ELASTOSTATIC PROBLEM FOR AN INTERFACE RIGID INCLUSION IN A PERIODIC TWO-LAYER SPACE

Andrzej KACZYŃSKI*, Wojciech KOZŁOWSKI*

*Faculty of Mathematics and Information Science, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw

akacz@mini.pw.edu.pl, wojthekk@wp.pl

Abstract: The article is devoted to the elastostatic three-dimensional problem of an interface sheet-like inclusion (anticrack) embedded into a periodic two-layered unbounded composite. An approximate analysis is carried out within the framework of the homogenized model with microlocal parameters. The formulation and the method of solving the general problem for an arbitrarily shaped inclusion is presented. As an example illustrating this method, the problem for a rigid circular inclusion under perpendicular tension is solved explicitly and discussed from the point of view of failure theory.

1. INTRODUCTION

Problems dealing with stress concentrations in deformable solids containing different kinds of defects attract the attention of specialists from many areas, such as geomechanics, metallurgy, materials science. In recent decades interest in the study of interface fracture phenomena has grown considerably (see, for example, the proceedings edited by Rossmanith (1997)). Rigid inclusions (called anticracks) are the counterpart of cracks. From the standpoint of inhomogeneities in solids, these defects are the two dangerous extreme cases, namely, for a rigid inclusion $\mu \to \infty$, and for a crack $\mu \to 0$, where μ is the shear modulus of the inhomogeneity phase. Interfacial inclusions play a significant role in the failure behavior. As well known, serious stress concentrations occur near the sharp edges of the inclusions, from which cracking, debonding, damage and so on may emanate. Therefore, the studies in this area with the aim of obtaining the theoretical solutions of the problems involving rigid inclusions under different loading conditions are important for structural integrity assessments. In comparison with crack problems, the investigation of anticracks problems is rather limited, and basic research has been performed on two-dimensional problems involving rigid line inclusions in elastic homogeneous media (see the monographs by Berezhnitskii et al. (1983) and Ting (1996)). The corresponding, more practical three-dimensional problems dealing with rigid sheet-like disc inclusions seem to remain inadequately treated and have been performed to a much lesser extent. Much of the past works related in this field can be found in Kassir and Sih (1968), Selvadurai (1982), Silovanyuk (1984), Podil'chuk (1997), Rahman (1999), and in the basic monographs by Mura (1981) and Panasyuk et al. (1986). The studies of 3D problems of rigid inclusions at the interface a bimaterial have been found only in Gladwell (1999), Selvadurai (2000), Li and Fan (2001), Chaudhuri (2006). The results show that the asymptotic stress elastic fields near the rigid inclusion front exhibit the oscillatory singularity similar to that for interface cracks. This physically anomalous behaviour does not occur in numerous problems of interface cracks or anticracks in a periodically layered space (see, for example, Kaczyński and Matysiak 1997, 1999) treated within the framework of linear thermoelasticity with microlocal parameters (Woźniak, 1987; Matysiak and Woźniak, 1988).

This paper is devoted to a three-dimensional static problem of an arbitrary shaped rigid inclusion lying on one of the interfaces in a periodic two-layer laminated space subjected to some external loads. An approximate analysis is based on the concept of microlocal homogenization that leads to a replacement of the considered periodic composite by some homogenized model with microlocal parameters. In Section 2 we review governing equations and formulate the anticrack problem within this model. Section 3 presents a general method of solving the resulting boundary value problem. As an illustration, a closed-form solution is given and discussed in Section 4 for a circular rigid inclusion subjected to tension at infinity.

2. GOVERNING EQUATIONS AND FORMULATION

The composite being considered is a periodic laminated space consisting of thin repeated fundamental layers of thickness δ which is composed of two bonded homogeneous isotropic layers denoted by 1 and 2 as shown in Fig. 1. In the following, all quantities (material constants, stresses, etc.) pertinent to these sublayers will be denoted with the index l or (l) taking the values 1 and 2, respectively. Let λ_1, μ_1 be the Lamé constants, and δ_l be the thicknesses of subsequent sublayers, thus $\delta = \delta_1 + \delta_2$.

Referring to the rectangular Cartesian coordinate system $OX_1X_2X_3$ with the X_3 – axis directed normal to the layering and the X_1X_2 – plane being one of the interfaces of the materials, denote at the typical point $x = (x_1, x_2, x_3)$ the components of the displacement vector and stress tensor by w_i and σ_{ij}, $i, j \in \{1,2,3\}$, respectively.

Suppose that a rigid sheet-like inclusion (anticrack) serving as a mechanical defect in this periodically layered composite occupies a domain S with smooth boundary at the interface $x_3 = 0$ and is subjected to some external loads.

To analyze the elastic field disturbed by this defect, a direct analytical approach becomes intricate because of the complicated geometry and complex boundary conditions. Therefore, the special homogenization procedure called microlocal modelling will be employed in order to seek an approximate solution within the homogenized model of the considered composite. Next, we recall only some relevant results from this approach (see Matysiak and Woźniak (1988), Kaczyński (1993) for more details).

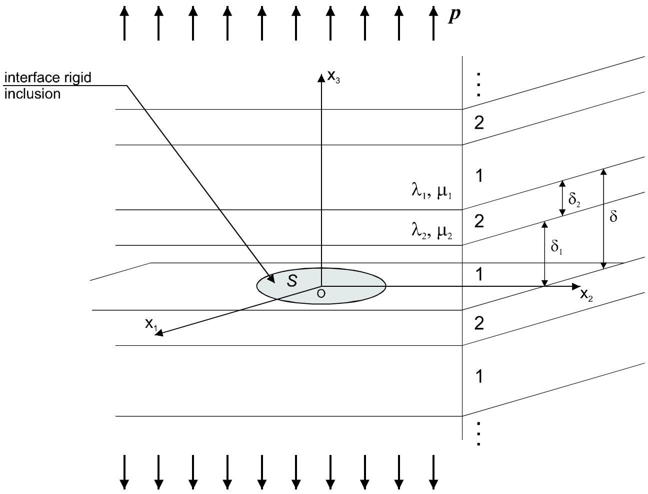

Fig. 1. Two-layer periodic space with an interface anticrack

In the subsequent considerations the following notation will be used: Latin subscripts always assume values 1, 2, 3 and the Greek ones 1,2. The Einstein summation convention holds and a comma followed by an index denotes the partial differentiation with respect to the corresponding coordinate variable.

