CONTACT STRENGTH OF A SYSTEM OF TWO ELASTIC HALF SPACES WITH AN AXIALLY SYMMETRIC RECESS UNDER COMPRESSION

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Abstract: Frictionless contact of two isotropic half spaces is considered one of which has a small smooth circular recess. A method of solving the corresponding boundary value problem of elasticity in axially symmetric case is presented via the function of gap height. The governing integral equation for this function is solved analytically by assuming a certain shape of the initial recess. On the basis of the closed-form solution obtained the strength analysis of a contact couple is performed and illustrated from the standpoint of fracture mechanics.

1. INTRODUCTION

The knowledge of the solutions (especially analytical solutions) to contact problems is a ground for the investigation of strength, durability, fatigue of contacting couples. The overwhelming majority of works devoted to strength of contacting joints utilizes the solutions to problems of penetration of rigid indenters into an elastic half-space (Hertzian contact). Extensive accounts can be found in the book by Kolesnikov and Morozov (1989). However, much research has been concerned with contact problems when conjugates solids touch at point or along the line before loading (contact with non-conformable boundaries, see a classification by Johnson, 1985). On the contrary, the contact interaction of bodies with conformable surfaces has been investigated much less. Approaches employing this kind of interaction take into account the existence of imperfections (recesses, pits, protrusions, concavities, etc.) of surfaces related to their small deviations from a flat onto local parts. Such perturbations lead to the local absence of contact, so the intercontact gaps are created.

The problem under study – compression of two semi-infinite isotropic elastic half spaces with an axially symmetric smooth recess – belongs to the class of non-classical contact problems involving contact interactions of solids with conformable boundaries. The purpose of the present work is to determine the stress distribution within the mated bodies and carry out a detailed analysis of strength of a contact couple from the point of view of fracture mechanics.

2. FORMULATION OF THE PROBLEM

Consider two isotropic elastic semi-infinite solids, being in frictionless contact due to uniform pressure p applied at infinity (see Fig. 1). The boundary of one (body 1) possesses a local deviation from the plane in the form of a small smooth circular recess with radius b. The shape of this imperfection is assumed to be axially symmetric and smooth. The boundary of the opposite body 2 is a plane.

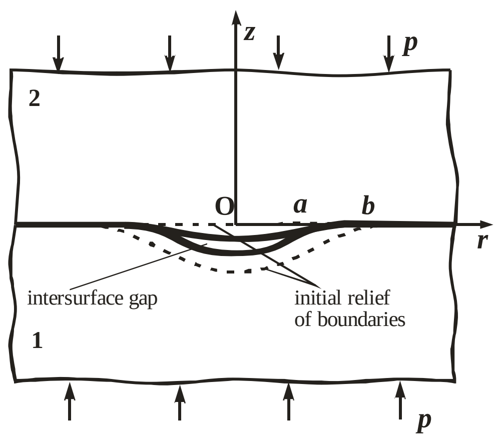

Fig. 1. Contact of two half-spaces with allowance for an intersurface gap

The problem is posed within the linear elasticity for axially symmetric case. In the cylindrical co-ordinate system the shape of the surface recess, occupying a circular region of radius b $\{(r, z = 0) : 0 \le r \le b\}$, is described by a function $f(r)$. This initial recess results in the formation of an intersurface gap of radius a that is unknown and depends on the pressure p (it is found in the process of solution of the problem). Thus, the nominal contact interface $z = 0$ is subdivided into two regions: the gap$\{(r, z = 0) : 0 \le r \le a\}$ and the region of body 1 – body 2 contact, defined by $\{(r, z = 0) : 0 \le r \le \infty\}$.

A method of solving the above problem is based on the use of the principle of superposition. Knowing a trivial solution corresponding to the basic stress and strain fields formed as a result of frictionless contact of half spaces with flat surfaces, we concentrate attention on the perturbed problem associated with the excited state caused by an initial geometrically perturbed surface (recess) and the created gap between the surfaces. For this problem, we get the following boundary conditions:

$$\text{at } z = \pm\infty: \quad \sigma_{zz}^{(i)} = 0, \quad \sigma_{zr}^{(i)} = 0, \tag{1}$$

at $z = 0$:

$$\begin{aligned} &\sigma_{rz}^{(i)} = 0, && 0 < r < \infty, \\ &\sigma_{zz}^{(1)} = \sigma_{zz}^{(2)}, && 0 < r < \infty, \\ &\sigma_{zz}^{(2)} = p, && 0 < r < a, \\ &u_z^{(1)} - u_z^{(2)} = f(r), && a < r < \infty . \end{aligned} \tag{1}$$

Here superscripts (i), i = 1, 2 refer the quantity to body 1 or 2, respectively.

