

## ANALYSIS AND APPLICATIONS OF COMPOSED FORMS OF CAPUTO FRACTIONAL DERIVATIVES

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**Abstract:** In this paper we consider two ordinary fractional differential equations with composition of the left and the right Caputo derivatives. Analytical solution of this type of equations is known for particular cases, having a complex form, and therefore is difficult in practical calculations. Here, we present two numerical schemes being dependent on a fractional order of equation. The results of numerical calculations are compared with analytical solutions and then we illustrate convergence of our schemes. Finally, we show an application of the considered equation.

### 1. INTRODUCTION

This study is devoted to the analysis of ordinary differential equations containing a composed form of left- and right-sided fractional derivatives, which are defined in any sense, i.e. the Riemann-Liouville and the Caputo ones. Moreover, we consider the equations in a restricted domain. The equations are obtained by modification the minimum action principle and the application of fractional integration by parts. It should be noted that many authors (Agrawal, 2002; Klimek, 2002; Riewe, 1996) elaborated fractional forms of the Euler-Lagrange equations. However, the equations contain only specific compositions of fractional derivatives, i.e. the arbitrary form of Riemann-Liouville (left- or right-sided) composed with the arbitrary form of Caputo (also left- or right-sided). Therefore, in the Euler-Lagrange equations a disadvantage in boundary conditions occurs. The disadvantage reveals an introduction of homogenous conditions for one boundary, where the Riemann-Liouville fractional derivative exists (Błaszczyk et al., 2011; Leszczyński and Błaszczyk, 2010). To omit such problems, we consider a composed form of fractional derivatives, where the left- and the right-sided Caputo operators are used. Moreover, we expect that a fractional differential equation containing the composition of two Caputo derivatives has physical meaning and will be useful in modelling complex processes in nature.

To obtain the analytical solution is one of the fundamental problem that arises from Euler-Lagrange equations. The results, based on the fixed point theorem (Klimek, 2007), are not capable in practice, because the solution is presented in the form of very complex series. Klimek (Klimek, 2008) proposed to use the Mellin transform in order to obtain the analytical solution. However, such solution has complex form, which includes series of special functions. For practical applications we cannot use the

analytical solution due to its useless in calculations. Therefore, we will construct some approximate solutions. Some numerical basics can be found in the studies (Błaszczyk, 2009; Błaszczyk, 2010; Błaszczyk & Ciesielski, 2010).

### 2. FORMULATION OF THE PROBLEM

We consider two ordinary fractional differential equations with composition of the left- and the right-sided Caputo derivatives, which have the following forms

$${}^C D_{b-}^\alpha {}^C D_{0+}^\alpha T(x) - \lambda T(x) = 0, \tag{1}$$

$${}^C D_{0+}^\alpha {}^C D_{b-}^\alpha T(x) - \lambda T(x) = 0, \tag{2}$$

where  $x \in [0, b]$  and operators  ${}^C D_{0+}^\alpha$ ,  ${}^C D_{b-}^\alpha$  are defined as (Kilbas et al., 2006)

$${}^C D_{0+}^\alpha T(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{T^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \text{ for } x > 0 \tag{3}$$

$${}^C D_{b-}^\alpha T(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{T^{(n)}(\tau)}{(\tau-x)^{\alpha-n+1}} d\tau, \text{ for } x < b \tag{4}$$

where  $n = [\alpha] + 1$ .

Here, we mean  ${}^C D_{0+}^\alpha$  as the left-sided Caputo derivative and  ${}^C D_{b-}^\alpha$  denotes the right-sided Caputo derivative.

For  $\alpha \in (0, 1)$  Eqns. (1) and (2) are supplemented by the adequate boundary conditions

$$T(0) = T_0, \quad T(b) = T_b, \tag{5}$$

Analytical solutions are known only for some type of Euler-Lagrange equations (Klimek, 2007; Klimek,

2008), and they have very complex form. To omit this problem we propose a numerical approach.