The microlocal modelling is based on the following displacement representations:

$$w_i(\boldsymbol{x}) = u_i(\boldsymbol{x}) + \underline{s(x_3)\, d_i(\boldsymbol{x})}. \tag{1}$$

Here the unknown functions u_i and d_i are interpreted as macro-displacements and microlocal parameters, respectively. Moreover, the postulated *a priori* function s, called the shape function, characterises the special approximate model of the treated composite. It is chosen to be sectionally linear, δ – periodic, defined as

$$s(x_3) = \begin{cases} x_3 - 0{,}5\delta_1, & x_3 \in \langle 0, \delta_1 \rangle \\ (\delta_1 - \eta x_3)/(1-\eta) - 0{,}5\delta_1, & x_3 \in \langle \delta_1, \delta \rangle \end{cases} ;\ \eta = \delta_1 / \delta. \tag{2}$$

The underlined term in Eq. (1) represents the micro-displacements due to the microperiodic material structure of the composite. Note, that for thin layers (δ is small) this term may be treated as small and can be neglected, but the derivative s' is a sectionally constant function (taking the value 1 if $l = 1$ and $-\eta/(1-\eta)$ if $l = 2$) that is not small even for small δ. Hence, the following approximations for the displacements and stresses (according to Hooke's law) hold:

$$\begin{aligned} & w_i \approx u_i, \quad w_{i,\alpha} \approx u_{i,\alpha}, \quad w_{i,3} \approx u_{i,3} + s' d_i, \\ & \sigma_{\alpha\beta} \approx \mu_l \left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) + \delta_{\alpha\beta}\, \lambda_l \left(u_{i,i} + s' d_3\right), \\ & \sigma_{\alpha 3} \approx \mu_l \left(u_{\alpha,3} + u_{3,\alpha} + s' d_\alpha\right), \\ & \sigma_{33} \approx \left(\lambda_l + 2\mu_l\right)\left(u_{3,3} + s' d_3\right) + \lambda_l u_{\gamma,\gamma}, \end{aligned} \tag{3}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

By using methods of the nonstandard analysis to the homogenization procedure, the asymptotic approach to the macro-modelling of the laminated space under study leads to the governing relations of certain macro-homogeneous medium (the homogenized model), given by means of macro-displacements (after eliminating the microlocal parameters) and taking the following form (in the absence of body forces and in the static case):

$$\begin{cases} \tfrac{1}{2}(c_{11} + c_{12}) u_{\gamma,\gamma\alpha} + \tfrac{1}{2}(c_{11} - c_{12}) u_{\alpha,\gamma\gamma} + \\ + c_{44} u_{\alpha,33} + (c_{13} + c_{44}) u_{3,3\alpha} = 0, \\ (c_{13} + c_{44}) u_{\gamma,\gamma 3} + c_{44} u_{3,\gamma\gamma} + c_{33} u_{3,33} = 0 \end{cases} \tag{4}$$

$$\begin{cases} \sigma_{\alpha 3} = c_{44}\left(u_{\alpha,3} + u_{3,\alpha}\right), \\ \sigma_{33} = c_{13} u_{\gamma,\gamma} + c_{33} u_{3,3}, \\ \sigma_{12}^{(l)} = \mu_l \left(u_{1,2} + u_{2,1}\right), \\ \sigma_{11}^{(l)} = d_{11}^{(l)} u_{1,1} + d_{12}^{(l)} u_{2,2} + d_{13}^{(l)} u_{3,3}, \\ \sigma_{22}^{(l)} = d_{12}^{(l)} u_{1,1} + d_{11}^{(l)} u_{2,2} + d_{13}^{(l)} u_{3,3}. \end{cases} \tag{5}$$

Positive coefficients appearing in the above equations, describing the material and geometrical characteristics of the subsequent layers, are given in Appendix A.

The advantage of the governing equations is their relatively simple form resembling fundamental equations for a transversely isotropic body. Moreover, the condition of perfect mechanical bonding between the layers (the continuity of the stress vector at the interfaces) is satisfied, so hereafter we shall omit the index (l) in the components σ_{i3}. Note, however, that the stress components $\sigma^{(l)}_{\alpha\beta}$ are discontinuous at the interfaces. Finally, putting $\lambda_1 = \lambda_2 \equiv \lambda$, $\mu_1 = \mu_2 \equiv \mu$ entails $c_{11} = c_{33} = d^{(l)}_{11} = \lambda + 2\mu$, $c_{11} = c_{33} = d^{(l)}_{11} = d^{(l)}_{13} = \lambda$, $c_{44} = \mu$, and Eqs (4) and (5) reduce to the well-known equations of elasticity for a homogeneous isotropic body with Lamé's constants λ, μ.

Within the scope of the presented homogenized model we are concerned with the problem of a rigid inclusion occupying a region S at the interface $x_3 = 0$ and subjected to external loadings. In order to satisfy the global mechanical boundary condition ensuring that the faces of inclusion are free from displacements, superposition is applied to separate this problem into two parts: the first one (attached by 0) relating to a basic state of the homogenized space with no inclusion subjected to the given loads and the second, corrective part (with tilde) in which the displacements along S are prescribed as the negative of those generated in the first part. In addition, the displacement and rotation of the inclusion as a rigid body ought to be taken into consideration. Thus, the complete field of the displacements u_i and stresses σ_{ij} in the composite with the inclusion can be represented in the form

$$u_i = \overset{0}{u}_i + \tilde{u}_i, \quad \sigma_{ij} = \overset{0}{\sigma}_{ij} + \tilde{\sigma}_{ij} \tag{6}$$

and in the following we assume that $\overset{0}{u}_i$ and $\overset{0}{\sigma}_{ij}$ are known from the solution to the first problem. In fact, only the values of $\overset{0}{u}_i(x_1, x_2, 0), \forall(x_1, x_2) \in S$ are needed in the subsequent analysis. Next, special attention is paid to the second non-trivial perturbed problem involving the local disturbance due to the presence of the anticrack S. The mathematical formulation of this boundary value problem is as follows: find fields $\tilde{u}_i$ and $\tilde{\sigma}_{ij}$, decaying at infinity, suitable smooth on $R^3 - S$, such that Eqs (4) and (5) hold subject to the boundary conditions on S

$$\begin{aligned} \tilde{u}_1 &= -\overset{0}{u}_1 + \varepsilon_1 - \omega_3 x_2, \\ \tilde{u}_2 &= -\overset{0}{u}_2 + \varepsilon_2 + \omega_3 x_1 \\ \tilde{u}_3 &= -\overset{0}{u}_3 + \varepsilon_3 - \omega_2 x_1 + \omega_1 x_2, \end{aligned} \tag{7}$$

where ε_i and ω_i are the unknown components of a small displacement vector and a small angle of rotation describing a motion of the inclusion as a rigid whole under the action of external loads. These parameters will be determined later in the course of solving the problem in hand from the equilibrium conditions of the anticrack (no resultant forces and zero-moments).