Note, that the radius of the gap a is an unknown parameter. It can be found from the condition of smooth passage of gap's faces

$$h'(a) = 0, \tag{2}$$

where

$$h(r) = f(r) + u_z^{(2)}(r,0) - u_z^{(1)}(r,0) \tag{3}$$

is the height of the gap.

3. METHOD OF SOLUTION

The method of the solution to contact problems for semi-infinite solids with allowance for geometric surface disturbances has been developed in series of papers by Martynyak and co-workers (Martynyak, 1985; Shvets et al., 1996; Kit and Martynyak, 1999; Martynyak, 2000, Kit et al., 2001). Some new results dealing with the local contact absence are given in works by Kaczyński and Monastyrskyy (2002,2005), Monastyrskyy and Martynyak (2003).

The main idea of this method consists in the following:
(i) construction of the representation of the stresses and displacements within the every of mated solids through the function of the gaps' height;
(ii) subsequent reduction of the problem to some integral equations for this function.

3.1. Representation of stresses and displacements

Following the approach in the axially symmetric problems, developed by Monastyrskyy (2002), the appropriate representations of the displacements and stresses through the Hankel transform $H(\xi)$ of the gap's function $h(r)$ ($H(\xi) = \int_0^\infty \rho h(\rho) J_0(\xi\rho) d\rho$) has been constructed. It was shown that the function $H(\xi)$ satisfies the dual integral equations

$$\begin{aligned} &\int_0^\infty \xi^2 H(\xi) J_0(\xi r) d\xi = -\frac{p}{M} + \\ &\qquad + \int_0^\infty \xi^2 F(\xi) J_0(\xi r) d\xi, \quad 0 < r < a, \\ &\int_0^\infty \xi H(\xi) J_0(\xi r) d\xi = 0, \qquad a < r < \infty. \end{aligned} \tag{4}$$

Moreover, the expressions for components of displacement vector $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ through the function $H(\xi)$ are given as

$$\begin{aligned} &\frac{u_r^{(i)}(r,z)\, m_i\, 2(1-\nu_i)}{M} = \\ &= \int_0^\infty \xi(1-2\nu_i - \xi|z|)(F(\xi) - H(\xi))e^{-\xi|z|} J_1(\xi r) d\xi, \\ &\frac{u_z^{(i)}(r,z)\, m_i\, 2(1-\nu_i)}{M} = \\ &= (-1)^{i+1} \int_0^\infty \xi(2(1-\nu_i) + \xi|z|)(F(\xi) - H(\xi))e^{-\xi|z|} J_0(\xi r) d\xi, \\ &\frac{\sigma_{rz}^{(i)}(r,z)}{M} = z \int_0^\infty \xi^3 (F(\xi) - H(\xi))e^{-\xi|z|} J_1(\xi r) d\xi, \\ &\frac{\sigma_{zz}^{(i)}(r,z)}{M} = \int_0^\infty \xi^2 \left[(1+\xi|z|)(F(\xi) - H(\xi))\right] e^{-\xi|z|} J_0(\xi r) d\xi, \\ &\frac{\sigma_{rr}^{(i)}(r,z)}{M} = \int_0^\infty \xi^2 \left[(1-\xi|z|)(F(\xi) - H(\xi))\right] e^{-\xi|z|} J_0(\xi r) d\xi - \\ &\qquad - \int_0^\infty \xi \left[(1-2\nu_i - \xi|z|)(F(\xi) - H(\xi))\right] e^{-\xi|z|} \frac{J_1(\xi r)}{r} d\xi, \\ &\frac{\sigma_{\theta\theta}^{(i)}(r,z)}{M} = 2\nu_i \int_0^\infty \xi^2 (F(\xi) - H(\xi))e^{-\xi|z|} J_0(\xi r) d\xi + \\ &\qquad + \int_0^\infty \xi \left[(1-2\nu_i - \xi|z|)(F(\xi) - H(\xi))\right] e^{-\xi|z|} \frac{J_1(\xi r)}{r} d\xi . \end{aligned} \tag{5}$$