### 3. NUMERICAL SCHEMES

In order to develop a discrete form of Eqns. (1) and (2), the homogenous grid of nodes is introduced as

$$0 = x_0 < x_1 < \dots < x_i < x_{i+1} < \dots < x_N = b, \quad (6)$$

where

$$x_i = x_0 + i \Delta x, \quad (7)$$

Function  $T$  determined at the point  $x_i$  is denoted as  $T_i = T(x_i)$ . We also assume  $\alpha \in (0, 1)$ .

#### 3.1. Discrete scheme for Eqns. (1) and (2)

We have introduced the discrete form of fractional derivatives for Eqn. (1). The value of the left-sided Caputo derivative at point  $x_i$  can be approximated as

$$\begin{aligned} {}^c D_{0+}^\alpha T(x) \Big|_{x=x_i} &= \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_i} \frac{T'(\tau)}{(x_i - \tau)^\alpha} d\tau \\ &\cong \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{T_{j+1} - T_j}{\Delta x} \int_{x_j}^{x_{j+1}} \frac{1}{(x_i - \tau)^\alpha} d\tau \quad (8) \\ &= (\Delta x)^{-\alpha} \sum_{j=0}^i T_j v(i, j) \end{aligned}$$

where

$$v(i, j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} (i-1)^{1-\alpha} - i^{1-\alpha} & \text{for } j=0 \\ (i-j+1)^{1-\alpha} - 2(i-j)^{1-\alpha} \\ + (i-j-1)^{1-\alpha} & \text{for } j=1, \dots, i-1 \\ 1 & \text{for } j=i \end{cases} \quad (9)$$

Next, denoting  $g(x) = {}^c D_{0+}^\alpha T(x)$  in Eqn. (1) we find the discrete form of the right-sided Caputo derivative

$$\begin{aligned} {}^c D_{b-}^\alpha {}^c D_{0+}^\alpha T(x) \Big|_{x=x_i} &= {}^c D_{b-}^\alpha g(x) \Big|_{x=x_i} \\ &= \frac{-1}{\Gamma(1-\alpha)} \int_{x_i}^{x_N} \frac{g'(\tau)}{(\tau - x_i)^\alpha} d\tau \\ &\cong \frac{-1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1} - g_j}{\Delta x} \int_{x_j}^{x_{j+1}} \frac{1}{(\tau - x_i)^\alpha} d\tau \\ &= (\Delta x)^{-\alpha} \sum_{j=i}^N g_j w(i, j) \end{aligned} \quad (10)$$

where

$$w(i, j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} 1 & \text{for } j=i \\ (j-i+1)^{1-\alpha} - 2(j-i)^{1-\alpha} \\ + (j-i-1)^{1-\alpha} & \text{for } j=i+1, \dots, N-1 \\ (N-i-1)^{1-\alpha} - (N-i)^{1-\alpha} & \text{for } j=N \end{cases} \quad (11)$$

Using formulas (8) and (10) we obtain a system containing the discrete form of Eqn. (1) and boundary conditions as

$$\begin{cases} T_0 = T(x_0) \\ (\Delta x)^{-2\alpha} \sum_{j=i}^N \left[ w(i, j) \sum_{k=0}^j v(j, k) T_k \right] - \lambda T_i = 0, \text{ for } i=1, \dots, N-1 \\ T_N = T(x_N) \end{cases} \quad (12)$$

Similarly to previous considerations, we write the discrete form of Eqn. (2) as

$$\begin{cases} T_0 = T(x_0) \\ (\Delta x)^{-2\alpha} \sum_{j=0}^i \left[ v(i, j) \sum_{k=j}^N w(j, k) T_k \right] - \lambda T_i = 0, \text{ for } i=1, \dots, N-1 \\ T_N = T(x_N) \end{cases} \quad (13)$$

To obtain full numerical solutions of Eqns. (1) and (2), we need to solve a system of algebraic equations (12) and (13) respectively.