To reduce the above problem to mixed boundary value problems of potential theory associated with a half-space region (say, at $x_3 \geq 0$) and further, to integral equations (Kaczyński, 1999), we invoke the relevant symmetry properties about the plane $x_3 = 0$ and can split the problem into two subproblems:

(A) – the antisymmetric problem with the mixed conditions

$$\begin{aligned} &\tilde{u}_3 = -\overset{0}{u}_3 + \varepsilon_3 - \omega_2 x_1 + \omega_1 x_2, && \forall(x_1, x_2) \in S, \\ &\tilde{u}_1 = \tilde{u}_2 = 0, && \forall(x_1, x_2) \in R^2, \\ &\tilde{\sigma}_{33} = 0, && \forall(x_1, x_2) \in R^2 - S \end{aligned} \tag{8}$$

and supplemented by the corresponding equilibrium conditions to determine ε_3, ω_α

$$\begin{aligned} &\iint_S \left[\tilde{\sigma}_{33}\left(x_1, x_2, 0^+\right) - \tilde{\sigma}_{33}\left(x_1, x_2, 0^-\right) \right] dx_1 dx_2 = 0, \\ &\iint_S x_{3-\alpha} \left[\tilde{\sigma}_{33}\left(x_1, x_2, 0^+\right) - \tilde{\sigma}_{33}\left(x_1, x_2, 0^-\right) \right] dx_1 dx_2 = 0 \end{aligned} \tag{9}$$

(B) – the symmetric problem with the mixed conditions

$$\begin{aligned} &\tilde{u}_1 = -\overset{0}{u}_1 + \varepsilon_1 - \omega_3 x_2, && \forall(x_1, x_2) \in S, \\ &\tilde{u}_2 = -\overset{0}{u}_2 + \varepsilon_2 + \omega_3 x_1, && \forall(x_1, x_2) \in S, \\ &\tilde{u}_3 = 0, && \forall(x_1, x_2) \in R^2, \\ &\tilde{\sigma}_{31} = \tilde{\sigma}_{32} = 0, && \forall(x_1, x_2) \in R^2 - S \end{aligned} \tag{10}$$

and additional equilibrium conditions to determine ε_α, ω_3

$$\begin{aligned} &\iint_S \left[\tilde{\sigma}_{3\alpha}\left(x_1, x_2, 0^+\right) - \tilde{\sigma}_{3\alpha}\left(x_1, x_2, 0^-\right) \right] dx_1 dx_2 = 0, \\ &\iint_S \Big\{ x_2 \left[\tilde{\sigma}_{31}\left(x_1, x_2, 0^+\right) - \tilde{\sigma}_{31}\left(x_1, x_2, 0^-\right) \right] \\ &\quad - x_1 \left[\tilde{\sigma}_{32}\left(x_1, x_2, 0^+\right) - \tilde{\sigma}_{32}\left(x_1, x_2, 0^-\right) \right] \Big\} dx_1 dx_2 = 0. \end{aligned} \tag{11}$$

3. SOLVING THE ANTICRACK PROBLEM

For the solution of the problems (A) and (B) we use the potential function approach based on representing the components of displacements $\tilde{u}_i$ in terms of quasi-harmonic functions that satisfy the governing equations (4) and are well suited to the mixed boundary conditions (8) an (9).

According to the results obtained by Kaczyński (1993) the potential displacement representation is dependent on the material constants of the sublayers. Hereafter, only the general case $u_1 \neq u_2$ will be considered in which the displacements and stresses are expressed in terms of three harmonic potentials $\phi_i(x_1, x_2, z_i)$, $z_i = t_i x_3$, $\forall i \in \{1, 2, 3\}$ as

$$\begin{aligned} &u_1 = \left(\phi_1 + \phi_2\right)_{,1} - \phi_{3,2}, \quad u_2 = \left(\phi_1 + \phi_2\right)_{,2} + \phi_{3,1} \\ &u_3 = \sum_{\alpha=1}^{2} m_\alpha t_\alpha \frac{\partial \phi_\alpha}{\partial z_\alpha}, \\ &\frac{\sigma_{31}}{c_{44}} = \left[\sum_{\alpha=1}^{2} \left(1 + m_\alpha\right) t_\alpha \frac{\partial \phi_\alpha}{\partial z_\alpha} \right]_{,1} - t_3 \frac{\partial^2 \phi_3}{\partial z_3 \partial x_2}, \\ &\frac{\sigma_{32}}{c_{44}} = \left[\sum_{\alpha=1}^{2} \left(1 + m_\alpha\right) t_\alpha \frac{\partial \phi_\alpha}{\partial z_\alpha} \right]_{,2} + t_3 \frac{\partial^2 \phi_3}{\partial z_3 \partial x_1}, \\ &\frac{\sigma_{33}}{c_{44}} = \sum_{\alpha=1}^{2} \left(1 + m_\alpha\right) \frac{\partial^2 \phi_\alpha}{\partial z_\alpha^2}, \end{aligned} \tag{12}$$

$$\begin{aligned} \sigma_{12}^{(l)} &= \mu_l \left[2\left(\phi_1 + \phi_2\right)_{,12} + \phi_{3,11} - \phi_{3,22} \right], \\ \sigma_{11}^{(l)} &= d_{11}^{(l)} \left(\phi_1 + \phi_2\right)_{,11} + d_{12}^{(l)} \left(\phi_1 + \phi_2\right)_{,22} \\ &\quad + d_{13}^{(l)} \sum_{\alpha=1}^{2} m_\alpha t_\alpha^2 \frac{\partial^2 \phi_\alpha}{\partial z_\alpha^2} - 2\mu_l \phi_{3,12}, \\ \sigma_{22}^{(l)} &= d_{12}^{(l)} \left(\phi_1 + \phi_2\right)_{,11} + d_{11}^{(l)} \left(\phi_1 + \phi_2\right)_{,22} \\ &\quad + d_{13}^{(l)} \sum_{\alpha=1}^{2} m_\alpha t_\alpha^2 \frac{\partial^2 \phi_\alpha}{\partial z_\alpha^2} + 2\mu_l \phi_{3,12}, \end{aligned} \tag{13}$$

where the constants t_i, m_α are defined in Appendix B.

We now proceed to construct the potentials separately in subproblems (A) and (B) with the aim of their reducing to some mixed problems of potential theory.