In the above, $F(\xi) = \int_0^\infty \rho f(\rho) J_0(\xi\rho) d\rho$ is the Hankel transform of the function of the initial recess shape $f(r)$, $m_i = \mu_i/(1 - \nu_i)$, where μ_i, ν_i stand for shear modulus and Poisson's ratio of the body denoted by i = 1,2 and $M = m_1 m_2/(m_1 + m_2)$. Hence the contact problem is reduced to solving the dual integral equations (5).

3.2. Integral equation and its solution

The obtained dual integral equation (5) is well studied in literature. The technique of its solution is known (Uflyand, 1977, Sneddon, 1966). Representing the sought function $H(\xi)$ as

$$H(\xi) = \xi^{-1} \int_0^a \gamma(\rho) \sin \xi\rho \, d\rho, \tag{6}$$

the equations (5) can be reduced to Abel's integral equation for function $\gamma(r)$

$$\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{\rho\,\gamma(\rho)\,d\rho}{\sqrt{r^2-\rho^2}} = g(r) \tag{7}$$

with the solution (Barber, 1983)

$$\gamma(r) = \frac{2}{\pi}\int_0^r \frac{\rho\, g(\rho)\,d\rho}{\sqrt{r^2-\rho^2}}, \tag{8}$$

where $g(r)$ stands for RHS of equation $(5)_1$.

To complete solving the problem in hand, it is necessary to determine the radius of the gap a. To do this, we utilize the condition (3) of smooth closure of the gap. Determining the height of the gap $h(r)$ via the function $\gamma(r)$

$$h(r) = \int_r^a \frac{\gamma(\rho)\,d\rho}{\sqrt{\rho^2-r^2}} \tag{9}$$

we see that the condition (3) is equivalent to the equation

$$\gamma(a) = 0\,. \tag{10}$$

Once the function $\gamma(r)$ and radius of the gap a are found, the stress and displacement fields within every solid can be recovered by virtue of relations (6) with the aid of (7) and (9).

4. EXAMPLE

As an example, assume that the shape of the initial recess is given by formula

$$f(r) = h_0\left(1 - r^2/b^2\right)^{3/2}. \tag{11}$$

For this case the Hankel transform $F(\xi)$ of $f(r)$ is

$$\begin{gathered} F(\xi) = h_0 b^2 (\xi b)^{-5}\left(-3(\xi b)^2\sin(\xi b) + \right. \\ \left. +9\sin(\xi b) - 9\xi b\cos(\xi b)\right). \end{gathered} \tag{12}$$

The function $\gamma(r)$, calculated from (9) is

$$\gamma(r) = -\frac{2}{\pi}\left[\frac{1}{M}p - \frac{3\pi}{4}\frac{h_0}{b}\left(1-\frac{r^2}{b^2}\right)\right]r\,. \tag{13}$$

The solution of equation (11), provided the function $\gamma(r)$ is given by (14), yields the following value of the radius of the gap a :

$$a = b\sqrt{1-\frac{p}{M}\frac{3\pi}{4}\frac{h_0}{b}}\,. \tag{14}$$

Now it follows that there is a certain level of external load, namely, $p = 4bM/3\pi h_0$, for which the radius of the gap becomes zero. It means that for this magnitude of the pressure the gap is closed and the contact of the solids is realized through the whole contact interface $z = 0$. The dependence $a = a(p)$ is shown in Fig.2. The following dimensionless parameters have been introduced: $\bar{a} = a/b$, $\bar{p} = p/M$, $\bar{h}_0 = h_0/b = 10^{-3}$.