#### 3.2. Convergence and error analysis

Including discrete forms of Eqns. (1) and (2) we analyse errors and convergence of the numerical schemes. Let us assume  $\alpha \in (0, 1)$ ,  $x \in [0, 1]$ ,  $\lambda = 0$  and boundary conditions as

$$T(0) = 0, \quad T(1) = 1 \quad (14)$$

Then, the solution of Eqn. (1) has the following form

$$T(x) = x^\alpha \quad (15)$$

Tab.1 shows errors generated by numerical scheme (12) being dependent on fractional order  $\alpha$  and step  $\Delta x$  which was assumed in calculations.

We determine experimental estimation of the convergence row (EOC) as

$$EOC = \log_2 \left( \frac{\text{error}[N]}{\text{error}[2N]} \right), \quad (16)$$

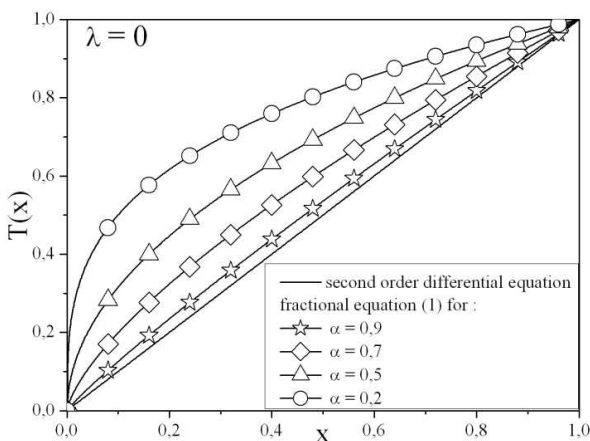
where

$$error[N] = \frac{\frac{1}{2}|T(x_0) - T_0| + \frac{1}{2}|T(x_N) - T_N| + \sum_{i=1}^{N-1}|T(x_i) - T_i|}{N}, \quad (17)$$

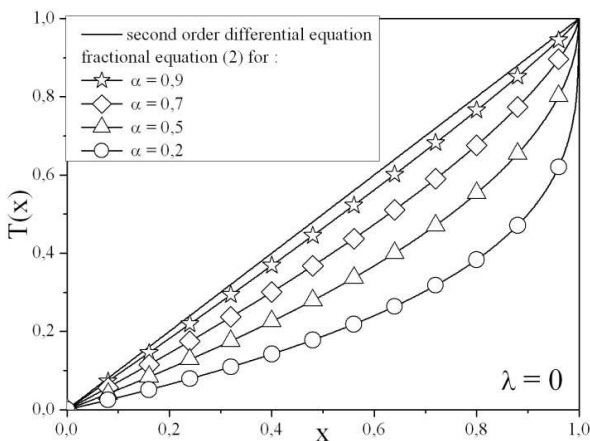
In the error calculations we take into account boundary conditions (14).

**Tab. 1.** Errors and experimental estimation of the convergence row (EOC) generated by the numerical scheme (12)

$\Delta x$	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	error	EOC	error	EOC	error	EOC
1/16	1.51e-2		1.46e-2		1.05e-2	
1/32	8.47e-3	0.83	8.19e-3	0.83	6.05e-3	0.79
1/64	4.60e-3	0.88	4.41e-3	0.89	3.34e-3	0.86
1/128	2.44e-3	0.91	2.32e-3	0.93	1.80e-3	0.89



**Fig. 1.** Numerical solutions of Eqn. (1)



**Fig. 2.** Numerical solutions of Eqn. (2)

When we solve Eqn. (2) numerically with boundary conditions

$$T(0) = 1, \quad T(1) = 0 \quad (18)$$

then we obtain identical table of errors. This is resulted by the effect of relation between considered equations and the reflection operator (Błaszczuk and Ciesielski, 2010).

Analyzing values of EOC in table 1 one can observe that the convergence of our numerical schemes is  $O(h)$  and does not depend of parameter  $\alpha$ .