Subproblem (A)

An appropriate displacement representation in terms of a single harmonic function $f(x_1, x_2, x_3)$ that frees the plane

x_3 = 0 of the displacement $\tilde{u}_\alpha$ is obtained by taking in the general solution (12)

$$\phi_\alpha(x_1, x_2, z_\alpha) = (-1)^\alpha f(x_1, x_2, z_\alpha), \quad \phi_3 \equiv 0. \tag{14}$$

Then the displacement and stress components are

$$\begin{aligned}
\tilde{u}_\alpha &= \sum_{\beta=1}^{2} (-1)^\beta \frac{\partial f(x_1, x_2, z_\beta)}{\partial x_\beta}, \\
\tilde{u}_3 &= \sum_{\beta=1}^{2} (-1)^\beta m_\beta t_\beta \frac{\partial f(x_1, x_2, z_\beta)}{\partial z_\beta}, \\
\tilde{\sigma}_{3\alpha} &= c_{44}\left[\sum_{\beta=1}^{2} (-1)^\beta (1+m_\beta) t_\beta \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial z_\beta \partial x_\alpha}\right], \\
\tilde{\sigma}_{33} &= c_{44}\left[\sum_{\beta=1}^{2} (-1)^\beta (1+m_\beta) \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial z_\beta^2}\right], \\
\tilde{\sigma}_{12}^{(l)} &= 2\mu_l \sum_{\beta=1}^{2} (-1)^\beta \frac{\partial f(x_1, x_2, z_\beta)}{\partial x_1 \partial x_2}, \\
\tilde{\sigma}_{11}^{(l)} &= \sum_{\beta=1}^{2} (-1)^\beta \left[d_{11}^{(l)} \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial x_1^2} + d_{12}^{(l)} \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial x_2^2}\right] \\
&\quad + d_{13}^{(l)} \sum_{\beta=1}^{2} (-1)^\beta m_\beta t_\beta^2 \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial z_\beta^2}, \\
\tilde{\sigma}_{22}^{(l)} &= \sum_{\beta=1}^{2} (-1)^\beta \left[d_{12}^{(l)} \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial x_1^2} + d_{11}^{(l)} \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial x_2^2}\right] \\
&\quad + d_{13}^{(l)} \sum_{\beta=1}^{2} (-1)^\beta m_\beta t_\beta^2 \frac{\partial^2 f(x_1, x_2, z_\beta)}{\partial z_\beta^2}.
\end{aligned} \tag{15}$$

Across the interface $x_3 = 0^\pm$ equations (15) become

$$\begin{aligned}
&\tilde{u}_1 = \tilde{u}_2 = 0, \\
&\tilde{u}_3^\pm = (m_2 t_2 - m_1 t_1)\left[f_{,3}(x_1, x_2, x_3)\right]_{x_3=0^\pm}, \\
&\tilde{\sigma}_{3\alpha}^\pm = c_{44}(t_- + m_2 t_2 - m_1 t_1)\left[f_{,3\alpha}(x_1, x_2, x_3)\right]_{x_3=0^\pm}, \\
&\tilde{\sigma}_{33}^\pm = c_{44}(m_2 - m_1)\left[f_{,33}(x_1, x_2, x_3)\right]_{x_3=0^\pm}, \\
&\tilde{\sigma}_{12} = 0, \\
&\tilde{\sigma}_{11}^\pm = \tilde{\sigma}_{22}^\pm = d_{13}^{(l)}\left(m_2 t_2^2 - m_1 t_1^2\right)\left[f_{,33}(x_1, x_2, x_3)\right]_{x_3=0^\pm} \\
&\left(\text{substitute } d_{13}^{(1)} \text{ for } x_3 = 0^+,\ d_{13}^{(2)} \text{ for } x_3 = 0^-\right).
\end{aligned} \tag{16}$$

The application of conditions (8) leads to the boundary conditions

– for $(x_1, x_2) \in S$

$$\left[f_{,3}(x_1, x_2, x_3)\right]_{x_3=0^+} = \frac{-\overset{0}{u}_3(x_1, x_2, 0^+) + \varepsilon_3 - \omega_2 x_1 + \omega_1 x_2}{m_2 t_2 - m_1 t_1},$$

– for $(x_1, x_2) \in R^2 - S$

$$\left[f_{,33}(x_1, x_2, x_3)\right]_{x_3=0^+} = 0. \tag{17}$$

The mixed boundary value problem posed by the above equations is regarded as the classical one appearing in typical electrostatic and punch problems (Sneddon, 1966). It is reduced to an integral equation by assuming the following representation for the unknown potential f and its derivative $f_{,3}$:

$$\begin{aligned}
f(\boldsymbol{x}) &= -\frac{1}{2\pi c_{44}(m_2 - m_1)} \iint_S q(\boldsymbol{\xi}) \ln\left(|\boldsymbol{x} - \boldsymbol{\xi}| + x_3\right) dS_{\boldsymbol{\xi}}, \\
f_{,3}(\boldsymbol{x}) &= -\frac{1}{2\pi c_{44}(m_2 - m_1)} \iint_S \frac{q(\boldsymbol{\xi})\, dS_{\boldsymbol{\xi}}}{|\boldsymbol{x} - \boldsymbol{\xi}|}.
\end{aligned} \tag{18}$$

Here, the unknown layer density $q(\xi_1, \xi_2)$ can be identified as the stress $\tilde{\sigma}_{33}(\xi_1, \xi_2, 0^+)$, $\forall (\xi_1, \xi_2) \in S$ and $|\boldsymbol{x} - \boldsymbol{\xi}|$ is a distance between the field point $\boldsymbol{x} = (x_1, x_2, x_3)$ and the integration point $\boldsymbol{\xi} = (\xi_1, \xi_2, 0)$. Due to the well-known properties of the potential of a simple layer (given by $(18)_2$), the last condition in (17) is satisfied, and the first one leads to the following integral equation for the stress $q \equiv \tilde{\sigma}_{33}|_{S^+}$:

$$H \iint_S \frac{\tilde{\sigma}_{33}(\xi_1, \xi_2)\, d\xi_1\, d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} = -\hat{u}_3(x_1, x_2), \tag{19}$$

where the following notations are used:

$$\begin{aligned}
H &= \frac{m_2 t_2 - m_1 t_1}{2\pi c_{44}(m_2 - m_1)}, \\
\hat{u}_3(x_1, x_2) &= -\overset{0}{u}_3(x_1, x_2, 0^+) + \varepsilon_3 - \omega_2 x_1 + \omega_1 x_2.
\end{aligned} \tag{20}$$

It is noteworthy that this governing equation has a similar form as that arising in the classical contact problem (Fabrikant, 1989). Generally, it can be solved by numerical methods. However, in the case where S is an ellipse and $\hat{u}_3$ is an arbitrary polynomial, a closed-form exact solution can be obtained (Rahman, 2002).