Whereas the function $\gamma(r)$ and the gap's radius a are known, the complete solution can be determined from relations (6) and (7). Thus, the solution to the contact problem can be rewritten through the corresponding integrals. After calculations we obtain

$$\begin{aligned}
\frac{\sigma_{rz}^{(i)}(r,z)}{M} &= z\frac{h_0}{b^3}\Big(-3b^2 Int_1(r,z,b) + 9Int_2(r,z,b) + \\
&\quad +3a^2 Int_1(r,z,a) - 9Int_2(r,z,a)\Big), \\
\frac{\sigma_{zz}^{(i)}(r,z)}{M} &= -p + \frac{h_0}{b^3}\Big(-3b^2 Int_3(r,z,b) + 9Int_4(r,z,b) + \\
&\quad +z\big(-3b^2 Int_5(r,z,b) + 9Int_6(r,z,b)\big) + \\
&\quad +3a^2 Int_3(r,z,a) - 9Int_4(r,z,a) + \\
&\quad +z\big(3a^2 Int_5(r,z,a) - 9Int_6(r,z,a)\big)\Big), \\
\frac{\sigma_{rr}^{(i)}(r,z)}{M} &= \\
&-\frac{h_0}{b^3}\Big((1-2\nu_i)\big(-3b^2 Int_7(r,z,b) + 9Int_8(r,z,b)\big) + \\
&\quad + z\big(-3b^2 Int_9(r,z,b) + 9Int_{10}(r,z,b)\big) + \\
&\quad +(1-2\nu_i)\big(3a^2 Int_7(r,z,a) - 9Int_8(r,z,a)\big) + \\
&\quad +z\big(3a^2 Int_9(r,z,a) - 9Int_{10}(r,z,a)\big)\Big) + \\
&+\frac{h_0}{b^3}\Big(-3b^2 Int_3(r,z,b) + 9Int_4(r,z,b) + \\
&\quad +z\big(-3b^2 Int_5(r,z,b) + 9Int_6(r,z,b)\big) + \\
&\quad +3a^2 Int_3(r,z,a) - 9Int_4(r,z,a) + \\
&\quad +z\big(3a^2 Int_5(r,z,a) - 9Int_6(r,z,a)\big)\Big), \\
\frac{\sigma_{\theta\theta}^{(i)}(r,z)}{M} &= \\
&\frac{h_0}{b^3}\Big((1-2\nu_i)\big(-3b^2 Int_7(r,z,b) + 9Int_8(r,z,b)\big) + \\
&\quad +z\big(-3b^2 Int_9(r,z,b) + 9Int_{10}(r,z,b)\big) + \\
&\quad +(1-2\nu_i)\big(3a^2 Int_7(r,z,a) - 9Int_8(r,z,a)\big) + \\
&\quad +z\big(3a^2 Int_9(r,z,a) - 9Int_{10}(r,z,a)\big)\Big) + \\
&+2\nu_i\frac{h_0}{b^3}\Big(-3b^2 Int_3(r,z,b) + 9Int_4(r,z,b) + \\
&\quad +3a^2 Int_3(r,z,a) - 9Int_4(r,z,a)\Big),
\end{aligned} \tag{15}$$

where $Int_k(r,z,b)$ ($k = \overline{1,10}$) stand for integrals given in Appendix.

The solution of the axially symmetric contact problem for semi-infinite solids with a surface recess given by formula (16) is found thus analytically.

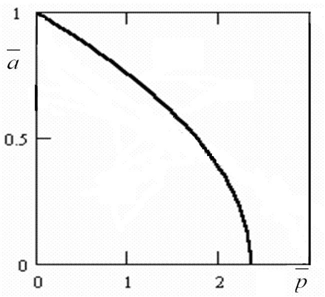

Fig. 2. Dependence of radius of the gap on the applied load

4.1. Contact strength

The obtained closed-form solution can be useful for analyzing the assessment of strength of the contacting couple. To estimate the strength of a system of two mated elastic half spaces allowing for unevenness of their boundaries, we shall use the classical criteria of fracture: the criterion of maximal principle stresses and the criterion of maximal shear stresses (Božydarnyk and Sulym, 1999).

It's worth noting that an analysis of the stress distribution within the every solid reveals that the stresses σ_{zz}, σ_{rr}, $\sigma_{\theta\theta}$ are the principle stresses at the contact boundary (z = 0). Moreover, the principle stresses achieve their extreme value at z = 0. That's why we pay our attention to the analysis of stresses at the contact interface.

