checking of this system we apply simple necessary and sufficient condition given in Lemma 7.

Computing the characteristic polynomial of the matrix (47) we obtain

$$\det(\lambda I - A) = \lambda^4 + 5.7\lambda^3 + 11.284\lambda^2 + 8.0684\lambda + 0.8373.$$

All coefficients of the above polynomial are positive. From Lemma 7 it follows that the positive fractional system (1) with matrix A of the form (47) is stable for any $\alpha \in (0,1]$.

The matrix (47) has the following eigenvalues:

$$\lambda_1 = -0.1239;\ \lambda_2 = -1.5683;\ \lambda_{3,4} = -2.0039 \pm j0.5404.$$

From (14) we have $\gamma = 2.8782$ and $\alpha_0 = 1.8323$. From Lemma 3 it follows that the system (1) with A of the form (47) is stable for any fractional order $\alpha \in (0,\ 1.8323)$.

5. CONCLUDING REMARKS

Review of the existing methods for stability analysis of the system (1) of fractional order $\alpha \in (0,2)$ is given and the new results are presented.

In particular, generalisation of the classical Mikhailov stability criterion to the class of fractional order systems (1) with $\alpha \in (0,2)$ is proposed.

Moreover, it has been shown that:

– the fractional system (1) is unstable for all $\alpha \in (0,2)$ if the matrix A has at least one non-negative real eigenvalue (Lemma 2);
– if A has no real non-negative eigenvalues, then the fractional system (1) is stable if and only if $\alpha \in (0, \alpha_0)$ where $\alpha_0 = 2\gamma/\pi$ and γ is computed from (14) (Lemma 3);
– the positive system (1) is stable for all $\alpha \in (0,1]$ if and only if all coefficients of the characteristic polynomial of the matrix A are positive (Lemma 7).

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