Subproblem (B)

The mathematical formulation satisfying the conditions (10) is more complex that than used in subproblem (A) and requires the introduction of two harmonic functions $G(x_1, x_2, x_3)$, $H(x_1, x_2, x_3)$ such that their relationships to ϕ_i in Eqs. (12) are

$$\phi_1(x_1,x_2,z_1) = -\frac{m_2t_2}{m_2t_2-m_1t_1}F_1(x_1,x_2,z_1),$$
$$\phi_2(x_1,x_2,z_2) = \frac{m_1t_1}{m_2t_2-m_1t_1}F_2(x_1,x_2,z_2), \quad (21)$$
$$\phi_3(x_1,x_2,z_3) = F_3(x_1,x_2,z_3),$$

where

$$F_1(x_1,x_2,z_1) = \frac{\partial G(x_1,x_2,z_1)}{\partial x_1} + \frac{\partial H(x_1,x_2,z_1)}{\partial x_2},$$
$$F_2(x_1,x_2,z_2) = \frac{\partial G(x_1,x_2,z_2)}{\partial x_1} + \frac{\partial H(x_1,x_2,z_2)}{\partial x_2}, \quad (22)$$
$$F_3(x_1,x_2,z_3) = \frac{\partial G(x_1,x_2,z_3)}{\partial x_2} - \frac{\partial H(x_1,x_2,z_3)}{\partial x_1}.$$

Inserting Eqs (21) into (12) yields the displacements

$$\tilde{u}_1 = \frac{[m_1t_1F_2-m_2t_2F_1]_{,1}}{m_2t_2-m_1t_1} - \frac{\partial F_3}{\partial x_2},$$
$$\tilde{u}_2 = \frac{[m_1t_1F_2-m_2t_2F_1]_{,2}}{m_2t_2-m_1t_1} + \frac{\partial F_3}{\partial x_1}, \quad (23)$$
$$\tilde{u}_3 = \frac{t_1t_2}{m_2t_2-m_1t_1}\left(\frac{\partial F_2}{\partial z_2} - \frac{\partial F_1}{\partial z_1}\right).$$

By the same procedures, the corresponding stress components are found from Eqs (13) as

$$\frac{\tilde{\sigma}_{31}}{c_{44}} = \frac{t_1t_2}{m_2t_2-m_1t_1}\left[(1+m_1)\frac{\partial^2F_2}{\partial x_1\partial z_2} - (1+m_2)\frac{\partial^2F_1}{\partial x_1\partial z_1}\right] - t_3\frac{\partial^2F_3}{\partial x_2\partial z_3}, \quad (24)$$

$$\frac{\tilde{\sigma}_{32}}{c_{44}} = \frac{t_1t_2}{m_2t_2-m_1t_1}\left[(1+m_1)\frac{\partial^2F_2}{\partial x_2\partial z_2} - (1+m_2)\frac{\partial^2F_1}{\partial x_2\partial z_1}\right] + t_3\frac{\partial^2F_3}{\partial x_1\partial z_3},$$

$$\frac{\tilde{\sigma}_{33}}{c_{44}} = \frac{1}{m_2t_2-m_1t_1}\left[t_1(1+m_1)\frac{\partial^2F_2}{\partial z_2^2} - t_2(1+m_2)\frac{\partial^2F_1}{\partial z_1^2}\right],$$

$$\frac{\tilde{\sigma}_{12}^{(l)}}{\mu_l} = \frac{2[m_1t_1F_2-m_2t_2F_1]_{,12}}{m_2t_2-m_1t_1} + F_{3,11} - F_{3,22},$$

$$\tilde{\sigma}_{11}^{(l)} = \frac{d_{11}^{(l)}[m_1t_1F_2-m_2t_2F_1]_{,11} + d_{12}^{(l)}[m_1t_1F_2-m_2t_2F_1]_{,22}}{m_2t_2-m_1t_1} + \frac{d_{13}^{(l)}t_1t_2}{m_2t_2-m_1t_1}\left[t_2\frac{\partial^2F_2}{\partial z_2^2} - t_1\frac{\partial^2F_1}{\partial z_1^2}\right] - 2\mu_lF_{3,12},$$

$$\tilde{\sigma}_{22}^{(l)} = \frac{d_{12}^{(l)}[m_1t_1F_2-m_2t_2F_1]_{,11} + d_{11}^{(l)}[m_1t_1F_2-m_2t_2F_1]_{,22}}{m_2t_2-m_1t_1} + \frac{d_{13}^{(l)}t_1t_2}{m_2t_2-m_1t_1}\left[t_2\frac{\partial^2F_2}{\partial z_2^2} - t_1\frac{\partial^2F_1}{\partial z_1^2}\right] + 2\mu_lF_{3,12}.$$

The above expressions simplify considerable on the plane $x_3 = 0$ (then $\forall i,\ z_i = 0,\ F_i(x_1,x_2,z_i) = F_i(x_1,x_2,x_3)$ $\frac{\partial F_i(x_1,x_2,z_i)}{\partial z_i} = \frac{\partial F_i(x_1,x_2,x_3)}{\partial x_3}$, and $F_1 = F_2 = G_{,1} + H_{,2}$, $F_3 = G_{,2} - H_{,1}$). Moreover, by letting

$$g(x_1,x_2,x_3) = \frac{\partial G(x_1,x_2,x_3)}{\partial x_3},$$
$$h(x_1,x_2,x_3) = \frac{\partial H(x_1,x_2,x_3)}{\partial x_3}, \quad (25)$$

Eqs (23) and (24) yield the displacement and stress components across the plane of symmetry $x_3 = 0$

$$\tilde{u}_1 = [g_{,3}]_{x_3=0},$$
$$\tilde{u}_2 = [h_{,3}]_{x_3=0},$$
$$\tilde{u}_3 = 0,$$
$$\frac{\tilde{\sigma}_{31}^{\pm}}{C^*} = [g_{,33} + \kappa g_{,22} - \kappa h_{,12}]_{x_3=0^{\pm}}, \quad (26)$$
$$\frac{\tilde{\sigma}_{32}^{\pm}}{C^*} = [h_{,33} + \kappa h_{,11} - \kappa g_{,12}]_{x_3=0^{\pm}},$$
$$\tilde{\sigma}_{33} = -c_{44}\left(1 + \frac{t_-}{m_2t_2-m_1t_2}\right)[g_{,31} + h_{,32}]_{x_3=0},$$
$$\tilde{\sigma}_{12}^{(l)} = \mu_l[g_{,32} + h_{,31}]_{x_3=0}.$$

Expressions for $\tilde{\sigma}_{11}^{(l)}$ and $\tilde{\sigma}_{22}^{(l)}$ have been omitted because of their complexity. The constants C^* and κ stand for

$$C^* = c_{44}\frac{t_1t_2(m_2-m_1)}{m_2t_2-m_1t_1}, \quad \kappa = 1 - \frac{t_3}{C^*}. \quad (27)$$

We see that the boundary value problem posed by Eqs (10) is equivalent to that of finding two harmonic functions g and h in $x_3 \geq 0$ such that their partial derivatives up to the third order vanish at infinity and satisfies the following mixed conditions on $x_3 = 0$:

– for $(x_1,x_2) \in S$

$$[g_{,3}(x_1,x_2,x_3)]_{x_3=0^+} = -\overset{0}{u}_1 + \varepsilon_1 - \omega_3x_2, \quad \forall(x_1,x_2) \in S,$$
$$[h_{,3}(x_1,x_2,x_3)]_{x_3=0^+} = -\overset{0}{u}_2 + \varepsilon_2 + \omega_3x_1, \quad \forall(x_1,x_2) \in S,$$

– for $(x_1,x_2) \in R^2 - S$ (28)

$$[g_{,33} + \kappa g_{,22} - \kappa h_{,12}]_{x_3=0^+} = 0,$$
$$[h_{,33} + \kappa h_{,11} - \kappa g_{,12}]_{x_3=0^+} = 0.$$