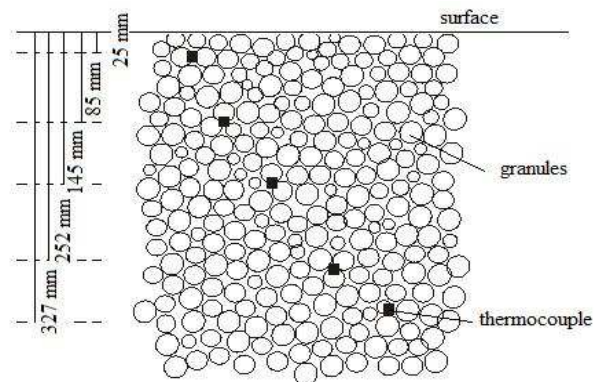
Next, we calculated some examples for different values of  $\alpha$  in order to show graphically how numerical solutions of Eqns. (1) and (2) behave.

In Figure 1 and 2 the solutions of Eqns. (1) and (2) for different values of the parameter  $\alpha$  are presented. One can see that both solutions are symmetrical. Analyzing the behavior of solutions we observe that  $T(x)$  tends to the solution of the classical second order ordinary differential equation for  $\alpha \rightarrow 1^-$ .

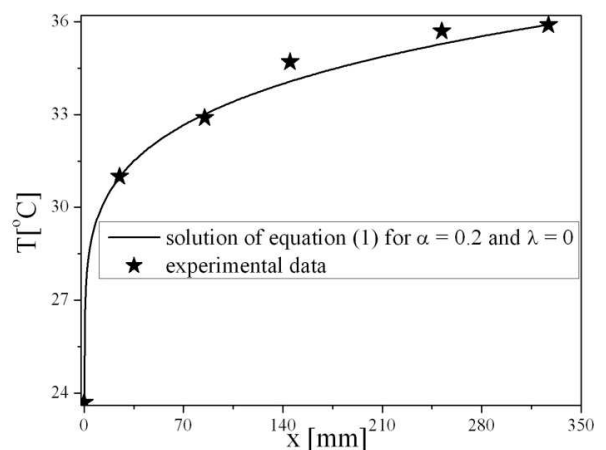
#### 4. APPLICATION

In order to show a practical application of Eqn. (1) we consider a steady state of heat transfer through the granular layer as presented by Fig. 3.

Using the idea presented in (US Department of Transportation, 2009) the experiment began with five thermocouples placed at depths 25 mm, 85 mm, 145 mm, 252 mm, 327 mm in the granular material which is used for road construction. Grains have specific parameters such as thermal conductivity, specific heat and density. It should be noted that the surface has been exposed to the weather conditions (irradiation, wind speed, relative humidity). Data from the thermocouples (US Department of Transportation, 2009) helped us to create a temperature profile.



**Fig. 3.** Experimental setup



**Fig. 4.** Comparison of Eqn. (1) with experiment data (US Department of Transportation, 2009)

In order to obtain the experimental results we approximate a temperature profile using the solution of the fractional Eqn. (1). Figure 4 presents comparison between experimental data and numerical results.

Analyzing changes in the temperature profile we can say that the nonlinear profile is observed. Additionally, we can observe a good agreement between the experimental data and the solution of the fractional Eqn. (1).

## 5. CONCLUSIONS

In this work the fractional differential equations with composition of the left- and the right-sided Caputo derivatives were considered. The analytical solutions of these equations are difficult to apply in practical calculations. Numerical solution is an alternative approach to the analytical one. In this study the numerical schemes were presented in order to obtain the solutions for considered equations. We show that the convergence row of our numerical schemes was  $O(h)$  and does not depend on parameter  $\alpha$ .

Our studies show that the model based on the fractional differential equation containing composition of the left- and the right-sided Caputo derivatives could reflect a steady state of the temperature profile in granular medium.

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