The maximum compressive stresses

Fig. 3. shows the distribution of contact normal stresses σ_{zz}, being the maximal compressive stresses. The stresses are zero within the gap region, then they increases, achieving the maximal value at $\bar{r}=1$ which corresponds to the initial recess' tip. Then they asymptotically approach to the magnitude of external pressure p.

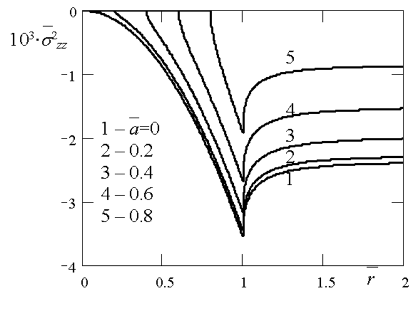

Fig. 3. The distribution of stresses σ_{zz} at the contact interface
($\bar{\sigma}_{zz} = \sigma_{zz}/M$)

According to the criteria of maximal principle stresses, from an analysis of stress distribution σ_{zz} one can conclude that the most dangerous zone is the vicinity of the recess' tip. Cracking of materials caused by the compressive stresses initiates most likely in the vicinity of the tip of the surface geometrical imperfection.

The maximum tensile stresses

Fig. 4 and 5 present the distribution of radial σ_{rr} and circular $\sigma_{\theta\theta}$ stresses at the contact interface. They reveal an interesting effect – the existence of tensile stresses at the contact interface. The stresses σ_{rr} and $\sigma_{\theta\theta}$ are: (*i*) tensile, (*ii*) constant and (*iii*) equal to each other within at the gap's faces. The magnitude of tensile stresses σ_{rr} and $\sigma_{\theta\theta}$ at the gap's faces is determined through the applied pressure and mechanical properties of the solids as

$$\sigma_{rr}^{(i)}(r,0) = \sigma_{\theta\theta}^{(i)}(r,0) = (1+2\nu_i)\, p/2, \quad 0 < r < a . \tag{16}$$

Thus, the cracking can be initiated by tensile stresses. The most dangerous region is the gap. Moreover, the possibilities of cracks initiating along radial and circular directions are equal.

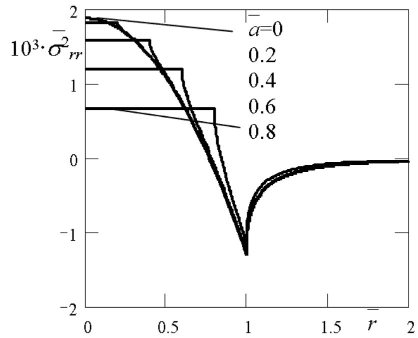

Fig. 4. The distribution of stresses σ_{rr} at the contact interface
($\bar{\sigma}_{rr} = \sigma_{rr}/M$)

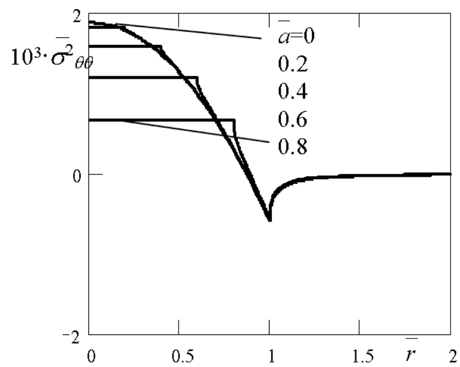

Fig. 5. The distribution of stresses $\sigma_{\theta\theta}$ at the contact interface
($\bar{\sigma}_{\theta\theta} = \sigma_{\theta\theta}/M$)

The maximum shear stresses

The analysis of maximal shear stresses at the contact interface has been carried out. Fig. 6 shows the distribution $\tau_{\max}$. Based on the criteria of maximal shear stresses, which are used for assessment of plastic zones initiated, the most dangerous zone is the vicinity of the recess' tip.