It can be observed that this formulation is dual to the well-known obtained for the shear loading crack problem (see Kassir and Sih, 1975). To solve the problem, we make use of the integral method developed by Kaczyński (1999). The harmonic functions g and h are expressed as Fourier's integrals (Sneddon, 1972)

$$\begin{bmatrix} g(\boldsymbol{x}) \\ h(\boldsymbol{x}) \end{bmatrix} = \iint_{R^2} \frac{\exp\left[-x_3|\boldsymbol{\xi}| + i(x_\alpha \xi_\alpha)\right]}{|\boldsymbol{\xi}|^2} \begin{bmatrix} A_g(\boldsymbol{\xi}) \\ A_h(\boldsymbol{\xi}) \end{bmatrix} dS_{\boldsymbol{\xi}}, \tag{29}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, 0) \in S$, $|\boldsymbol{\xi}| = \sqrt{\xi_1^2 + \xi_2^2}$ and the unknown functions A_g and A_h, in view of (28), must satisfy the following system

$$\begin{bmatrix} 1 - \dfrac{\kappa \xi_2^2}{|\boldsymbol{\xi}|^2} & \dfrac{\kappa \xi_1 \xi_2}{|\boldsymbol{\xi}|^2} \\ \dfrac{\kappa \xi_1 \xi_2}{|\boldsymbol{\xi}|^2} & 1 - \dfrac{\kappa \xi_1^2}{|\boldsymbol{\xi}|^2} \end{bmatrix} \begin{bmatrix} A_g(\boldsymbol{\xi}) \\ A_h(\boldsymbol{\xi}) \end{bmatrix} = \frac{1}{4\pi^2 C^*} \iint_S \begin{bmatrix} \sigma_{31}(\eta_1, \eta_2) \\ \sigma_{32}(\eta_1, \eta_2) \end{bmatrix} \exp\left[-i(\eta_\alpha \xi_\alpha)\right] dS_{\boldsymbol{\eta}}. \tag{30}$$

Its solution is

$$4\pi^2 C^* (1-\kappa) \begin{bmatrix} A_g(\boldsymbol{\xi}) \\ A_h(\boldsymbol{\xi}) \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{\kappa \xi_1^2}{|\boldsymbol{\xi}|^2} & -\dfrac{\kappa \xi_1 \xi_2}{|\boldsymbol{\xi}|^2} \\ -\dfrac{\kappa \xi_1 \xi_2}{|\boldsymbol{\xi}|^2} & 1 - \dfrac{\kappa \xi_2^2}{|\boldsymbol{\xi}|^2} \end{bmatrix} \iint_S \begin{bmatrix} \sigma_{31}(\boldsymbol{\eta}) \\ \sigma_{32}(\boldsymbol{\eta}) \end{bmatrix} \exp\left[-i(\eta_\alpha \xi_\alpha)\right] dS_{\boldsymbol{\eta}}. \tag{31}$$

Now, making use of these expressions it follows from Eqs (29) that (see Silovanyuk, 1984 and Kaczyński, 1999 for more details)

$$\begin{aligned} C g_{,3}(\boldsymbol{x}) &= \iint_S \frac{\tilde{\sigma}_{31}(\boldsymbol{\xi}) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x} - \boldsymbol{\xi}|} - \kappa \iint_S \frac{\tilde{\sigma}_{31}(\boldsymbol{\xi})(x_2 - \xi_2)^2 dS_{\boldsymbol{\xi}}}{|\boldsymbol{x} - \boldsymbol{\xi}|^3} + \kappa \iint_S \frac{\tilde{\sigma}_{32}(\boldsymbol{\xi})(x_1 - \xi_1)(x_2 - \xi_2) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x} - \boldsymbol{\xi}|^3}, \\ C h_{,3}(\boldsymbol{x}) &= \iint_S \frac{\tilde{\sigma}_{32}(\boldsymbol{\xi}) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x} - \boldsymbol{\xi}|} - \kappa \iint_S \frac{\tilde{\sigma}_{32}(\boldsymbol{\xi})(x_1 - \xi_1)^2 dS_{\boldsymbol{\xi}}}{|\boldsymbol{x} - \boldsymbol{\xi}|^3} + \kappa \iint_S \frac{\tilde{\sigma}_{31}(\boldsymbol{\xi})(x_1 - \xi_1)(x_2 - \xi_2) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x} - \boldsymbol{\xi}|^3} \end{aligned} \tag{32}$$

where

$$C = -2\pi(1-\kappa)C^* = -2\pi t_3. \tag{33}$$

Finally, from the first conditions of (28), one obtains the following integral equations for the stresses $\tilde{\sigma}_{3\alpha}|_{S^+}$:

$$\begin{aligned} &\iint_S \frac{\tilde{\sigma}_{31}(\boldsymbol{\xi}) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x}^* - \boldsymbol{\xi}|} - \kappa \iint_S \frac{\tilde{\sigma}_{31}(\boldsymbol{\xi})(x_2 - \xi_2)^2 dS_{\boldsymbol{\xi}}}{|\boldsymbol{x}^* - \boldsymbol{\xi}|^3} + \kappa \iint_S \frac{\tilde{\sigma}_{32}(\boldsymbol{\xi})(x_1 - \xi_1)(x_2 - \xi_2) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x}^* - \boldsymbol{\xi}|^3} = -2\pi t_3 \hat{u}_1(x_1, x_2), \\ &\iint_S \frac{\tilde{\sigma}_{32}(\boldsymbol{\xi}) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x}^* - \boldsymbol{\xi}|} - \kappa \iint_S \frac{\tilde{\sigma}_{32}(\boldsymbol{\xi})(x_1 - \xi_1)^2 dS_{\boldsymbol{\xi}}}{|\boldsymbol{x}^* - \boldsymbol{\xi}|^3} + \kappa \iint_S \frac{\tilde{\sigma}_{31}(\boldsymbol{\xi})(x_1 - \xi_1)(x_2 - \xi_2) dS_{\boldsymbol{\xi}}}{|\boldsymbol{x}^* - \boldsymbol{\xi}|^3} = -2\pi t_3 \hat{u}_2(x_1, x_2), \end{aligned} \tag{34}$$

where the following notations were introduced:

$$\begin{aligned} &\boldsymbol{x}^* = (x_1, x_2) \in S, \ \boldsymbol{\xi} = (\xi_1, \xi_2) \in S, \\ &\hat{u}_1(x_1, x_2) = -\overset{0}{u}_1(x_1, x_2, 0^+) + \varepsilon_1 - \omega_3 x_2, \\ &\hat{u}_2(x_1, x_2) = -\overset{0}{u}_2(x_1, x_2, 0^+) + \varepsilon_2 + \omega_3 x_1. \end{aligned} \tag{35}$$

Note that the form of (34) is similar to that given for the corresponding homogeneous isotropic problem. Moreover, it is verified that the derived governing integral equations are in agreement with those achieved by Silovanyuk (1984) in the homogeneous case. Knowing the stresses $\tilde{\sigma}_{3\alpha}(x_1, x_2)$ acting on the side S^+ of the rigid inclusion from the solution of Eqs (34), the stress and displacement fields can be found from the main potentials g and h, determined by means of (32).

4. EXAMPLE: ANTICRACK UNDER TENSION

For illustration, presented is a solution to the problem of a rigid circularly-shaped interface inclusion (such that $S = \left\{(x_1, x_2, 0) : r \equiv \sqrt{x_1^2 + x_2^2} \le a\right\}$) in a periodic two-layer laminated space subjected to a constant normal stress p at infinity (see Fig. 1), i.e.

$$\sigma_{33}(\infty) = p, \ \sigma_{31}(\infty) = \sigma_{32}(\infty) = 0. \tag{36}$$

The results for the 0-displacements of the inclusion-free problem involving the solution of the basic equations (4) and (5) with conditions (36) are readily obtained to be

$$\begin{aligned} &\overset{0}{u}_1 = -A p x_1, \quad \overset{0}{u}_2 = -A p x_2, \\ &\overset{0}{u}_3 = A_3 p x_3, \end{aligned} \tag{37}$$

where

$$A = \frac{c_{13}}{c_{33}\left(c_{11}+c_{12}\right)-2c_{13}^2}\ ,\qquad A_3 = \frac{c_{11}+c_{12}}{c_{33}\left(c_{11}+c_{12}\right)-2c_{13}^2}\ . \tag{38}$$

Now invoking the displacements in Eqs (37) on the plane $x_3 = 0$, we deduce by consideration of symmetry in (8) that $\varepsilon_3 = \omega_1 = \omega_2 = 0$. Thus, we proceed to solving the sub-problem B (cf (10) and (11)) in which the appropriate conditions are

$$\begin{aligned}
&\tilde{u}_1\left(x_1,x_2,0^+\right) = A\,p\,x_1+\varepsilon_1-\omega_3x_2,\quad \forall\left(x_1,x_2\right)\in S,\\
&\tilde{u}_2\left(x_1,x_2,0^+\right) = A\,p\,x_2+\varepsilon_2+\omega_3x_1,\ \forall\left(x_1,x_2\right)\in S,\\
&\tilde{u}_3\left(x_1,x_2,0^+\right) = 0,\ \ \forall\left(x_1,x_2\right)\in R^2,\\
&\tilde{\sigma}_{31}\left(x_1,x_2,0^+\right) = \tilde{\sigma}_{32}\left(x_1,x_2,0^+\right) = 0, \forall\left(x_1,x_2\right)\in R^2 - S\ ,\\
&\tilde{u}_i = \mathrm{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\ \text{ as }\ |\boldsymbol{x}|\to\infty\ .
\end{aligned} \tag{39}$$

This problem is reduced to the set of coupled two-dimensional integral equations (34) in which the right sides are determined by the following polynomials:

$$\begin{aligned}
\hat{u}_1\left(x_1,x_2\right) &= \varepsilon_1 + A\,p\,x_1-\omega_3x_2\ ,\\
\hat{u}_2\left(x_1,x_2\right) &= \varepsilon_2 + \omega_3x_1 + A\,p\,x_2\ .
\end{aligned} \tag{40}$$

An exact solution of these integral equations, obtained by using Galin's theorem, has the form

$$\tilde{\sigma}_{3\alpha}\left(x_1,x_2\right) = \frac{b_{0\alpha}+b_{1\alpha}x_1+b_{2\alpha}x_2}{\sqrt{a^2-r^2}}\ ,\ \forall\alpha\in\{1,2\}, \tag{41}$$

where $b_{i\alpha},\ i\in\{0,1,2\},\alpha\in\{1,2\}$ are the unknown constants to be determined. Putting (41) into (34) and calculating the resulting integrals (see Vorovich et al.,1974), we arrive at the equalities of two polynomials. Hence, a system of algebraic equations for $b_{i\alpha}$ can be obtained, and their solving yields

$$\begin{aligned}
b_{0\alpha} &= -\frac{4t_3}{\pi(2-k)}\,\varepsilon_\alpha,\ \ \forall\ \alpha\in\{1,2\},\\
b_{12} &= b_{21} = \frac{4t_3}{\pi}\,\omega_3,\\
b_{11} &= b_{22} = -\frac{4}{\pi}AC^*p\ .
\end{aligned} \tag{42}$$

If we now make use of the equilibrium conditions (11) on the anticrack we find (as might be expected) that

$$\varepsilon_1 = \varepsilon_2 = \omega_3 = 0 \tag{43}$$

and the solution given by (41) can be written in the simple form

$$\tilde{\sigma}_{3\alpha}\left(x_1,x_2\right) = B\,\frac{x_\alpha}{\sqrt{a^2-r^2}}\ ,\ \left(x_1,x_2,0^+\right)\in S, \tag{44}$$

where

$$B = -\frac{4}{\pi}AC^*p\,. \tag{45}$$

Accordingly, the problem is axially symmetric and the full elastic field is determined if we find the main potentials g and h (see (21) - (25)). Substituting (44) into (32) we get

$$\begin{aligned}
g_{,3} &= \frac{B}{C}\left(\varphi+\kappa x_2\,\varphi_{,2}-\kappa x_1\,\psi_{,2}\right),\\
h_{,3} &= \frac{B}{C}\left(\psi+\kappa x_1\,\psi_{,2}-\kappa x_2\,\varphi_{,1}\right).
\end{aligned} \tag{46}$$

Here φ and ψ are the potentials of simple layers defined as

$$\begin{aligned}
\varphi(\boldsymbol{x}) &= \iint_S \frac{\xi_1\,d\xi_1\,d\xi_2}{\sqrt{\left(x_1-\xi_1\right)^2+\left(x_2-\xi_2\right)^2+x_3^2}\ \sqrt{a^2-\xi_1^2-\xi_2^2}}\ ,\\
\psi(\boldsymbol{x}) &= \iint_S \frac{\xi_2\,d\xi_1\,d\xi_2}{\sqrt{\left(x_1-\xi_1\right)^2+\left(x_2-\xi_2\right)^2+x_3^2}\ \sqrt{a^2-\xi_1^2-\xi_2^2}}\ ,
\end{aligned} \tag{47}$$

for which the method of Fabrikant (1989) yields the explicit results in elementary functions as follows

$$\begin{aligned}
\varphi(\boldsymbol{x}) &= \pi\,x_1\left(\arcsin\frac{a}{l_2}-\frac{a\sqrt{l_2^2-a^2}}{l_2^2}\right),\\
\psi(\boldsymbol{x}) &= \pi\,x_2\left(\arcsin\frac{a}{l_2}-\frac{a\sqrt{l_2^2-a^2}}{l_2^2}\right),
\end{aligned} \tag{48}$$

where in his notation

$$\begin{aligned}
l_1 &\equiv l_1\left(a,r,x_3\right) = \frac{1}{2}\left[\sqrt{(r+a)^2+x_3^2}-\sqrt{(r-a)^2+x_3^2}\right],\\
l_2 &\equiv l_2\left(a,r,x_3\right) = \frac{1}{2}\left[\sqrt{(r+a)^2+x_3^2}+\sqrt{(r-a)^2+x_3^2}\right].
\end{aligned} \tag{49}$$

All the necessary partial derivatives or some integrals of potentials (48) can be found in Appendix 5 of the book by Fabrikant (1991), which allows us to write a complete solution to problem under study.