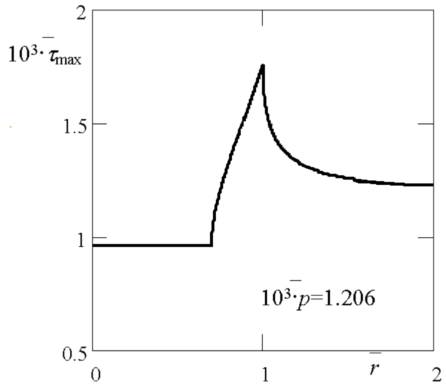

Fig. 6. The distribution of maximal shear stresses τ_{max} at the contact interface

5. CONCLUSIONS

The found closed-form solution to the contact problem has served as a theoretical basis for an analysis of strength of the contacting couple with a small surface recess. The analysis has been carried out by utilizing classical fracture criteria, namely, the criteria of maximal principle stresses and the criterion of maximal shear stresses.

The cracking of the material of the mated solids can be caused by both compressive and tensile stresses. In former case the most possible region where the cracks can be initiated is the vicinity of the recess tip. The compressive stresses achieve their maximal value at the contact interface at the tip of the recess. On other hand, the tensile stresses σ_{rr} and $\sigma_{\theta\theta}$ in the vicinity of the gap, appear at the interface. Moreover, the maximal value is achieved at the gap's faces, where σ_{rr} and $\sigma_{\theta\theta}$ are constant and equal to each other. The magnitude of the maximal tensile stress depends on Poisson's ratio of the material and lies in the range between 50% and 100% of the value of applied load at infinity. Two directions of cracking along the radial and circular co-ordinate lines are equally possible.

According to the criterion of the maximal shear stresses the most possible region where plastic zones can be initiated is the vicinity of the recess tip.

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APPENDIX

The values of the integrals appearing in (16) are as follows:

$$Int_1(r,z,b)=\int_0^\infty e^{-\xi z}\sin(\xi b)J_1(\xi r)d\xi=\begin{cases}\dfrac{b\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{2zb}{z^2-b^2+r^2}\right)\right)-z\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{2zb}{z^2-b^2+r^2}\right)\right)}{r\sqrt[4]{\left(z^2-b^2+r^2\right)^2+\left(2zb\right)^2}}, & z\neq 0;\\ 0, \quad z=0, \quad r\le b; & \\ \dfrac{b}{r\sqrt{r^2-b^2}}, \quad z=0, \; r>b. & \end{cases}$$

$$Int_2(r,z,b)=\int_0^\infty e^{-\xi z}\frac{\sin(\xi b)-\xi b\cos(\xi b)}{\xi^2}J_1(\xi r)d\xi=\begin{cases}\int_0^b \beta Int_1(r,z,\beta)d\beta, & z\neq 0;\\ \frac{\pi r}{4}, & z=0,\quad r\le b;\\ -\frac{b\sqrt{r^2-b^2}}{2r}+\frac{r}{2}\sin^{-1}\left(\frac{b}{r}\right), & z=0,\ r>b.\end{cases}$$

$$Int_3(r,z,b)=\int_0^\infty e^{-\xi z}\frac{\sin(\xi b)}{\xi}J_0(\xi r)d\xi=\begin{cases}\sin^{-1}\left(\frac{2b}{\sqrt{(b+r)^2+z^2}+\sqrt{(b-r)^2+z^2}}\right), & z\neq 0;\\ \frac{\pi}{2}, & z=0,\quad r\le b;\\ \sin^{-1}\left(\frac{b}{r}\right), & z=0,\ r>b.\end{cases}$$

$$Int_4(r,z,b)=\int_0^\infty e^{-\xi z}\frac{\sin(\xi b)-\xi b\cos(\xi b)}{\xi^3}J_0(\xi r)d\xi=\begin{cases}\int_0^b \beta Int_3(r,z,\beta)d\beta, & z\neq 0;\\ \frac{\pi\left(2b^2-r^2\right)}{8}, & z=0,\quad r\le b;\\ \frac{1}{4}\left(b\sqrt{r^2-b^2}+\left(2b^2-r^2\right)\sin^{-1}\left(\frac{b}{r}\right)\right), & z=0,\ r>b.\end{cases}$$

$$Int_5(r,z,b)=\int_0^\infty e^{-\xi z}\sin(\xi b)J_0(\xi r)d\xi=\begin{cases}\frac{2bz}{\sqrt{\left(\sqrt{(b+r)^2+z^2}+\sqrt{(b-r)^2+z^2}\right)^2-b^2}\sqrt{(b+r)^2+z^2}\sqrt{(b-r)^2+z^2}}, & z\neq 0;\\ \frac{1}{\sqrt{b^2-r^2}}, & z=0,\quad r<b;\\ 0, & z=0,\quad r>b.\end{cases}$$

$$Int_6(r,z,b)=\int_0^\infty e^{-\xi z}\frac{\sin(\xi b)-\xi b\cos(\xi b)}{\xi^2}J_0(\xi r)d\xi=\begin{cases}\int_0^b \beta Int_5(r,z,\beta)d\beta, & z\neq 0;\\ \sqrt{b^2-r^2}, & z=0,\quad r\le b;\\ 0, & z=0,\quad r>b.\end{cases}$$

$$Int_7(r,z,b)=\int_0^\infty \mathrm{e}^{-\xi z}\frac{\sin(\xi b)}{\xi^2}\frac{J_1(\xi r)}{r}d\xi=$$

$$=\begin{cases}\dfrac{1}{r^2}\left(-zb+\dfrac{1}{2\sqrt{2}}\left(z\sqrt{\sqrt{\left(z^2-b^2+r^2\right)^2+4z^2b^2}-\left(z^2-b^2+r^2\right)}+b\sqrt{\sqrt{\left(z^2-b^2+r^2\right)^2+4z^2b^2}+\left(z^2-b^2+r^2\right)}\right)\right)+\\ \quad +\dfrac{1}{2}\tan^{-1}\left(\dfrac{\sqrt{2}b+\sqrt{\sqrt{\left(z^2-b^2+r^2\right)^2+4z^2b^2}-\left(z^2-b^2+r^2\right)}}{\sqrt{2}z+\sqrt{\sqrt{\left(z^2-b^2+r^2\right)^2+4z^2b^2}+\left(z^2-b^2+r^2\right)}}\right), & z\neq 0;\\ \dfrac{\pi}{4}, & z=0,\quad r<b;\\ \dfrac{b}{2r^2}\sqrt{r^2-b^2}+\dfrac{1}{2}\tan^{-1}\left(\dfrac{b}{\sqrt{r^2-b^2}}\right), & z=0,\quad r>b.\end{cases}$$

$$Int_8(r,z,b)=\int_0^\infty \mathrm{e}^{-\xi z}\frac{\sin(\xi b)-\xi b\cos(\xi b)}{\xi^4}\frac{J_1(\xi r)}{r}d\xi=\begin{cases}\int_0^b \beta Int_7(r,z,\beta)d\beta, & z\neq 0;\\ \dfrac{\pi\left(4b^2-r^2\right)}{32}, & z=0,\quad r\leq b;\\ \dfrac{1}{16r^2}\left(b\left(2b^2+r^2\right)\sqrt{r^2-b^2}+\left(4b^2r^2-r^4\right)\sin^{-1}\left(\dfrac{b}{r}\right)\right), & z=0,\ r>b.\end{cases}$$

$$Int_9(r,z,b)=\int_0^\infty \mathrm{e}^{-\xi z}\frac{\sin(\xi b)}{\xi}\frac{J_1(\xi r)}{r}d\xi=\begin{cases}\dfrac{1}{r^2}\left(b-\dfrac{1}{\sqrt{2}}\sqrt{\sqrt{\left(z^2-b^2+r^2\right)^2+4z^2b^2}-\left(z^2-b^2+r^2\right)}\right), & z\neq 0;\\ \dfrac{1}{b+\sqrt{b^2-r^2}}, & z=0,\quad r<b;\\ \dfrac{b}{r^2}, & z=0,\quad r>b.\end{cases}$$

$$Int_{10}(r,z,b)=\int_0^\infty \mathrm{e}^{-\xi z}\frac{\sin(\xi b)-\xi b\cos(\xi b)}{\xi^3}\frac{J_1(\xi r)}{r}d\xi=\begin{cases}\int_0^b \beta Int_9(r,z,\beta)d\beta, & z\neq 0;\\ \dfrac{1}{3}\dfrac{1}{b+\sqrt{b^2-r^2}}\left(2b^2-r^2+b\sqrt{b^2-r^2}\right), & z=0,\quad r\leq b;\\ \dfrac{b^3}{3r^2}, & z=0,\ r>b.\end{cases}$$