It is of interest to record and discuss the relevant interfacial stresses in the plane of the anticrack. They are given below:

$$\tilde{\sigma}_{3r}\left(r,0^\pm\right) = \begin{cases} \mp\dfrac{4\beta_r\ p\,r}{\pi\sqrt{a^2-r^2}}\ , & 0\le r<a,\\ 0, & r>a, \end{cases} \tag{50}$$

$$\hat{\sigma}_{33}\left(r,0^\pm\right) = \begin{cases} -\beta_3\,p, & 0\le r<a,\\ \dfrac{2\beta_3\ p}{\pi}\left(\dfrac{a}{\sqrt{r^2-a^2}}-\arcsin\dfrac{a}{r}\right), & r>a\ , \end{cases} \tag{51}$$

where (see Appendices)

$$\beta_r = AC^* = \frac{c_{11}\,c_{33}\,c_{44}\,t_+}{\left[c_{33}\left(c_{11}+c_{12}\right)-2c_{13}^2\right]\left(c_{11}+c_{44}\right)}, \qquad (52)$$

$$\beta_3 = \frac{2c_{13}\,c_{44}\left(\sqrt{c_{11}\,c_{33}}-c_{13}\right)}{\left(\sqrt{c_{11}\,c_{33}}+c_{44}\right)\left[c_{33}\left(c_{11}+c_{12}\right)-2c_{13}^2\right]}\,. \qquad (53)$$

Now, it is significant to observe that the singularity of the stresses close to the edge of the anticrack has the order $r^{-1/2}$, contrary to oscillatory type observed in the elastic fields relating to bimaterial interfaces. From the standpoint of classical fracture mechanics, two failure mechanisms are possible:

– separation of the material from the inclusion characterized by the stress singularity coefficients

$$S_{\mathrm{II}}^{\pm} = \lim_{r\to a^-}\sqrt{2\pi\left(a-r\right)}\,\tilde{\sigma}_{3r}\left(r,0^{\pm}\right) = \mp\frac{4\beta_r\,p\sqrt{a}}{\sqrt{\pi}} \qquad (54)$$

– mode I (edge-opening) deformation characterized by the stress intensity factor

$$K_{\mathrm{I}} = \lim_{r\to a^+}\sqrt{2\pi\left(r-a\right)}\,\sigma_{33}\left(r,0\right) = \frac{2\beta_3 p\sqrt{a}}{\sqrt{\pi}}\,. \qquad (55)$$

These parameters may be used to the determination of the limiting equilibrium of the considered composite weakened by the anticrack (see, e.g. Rahman, 2002).

Finally, the solution to the corresponding homogeneous material problem is the special case when $\lambda_1=\lambda_2\equiv\lambda,\ \mu_1=\mu_2\equiv\mu$, and hence $c_{11}=c_{33}=\lambda+2\mu$, $c_{12}=c_{13}=\lambda$, $c_{44}=\mu$. Then Eqs (52) and (53) become

$$\beta_r = \frac{\lambda\left(\lambda+2\mu\right)}{\left(\lambda+3\mu\right)\left(3\lambda+2\mu\right)} = \frac{2\nu\left(1-\nu\right)}{\left(1+\nu\right)\left(3-4\nu\right)},$$
$$\beta_3 = \frac{2\lambda\mu}{\left(\lambda+3\mu\right)\left(3\lambda+2\mu\right)} = \frac{2\nu\left(1-2\nu\right)}{\left(1+\nu\right)\left(3-4\nu\right)} \qquad (56)$$

with $\nu = \dfrac{\lambda}{2\left(\lambda+\mu\right)}$ being Poisson's ratio, and the results are in agreement with those obtained differently by Kassir and Sih (1968).

APPENDIX A

Denoting by

$$b_l = \lambda_l + 2\mu_l \quad \left(l=1,2\right), \quad b = \left(1-\eta\right)b_1 + \eta\, b_2\,,$$

the positive coefficients in governing equations (4) and (5) are given by the following formulas:

$$c_{33} = b_1 b_2 / b\,,$$
$$c_{11} = c_{33} + \left[4\eta\left(1-\eta\right)\left(\mu_1-\mu_2\right)\left(\lambda_1-\lambda_2+\mu_1-\mu_2\right)\right]/b\,,$$
$$c_{13} = \left[\left(1-\eta\right)\lambda_2\, b_1 + \eta\,\lambda_1\, b_2\right]/b\,,$$
$$c_{12} = \left\{\lambda_1\lambda_2 + 2\left[\eta\,\mu_2+\left(1-\eta\right)\mu_1\right]\left[\eta\,\lambda_1+\left(1-\eta\right)\lambda_2\right]\right\}/b\,,$$
$$c_{44} = \mu_1\mu_2 / \left[\left(1-\eta\right)\mu_1+\eta\,\mu_2\right],$$
$$d_{11}^{(l)} = \left[4\mu_l\left(\lambda_l+\mu_l\right)+\lambda_l\, c_{13}\right]/b_l\,,$$
$$d_{12}^{(l)} = \left[2\mu_l\,\lambda_l+\lambda_l\, c_{13}\right]/b_l\,,$$
$$d_{13}^{(l)} = \lambda_l\, c_{33}/b_l\,.$$

APPENDIX B

The constants appearing in Eqs (12) and (13) are given as follows:

$$t_1 = \tfrac{1}{2}\left(t_+ - t_-\right), \qquad t_2 = \tfrac{1}{2}\left(t_+ + t_-\right),$$
$$t_3 = \sqrt{\left(c_{11}-c_{12}\right)/2c_{44}}\,,$$
$$m_\alpha = \frac{c_{11}\,t_\alpha^{-2}-c_{44}}{c_{13}+c_{44}}, \quad \forall\alpha\in\left\{1,2\right\},$$

where

$$t_\pm = \sqrt{\frac{\left(A_\pm \pm 2c_{44}\right)A_\mp}{c_{33}\,c_{44}}}\,,$$
$$A_\pm = \sqrt{c_{11}\,c_{33}} \pm c_{13}\,.$$

Note that $t_1 t_2 = \sqrt{c_{11}/c_{33}}$, $m_1 m_2 = 1$.

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